FACULTY OF ENGINEERING
AND ARCHITECTURE

DEPARTMENT OF INFORMATION TECHNOLOGY
ELECTROMAGNETICS RESEARCH GROUP

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A TIME-DOMAIN FEM-BEM SYMMETRIC COUPLING FOR TRANSIENT ELECTROMAGNETIC SCATTERING*

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IDLab
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imec

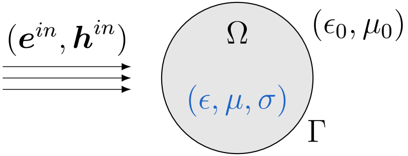
*A joint work with Prof. Kristof Cools.

Why Time-Domain FEM-BEM?

Goal: to investigate potential issues of the time-domain FEM-BEM coupling.

Why time-domain formulation?

- ✓ to model broadband systems
- ✓ to allow coupling with non-linear systems.



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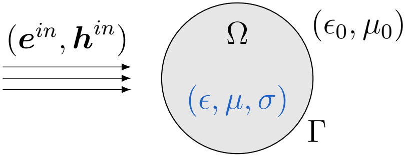

Why FEM-BEM coupling?

- ✓ FEM to capture inhomogeneous interior coefficients
- ✓ BEM to model homogeneous exterior material
- ✓ to easily accommodate lossy medium.

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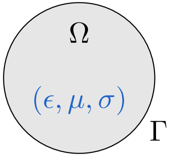
Interior (FEM) formulation

Maxwell's equation for interior electric field $\boldsymbol{e}$

$$\nabla \times \mu^{-1} \nabla \times \boldsymbol{e} + \sigma\,\partial_t \boldsymbol{e} + \epsilon\,\partial_{tt} \boldsymbol{e} = \mathbf{0} \qquad \text{in} \quad \Omega.$$



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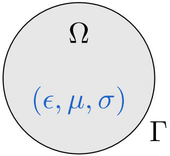

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Variational formulation: for any $\boldsymbol{\varphi}(t) \in \mathbf{H}(\mathbf{curl}, \Omega)$

$$\big(\nabla \times \boldsymbol{\varphi}, \mu^{-1} \nabla \times \boldsymbol{e}\big)_\Omega + (\boldsymbol{\varphi}, \sigma\, \partial_t \boldsymbol{e})_\Omega + (\boldsymbol{\varphi}, \epsilon\, \partial_{tt} \boldsymbol{e})_\Omega - \big\langle (\mathbf{n} \times \boldsymbol{\varphi}) \times \mathbf{n}, \mathbf{n} \times \partial_t \boldsymbol{h}^- \big\rangle_\Gamma = 0,$$

where $\mathbf{n}$ is the outward unit normal of Γ, and

$$(\boldsymbol{f}, \boldsymbol{g})_\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{g}\, \mathrm{d}\boldsymbol{x}, \qquad \langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Gamma = \int_\Gamma \boldsymbol{f} \cdot \boldsymbol{g}\, \mathrm{d}s_{\boldsymbol{x}}.$$

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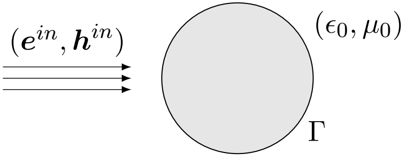
Exterior (BEM) formulation

Maxwell's equations for exterior $\boldsymbol{e}$ and $\boldsymbol{h}$

$$\mu_0^{-1}\nabla \times \nabla \times \boldsymbol{e} + \epsilon_0\, \partial_{tt}\boldsymbol{e} = \boldsymbol{0},$$
$$\epsilon_0^{-1}\nabla \times \nabla \times \boldsymbol{h} + \mu_0\, \partial_{tt}\boldsymbol{h} = \boldsymbol{0}.$$



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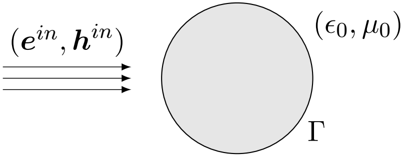

Calderón formula

$$\begin{pmatrix}-\mathbf{n}\times\boldsymbol{e}^{+}\\ \mathbf{n}\times\boldsymbol{h}^{+}\end{pmatrix}=\begin{pmatrix}\mathcal{K}+\frac{1}{2} & -\eta_0\mathcal{T}\\ \frac{1}{\eta_0}\mathcal{T} & \mathcal{K}+\frac{1}{2}\end{pmatrix}\begin{pmatrix}-\mathbf{n}\times\boldsymbol{e}^{+}\\ \mathbf{n}\times\boldsymbol{h}^{+}\end{pmatrix}+\begin{pmatrix}-\mathbf{n}\times\boldsymbol{e}^{in}\\ \mathbf{n}\times\boldsymbol{h}^{in}\end{pmatrix},$$

where $\eta_0=\sqrt{\mu_0/\epsilon_0},\ c_0=1/\sqrt{\mu_0\epsilon_0}$.

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Exterior (BEM) formulation

Time-domain boundary integral operators

$$\mathcal{K}\boldsymbol{j} = \mathbf{n} \times \mathbf{curl}\,(G * \boldsymbol{j}),$$

$$\mathcal{T}\boldsymbol{j} = -\frac{1}{c_0}\,\mathbf{n} \times \partial_t(G * \boldsymbol{j}) + c_0\,\mathbf{n} \times \mathbf{grad} \int_0^t (G * \mathrm{div}_\Gamma \boldsymbol{j})\,\mathrm{d}t,$$

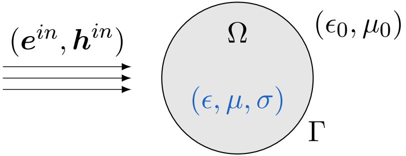
where the kernel

$$(G * \boldsymbol{j})(\boldsymbol{x}, t) := \int_\Gamma \frac{\boldsymbol{j}(\boldsymbol{y}, t - |\boldsymbol{x} - \boldsymbol{y}|/c_0)}{4\pi\,|\boldsymbol{x} - \boldsymbol{y}|}\,\mathrm{d}s_{\boldsymbol{y}}.$$

Costabel's symmetric coupling

Boundary conditions

$$\mathbf{n} \times \boldsymbol{e}^+ = \mathbf{n} \times \boldsymbol{e}^- \quad \text{on} \quad \Gamma,$$
$$\mathbf{n} \times \boldsymbol{h}^+ = \mathbf{n} \times \boldsymbol{h}^- \quad \text{on} \quad \Gamma.$$



Volumetric variational formulation

$$\left(\nabla \times \boldsymbol{\varphi}, \mu^{-1} \nabla \times \boldsymbol{e}\right)_\Omega + (\boldsymbol{\varphi}, \sigma\, \partial_t \boldsymbol{e})_\Omega + (\boldsymbol{\varphi}, \epsilon\, \partial_{tt} \boldsymbol{e})_\Omega - \left\langle (\mathbf{n} \times \boldsymbol{\varphi}) \times \mathbf{n}, \mathbf{n} \times \partial_t \boldsymbol{h}^- \right\rangle_\Gamma = 0.$$

Boundary integral formulation

$$\begin{pmatrix} -\mathbf{n} \times \boldsymbol{e}^+ \\ \mathbf{n} \times \boldsymbol{h}^+ \end{pmatrix} = \begin{pmatrix} \mathcal{K} + \frac{1}{2} & -\eta_0 \mathcal{T} \\ \frac{1}{\eta_0} \mathcal{T} & \mathcal{K} + \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\mathbf{n} \times \boldsymbol{e}^+ \\ \mathbf{n} \times \boldsymbol{h}^+ \end{pmatrix} + \begin{pmatrix} -\mathbf{n} \times \boldsymbol{e}^{in} \\ \mathbf{n} \times \boldsymbol{h}^{in} \end{pmatrix}.$$

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Costabel's symmetric coupling

Set $\boldsymbol{j} = \mathbf{n} \times \boldsymbol{h}$. Coupled FEM-BEM system reads

$$a_\Omega(\boldsymbol{\varphi}, \boldsymbol{e}) + \frac{1}{\eta_0} \langle (\mathbf{n} \times \boldsymbol{\varphi}) \times \mathbf{n}, \partial_t \mathcal{T} (\mathbf{n} \times \boldsymbol{e}) \rangle_\Gamma - \left\langle (\mathbf{n} \times \boldsymbol{\varphi}) \times \mathbf{n}, \left(\partial_t \mathcal{K} + \frac{1}{2} \partial_t \right) \boldsymbol{j} \right\rangle_\Gamma = - \langle \mathbf{n} \times \boldsymbol{\varphi}, \partial_t \boldsymbol{h}^{in} \rangle_\Gamma ,$$

$$\left\langle \mathbf{n} \times \boldsymbol{\lambda}, \left(\partial_t \mathcal{K} - \frac{1}{2} \partial_t \right) (\mathbf{n} \times \boldsymbol{e}) \right\rangle_\Gamma + \eta_0 \langle \mathbf{n} \times \boldsymbol{\lambda}, \partial_t \mathcal{T} \boldsymbol{j} \rangle_\Gamma = \langle \boldsymbol{\lambda}, \partial_t \boldsymbol{e}^{in} \rangle_\Gamma ,$$

for any $\boldsymbol{\varphi} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}(\text{div}, \Gamma)$, where

$$a_\Omega(\boldsymbol{\varphi}, \boldsymbol{e}) := (\nabla \times \boldsymbol{\varphi}, \mu^{-1} \nabla \times \boldsymbol{e})_\Omega + (\boldsymbol{\varphi}, \sigma \, \partial_t \boldsymbol{e})_\Omega + (\boldsymbol{\varphi}, \epsilon \, \partial_{tt} \boldsymbol{e})_\Omega .$$

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Costabel's symmetric coupling

In the matrix form

$$\left\langle \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\lambda} \end{pmatrix}, \begin{pmatrix} \mathcal{A}_\Omega + \frac{1}{\eta_0}\gamma_t^* \circ \partial_t \mathcal{T} \circ \gamma_D & -\gamma_t^* \circ \left(\partial_t \mathcal{K} + \frac{1}{2}\partial_t\right) \\ \gamma_D^* \circ \left(\partial_t \mathcal{K} - \frac{1}{2}\partial_t\right) \circ \gamma_D & \eta_0 \gamma_D^* \circ \partial_t \mathcal{T} \end{pmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{j} \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\lambda} \end{pmatrix}, \begin{pmatrix} \gamma_t^* \gamma_D \partial_t \boldsymbol{h}^{in} \\ \partial_t \boldsymbol{e}^{in} \end{pmatrix} \right\rangle,$$

where $\gamma_t \boldsymbol{\varphi} = (\mathbf{n} \times \boldsymbol{\varphi}) \times \mathbf{n}$ and $\gamma_D \boldsymbol{\varphi} = \mathbf{n} \times \boldsymbol{\varphi}$.

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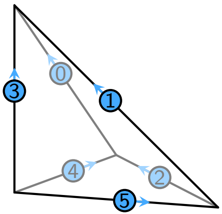
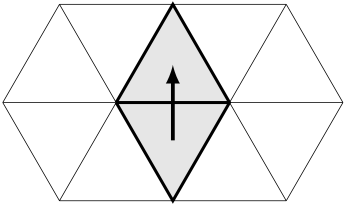
Discretization

Spatial and temporal trial functions

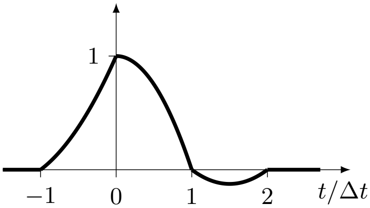
Nédélec element $\boldsymbol{f}(\boldsymbol{x})$



RWG function $\boldsymbol{g}(\boldsymbol{x})$



Temporal function $q(t)$



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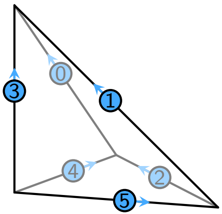

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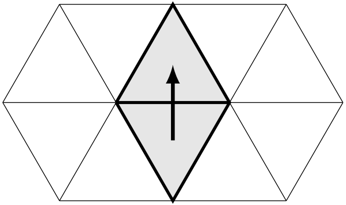

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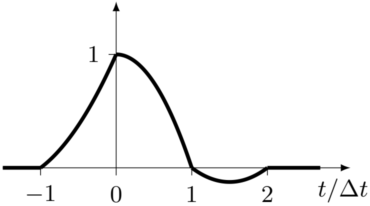

$$\boldsymbol{e}(\boldsymbol{x},t) \approx \sum_{i}^{N_T} \sum_{n}^{N_E^\Omega} [\mathrm{e}_i]_n \, q_i(t) \, \boldsymbol{f}_n(\boldsymbol{x}), \qquad \boldsymbol{j}(\boldsymbol{x},t) \approx \sum_{i}^{N_T} \sum_{n}^{N_E^\Gamma} [\mathrm{j}_i]_n \, q_i(t) \, \boldsymbol{g}_n(\boldsymbol{x}).$$

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Discretization

Collocation-in-time Galerkin-in-space testing

Test functions

$$\boldsymbol{\varphi}_{j,m} = \delta_j(t)\,\boldsymbol{f}_m(\boldsymbol{x}), \qquad \boldsymbol{\lambda}_{j,m} = \delta_j(t)\,\boldsymbol{g}_m(\boldsymbol{x}).$$

Testing scheme results in the marching-on-in-time system

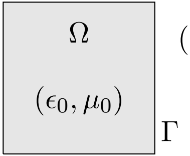
$$\begin{pmatrix} \mathbf{L}_0 & & & \\ \mathbf{L}_1 & \mathbf{L}_0 & & \\ \vdots & \vdots & \ddots & \\ \mathbf{L}_{N_T-1} & \mathbf{L}_{N_T-2} & \dots & \mathbf{L}_0 \end{pmatrix} \begin{pmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \\ \vdots \\ \mathrm{u}_{N_T} \end{pmatrix} = \begin{pmatrix} \mathrm{r}_1 \\ \mathrm{r}_2 \\ \vdots \\ \mathrm{r}_{N_T} \end{pmatrix},$$

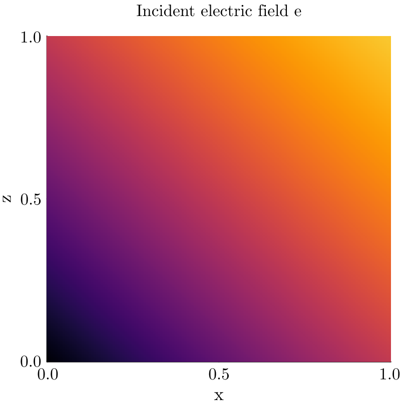
where the matrices

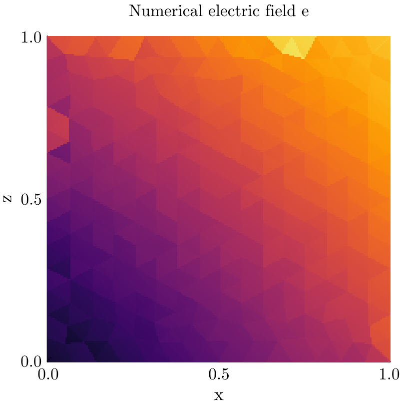
$$\mathbf{L}_i = \begin{pmatrix} \mathbf{A}_i + \eta_0^{-1}\dot{\tilde{\mathbf{T}}}_i & -\dot{\tilde{\mathbf{K}}}_i + \frac{1}{2}\dot{\tilde{\mathbf{I}}}_i \\ \dot{\hat{\mathbf{K}}}_i - \frac{1}{2}\dot{\hat{\mathbf{I}}}_i & -\eta_0\dot{\mathbf{T}}_i \end{pmatrix}, \qquad \mathrm{u}_i = \begin{pmatrix} \mathrm{e}_i \\ \mathrm{j}_i \end{pmatrix}.$$

Numerical results

Zero-contrast dielectric scattering







Plane-wave excitation:

- Direction: $(\hat{x} + \hat{z})/\sqrt{2}$
- Polarization: $\hat{y}$
- Gaussian-in-time.

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Numerical results

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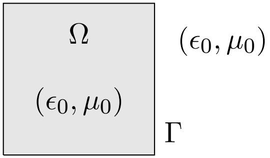
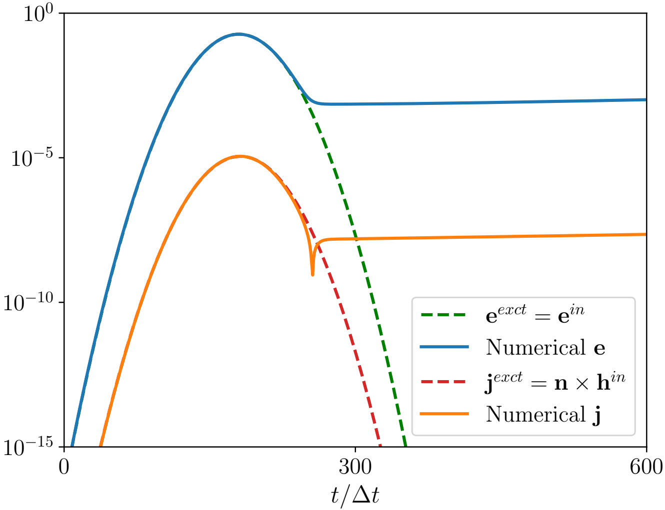


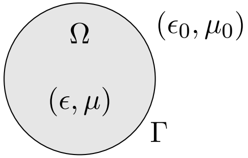


Properties of the system:

- ✓ Likely free of resonance
- × Ill-conditioned
- × Unstable at late times.

Numerical results

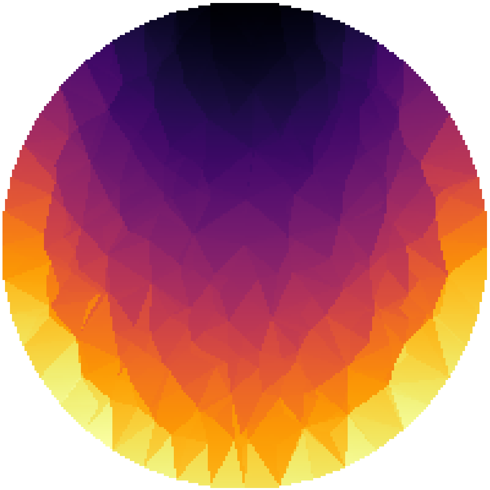
Scattering by Luneburg lens



Material coefficients:

- Permeability: $\mu = \mu_0$
- Permittivity:

$$\epsilon = \epsilon_0 \left(2 - \frac{r^2}{R^2}\right).$$



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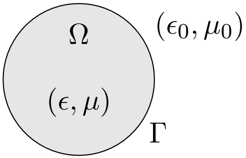

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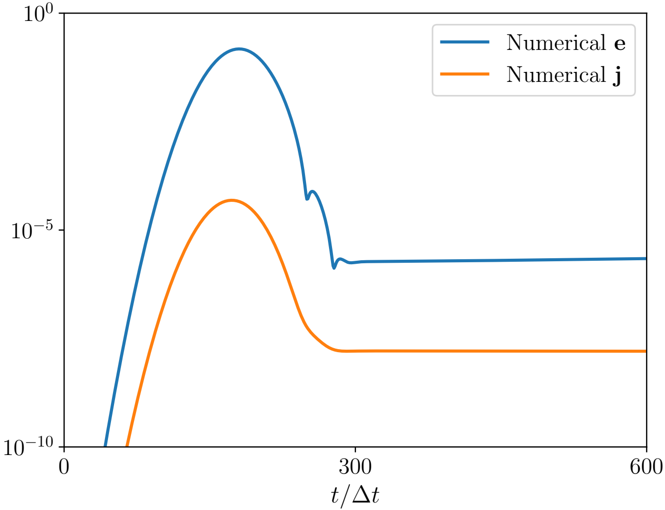
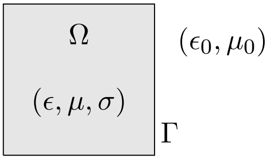


Numerical results

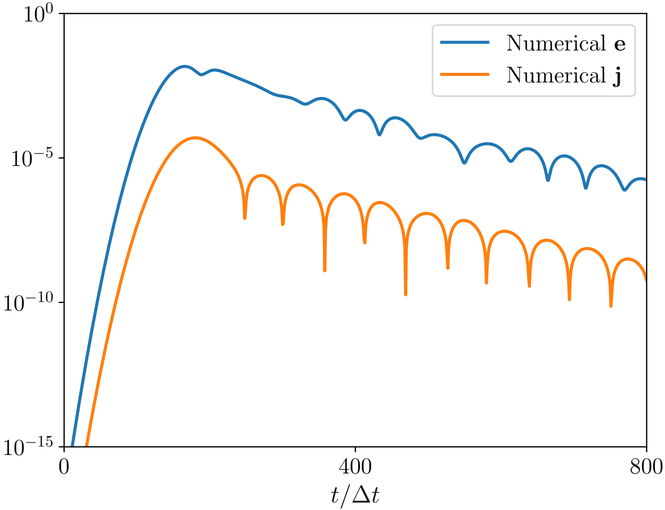
Scattering by lossy medium



Material coefficients:

- Permeability: $\mu = \mu_0$
- Permittivity: $\epsilon = 81\epsilon_0$
- Conductivity: $\sigma = 0.05\,\mathrm{S/m}$.

Property: slowly decaying.



Conclusions and future work

We have propose a time-domain FEM-BEM coupling scheme to

- ✓ model broadband and non-linear systems
- ✓ accommodate inhomogeneous dielectric and lossy medium
- ✓ serve as a benchmark for other solvers for lossy electromagnetic problems[1].

Future work:

- to further investigate the late-time behaviour of the solution
- to propose efficient preconditioners for the FEM-BEM coupling.

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[1]V. Giunzioni, A. Scazzola, A. Merlini, F. P. Andriulli, Low-frequency stabilizations of the PMCHWT equation for dielectric and conductive media: On a full-wave alternative to eddy-current solvers, *IEEE Trans. Antennas Propag.*, 2025.

References and acknowledgment

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