FACULTY OF ENGINEERING AND ARCHITECTURE

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A BOUNDARY ELEMENT METHOD FOR THE MAGNETIC FIELD INTEGRAL EQUATION IN ELECTROMAGNETICS*

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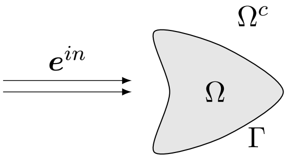
*A joint work with Prof. Kristof Cools

Electromagnetic scattering by perfect conductors

The scattered electric field $\boldsymbol{e}$ satisfies

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e} - \kappa^2 \boldsymbol{e} &= \mathbf{0} && \text{in} \quad \Omega^c, \\ \boldsymbol{e} \times \mathbf{n} &= -\boldsymbol{e}^{in} \times \mathbf{n} && \text{on} \quad \Gamma, \end{aligned}$$



and the Silver-Müller radiation condition

$$\lim_{r\to\infty} \int_{\partial B_r} \left| \mathbf{curl}\,\boldsymbol{e} \times \mathbf{n} + i\kappa(\mathbf{n} \times \boldsymbol{e}) \times \mathbf{n} \right|^2 \mathrm{d}s = 0.$$

Relich's lemma: The exterior problem has a unique solution for all $\kappa > 0$.

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Electric vs. magnetic field integral equations

EFIE (first-kind):

- ✓ satisfies the strong ellipticity
- × produces ill-conditioned matrices
- × suffers from resonant instability.

MFIE (second-kind):

- ✓ produces well-conditioned matrices
- × does not satisfy the strong ellipticity
- × suffers from resonant instability.

Why MFIE? Both EFIE and MFIE are crucial in

- combined field integral equations
- Maxwell transmission problems.

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Outline

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Function spaces

The Dirichlet and Neumann trace operators

$$\gamma_D \boldsymbol{u} := \boldsymbol{u} \times \mathbf{n}, \qquad \gamma_N := \gamma_D \circ \mathbf{curl}.$$

The trace space

$$\mathbf{H}^s_\times(\Gamma) := \gamma_D(\mathbf{H}^{s+1/2}(\Omega)), \qquad s \in (0,1),$$

and its dual $\mathbf{H}^{-s}_\times(\Gamma)$, whose elements are identified via

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\times,\Gamma} := \int_\Gamma (\boldsymbol{u} \times \mathbf{n}) \cdot \boldsymbol{v} \, \mathrm{d}s.$$

The natural trace space

$$\mathbf{H}^{-1/2}_\times(\mathrm{div}_\Gamma, \Gamma) := \left\{ \boldsymbol{u} \in \mathbf{H}^{-1/2}_\times(\Gamma) : \mathrm{div}_\Gamma \boldsymbol{u} \in \mathrm{H}^{-1/2}(\Gamma) \right\}.$$

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Potential operators

Let $\sigma = \kappa$ or $\sigma = i\kappa'$, with $\kappa, \kappa' > 0$. The fundamental solution associated with $\Delta + \sigma^2$

$$G_\sigma(\boldsymbol{x}, \boldsymbol{y}) := \frac{\exp(i\sigma\,|\boldsymbol{x} - \boldsymbol{y}|)}{4\pi\,|\boldsymbol{x} - \boldsymbol{y}|}, \qquad \boldsymbol{x} \neq \boldsymbol{y}.$$

The scalar and vectorial single layer potentials

$$\Psi_V^\sigma(\varphi)(\boldsymbol{x}) := \int_\Gamma \varphi(\boldsymbol{y}) G_\sigma(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}), \qquad \boldsymbol{\Psi}_{\boldsymbol{A}}^\sigma(\boldsymbol{u})(\boldsymbol{x}) := \int_\Gamma \boldsymbol{u}(\boldsymbol{y}) G_\sigma(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}).$$

The Maxwell single and double layer potentials

$$\boldsymbol{\Psi}_{SL}^\sigma(\boldsymbol{u}) := \boldsymbol{\Psi}_{\boldsymbol{A}}^\sigma(\boldsymbol{u}) + \frac{1}{\sigma^2}\,\mathbf{grad}\,\Psi_V^\sigma(\mathrm{div}_\Gamma \boldsymbol{u}), \qquad \boldsymbol{\Psi}_{DL}^\sigma(\boldsymbol{u}) := \mathbf{curl}\,\boldsymbol{\Psi}_{\boldsymbol{A}}^\sigma(\boldsymbol{u}).$$

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Potential operators (cont.)

For any $\boldsymbol{u} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, the potentials $\boldsymbol{\Psi}^{\sigma}_{SL}(\boldsymbol{u})$ and $\boldsymbol{\Psi}^{\sigma}_{DL}(\boldsymbol{u})$ are solutions to

$$\begin{aligned} &\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e} - \sigma^2 \boldsymbol{e} = \mathbf{0} && \text{in} \quad \Omega \cup \Omega^c, \\ &\lim_{r\to\infty} \int_{\partial B_r} \left|\mathbf{curl}\,\boldsymbol{e} \times \mathbf{n} + i\sigma(\mathbf{n} \times \boldsymbol{e}) \times \mathbf{n}\right|^2 \,\mathrm{d}s = 0. \end{aligned}$$

Every solution $\boldsymbol{e} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega \cup \Omega^c)$ satisfies the Stratton-Chu representation formula

$$\boldsymbol{e}(\boldsymbol{x}) = -\boldsymbol{\Psi}^{\sigma}_{DL}([\gamma_D]_\Gamma\, \boldsymbol{e})(\boldsymbol{x}) - \boldsymbol{\Psi}^{\sigma}_{SL}([\gamma_N]_\Gamma\, \boldsymbol{e})(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \cup \Omega^c,$$

where $[\gamma]_\Gamma := \gamma^+ - \gamma^-$.

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Boundary integral operators

Definitions

Let us define

$$\begin{aligned} S_\sigma &:= \{\gamma_D\}_\Gamma \circ \boldsymbol{\Psi}^\sigma_{SL} &&: \mathbf{H}_\times^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \to \mathbf{H}_\times^{-1/2}(\mathsf{div}_\Gamma, \Gamma), \\ C_\sigma &:= \{\gamma_N\}_\Gamma \circ \boldsymbol{\Psi}^\sigma_{SL} &&: \mathbf{H}_\times^{-1/2}(\mathsf{div}_\Gamma, \Gamma) \to \mathbf{H}_\times^{-1/2}(\mathsf{div}_\Gamma, \Gamma). \end{aligned}$$

Taking into account the jump relation

$$[\gamma_D]_\Gamma \circ \boldsymbol{\Psi}^\sigma_{SL} = 0, \qquad [\gamma_N]_\Gamma \circ \boldsymbol{\Psi}^\sigma_{SL} = -Id,$$

we end up with

$$\begin{aligned} \gamma_D^\pm \circ \boldsymbol{\Psi}^\sigma_{SL} &= \frac{1}{\sigma^2}\gamma_N^\pm \circ \boldsymbol{\Psi}^\sigma_{DL} = S_\sigma, \\ \gamma_N^\pm \circ \boldsymbol{\Psi}^\sigma_{SL} &= \gamma_D^\pm \circ \boldsymbol{\Psi}^\sigma_{DL} = \mp\frac{1}{2}Id + C_\sigma. \end{aligned}$$

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Boundary integral operators

Properties of $S_{i\kappa'}$

Symmetry:

$$\langle \boldsymbol{v}, S_{i\kappa'} \boldsymbol{u} \rangle_{\times,\Gamma} = \langle \boldsymbol{u}, S_{i\kappa'} \boldsymbol{v} \rangle_{\times,\Gamma}\,, \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma).$$

Ellipticity:

$$\langle \boldsymbol{u}, S_{i\kappa'} \overline{\boldsymbol{u}} \rangle_{\times,\Gamma} \geq C \, \|\boldsymbol{u}\|^2_{\mathbf{H}_{\times}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma)}\,, \qquad \forall \boldsymbol{u} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma).$$

Consequently, $\mathbf{H}_{\times}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma)$ is a Hilbert space equipped with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{X}_{\kappa'}} := \langle \boldsymbol{u}, S_{i\kappa'} \overline{\boldsymbol{v}} \rangle_{\times,\Gamma}\,, \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma).$$

Its induced norm is denoted by $\|\cdot\|_{\mathbf{X}_{\kappa'}}$.

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Boundary integral operators

Properties of $C_{i\kappa'}$

Compactness: the operator $C_\kappa - C_{i\kappa'}$ is compact.

Symmetry:

$$\langle \boldsymbol{v}, C_{i\kappa'}\boldsymbol{u}\rangle_{\times,\Gamma} = \langle \boldsymbol{u}, C_{i\kappa'}\boldsymbol{v}\rangle_{\times,\Gamma}\,, \qquad \forall \boldsymbol{u},\boldsymbol{v} \in \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma,\Gamma).$$

Contraction:

$$\left\|C_{i\kappa'}\boldsymbol{u}\right\|_{\mathbf{X}_{\kappa'}} \leq \beta \left\|\boldsymbol{u}\right\|_{\mathbf{X}_{\kappa'}}, \qquad \forall \boldsymbol{u} \in \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma,\Gamma),$$

where

$$\beta(\kappa',\Gamma) := \sqrt{\frac{1}{4} - \frac{{\kappa'}^2}{C_S^2}} < \frac{1}{2}, \qquad C_S(\kappa',\Gamma) := \sup_{\boldsymbol{u}\in\mathbf{X}_{\kappa'}} \frac{\left\|S_{i\kappa'}^{-1}\boldsymbol{u}\right\|_{\mathbf{X}_{\kappa'}}}{\left\|\boldsymbol{u}\right\|_{\mathbf{X}_{\kappa'}}}.$$

Sketch of proof: to use the Calderón projection formulas.

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Boundary integral operators

Properties of $\frac{1}{2}Id \pm C_{i\kappa'}$

Contraction:

$$\left(\frac{1}{2} - \beta\right) \|\boldsymbol{u}\|_{\mathbf{X}_{\kappa'}} \leq \left\| \left(\frac{1}{2}Id \pm C_{i\kappa'}\right) \boldsymbol{u} \right\|_{\mathbf{X}_{\kappa'}} \leq \left(\frac{1}{2} + \beta\right) \|\boldsymbol{u}\|_{\mathbf{X}_{\kappa'}} .$$

$S_{i\kappa'}$**-coercivity**:

$$\left\langle \left(\frac{1}{2}Id \pm C_{i\kappa'}\right) \boldsymbol{u}, S_{i\kappa'}\overline{\boldsymbol{u}} \right\rangle_{\times,\Gamma} \geq \left(\frac{1}{4} - \beta^2\right) \|\boldsymbol{u}\|^2_{\mathbf{X}_{\kappa'}}, \qquad \forall \boldsymbol{u} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma).$$

Proof:

$$\left(\left(\frac{1}{2}Id \pm C_{i\kappa'}\right) \boldsymbol{u}, \boldsymbol{u}\right)_{\mathbf{X}_{\kappa'}} \geq \frac{1}{2} \|\boldsymbol{u}\|^2_{\mathbf{X}_{\kappa'}} - \|C_{i\kappa'}\boldsymbol{u}\|_{\mathbf{X}_{\kappa'}} \|\boldsymbol{u}\|_{\mathbf{X}_{\kappa'}}$$

$$\geq \frac{1}{2} \|\boldsymbol{u}\|^2_{\mathbf{X}_{\kappa'}} - \frac{1}{4} \|\boldsymbol{u}\|^2_{\mathbf{X}_{\kappa'}} - \|C_{i\kappa'}\boldsymbol{u}\|^2_{\mathbf{X}_{\kappa'}} .$$

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The magnetic field integral equation (MFIE)

The Stratton-Chu representation formula

$$\boldsymbol{e}(\boldsymbol{x}) = -\boldsymbol{\Psi}^{\kappa}_{DL}([\gamma_D]_\Gamma\, \boldsymbol{e})(\boldsymbol{x}) - \boldsymbol{\Psi}^{\kappa}_{SL}([\gamma_N]_\Gamma\, \boldsymbol{e})(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \cup \Omega^c.$$

Taking the exterior Neumann trace γ_N^+ gives the MFIE

$$\left(\frac{1}{2} Id + C_\kappa\right)(\boldsymbol{u}) = \boldsymbol{f},$$

where $\boldsymbol{u} := \gamma_N^+ \boldsymbol{e} \in \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is the unknown, and

$$\boldsymbol{f} := \kappa^2 S_\kappa(\gamma_D^+ \boldsymbol{e}^{in}).$$

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Variational formulation

Flawed idea: using the $\mathbf{X}_{\kappa'}$-inner product

$$\left\langle \left(\frac{1}{2}Id + C_\kappa\right)\boldsymbol{u}, S_{i\kappa'}\overline{\boldsymbol{v}}\right\rangle_{\times,\Gamma} = \langle \boldsymbol{f}, S_{i\kappa'}\overline{\boldsymbol{v}}\rangle_{\times,\Gamma}\,, \qquad \forall \boldsymbol{v} \in \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma).$$

- ✓ Advantage: forming a compact perturbation of a $\mathbf{X}_{\kappa'}$-elliptic operator.
- × Drawback: involving three boundary integrals, making it undesirable in practice.

Appropriate form: using the duality pairing $\langle\cdot,\cdot\rangle_{\times,\Gamma}$

$$b(\boldsymbol{u},\boldsymbol{v}) := \left\langle \left(\frac{1}{2}Id + C_\kappa\right)\boldsymbol{u}, \overline{\boldsymbol{v}}\right\rangle_{\times,\Gamma} = \langle \boldsymbol{f}, \overline{\boldsymbol{v}}\rangle_{\times,\Gamma}\,, \qquad \forall \boldsymbol{v} \in \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma).$$

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Unique solvability

Assumption 1: κ^2 is bounded away from the spectrum of the interior Maxwell's problem.

$\Rightarrow$ the MFIE operator is injective.

For any $\kappa, \kappa' > 0$, the following generalized Gårding inequality holds

$$|b(\boldsymbol{u}, S_{i\kappa'}\boldsymbol{u}) - t(\boldsymbol{u}, S_{i\kappa'}\boldsymbol{u})| \geq \left(\frac{1}{4} - \beta^2\right) \|\boldsymbol{u}\|^2_{\mathbf{X}_{\kappa'}}, \qquad \forall \boldsymbol{u} \in \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma),$$

where the compact sesquilinear form

$$t(\boldsymbol{u}, \boldsymbol{v}) := \langle (C_\kappa - C_{i\kappa'})\, \boldsymbol{u}, \overline{\boldsymbol{v}} \rangle_{\times,\Gamma}\,.$$

Theorem (Unique solvability)

Let Assumption 1 be satisfied. Then, there exists a unique $\boldsymbol{u} \in \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ that solves the variational problem.

Boundary element method

Let us consider the boundary element space

$$\mathbf{U}_h := \left\{ \boldsymbol{u}_h \in \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma) : \boldsymbol{u}_h|_T \in \mathrm{RT}_0(T), \forall T \in \mathcal{T}_h \right\}.$$

Flawed idea: find $\boldsymbol{u}_h \in \mathbf{U}_h$ such that

$$\left\langle \left(\frac{1}{2} Id + C_\kappa \right) \boldsymbol{u}_h, \overline{\boldsymbol{w}_h} \right\rangle_{\times,\Gamma} = \langle \boldsymbol{f}, \overline{\boldsymbol{w}_h} \rangle_{\times,\Gamma}, \qquad \forall \boldsymbol{w}_h \in \mathbf{U}_h .$$

There exists $\mathbf{W}_h \subset \mathbf{U}_h$ such that

$$\sup_{\boldsymbol{u}_h \in \mathbf{U}_h} \sup_{\boldsymbol{w}_h \in \mathbf{W}_h} \frac{\left| \langle \boldsymbol{u}_h, \overline{\boldsymbol{w}_h} \rangle_{\times,\Gamma} \right|}{\left\| \boldsymbol{u}_h \right\|_{\mathbf{X}_{\kappa'}} \left\| S_{i\kappa'}^{-1} \boldsymbol{w}_h \right\|_{\mathbf{X}_{\kappa'}}} \leq C h^{1/2}.$$

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Boundary element method (cont.)

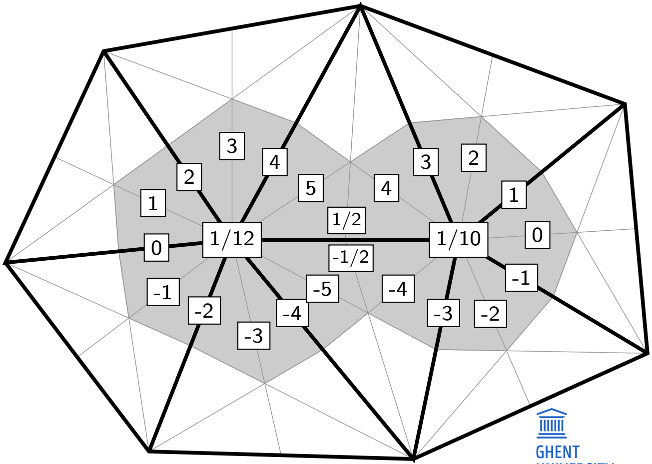
We choose $\mathbf{V}_h \subset \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ that

$$\inf_{\boldsymbol{u}_h \in \mathbf{U}_h} \sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{\left| \langle \boldsymbol{u}_h, \overline{\boldsymbol{v}_h} \rangle_{\times,\Gamma} \right|}{\|\boldsymbol{u}_h\|_{\mathbf{X}_{\kappa'}} \left\| S_{i\kappa'}^{-1} \boldsymbol{v}_h \right\|_{\mathbf{X}_{\kappa'}}} \geq \alpha,$$

and $\dim \mathbf{V}_h = \dim \mathbf{U}_h$.

$\Rightarrow$ $\mathbf{V}_h$ is the space of Buffa-Christiansen basis functions.

Please note that $\alpha(\kappa', \Gamma) \leq 1$.



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Petrov-Galerkin discretization

Find $\boldsymbol{u}_h \in \mathbf{U}_h$ such that[1]

$$b(\boldsymbol{u}_h, \boldsymbol{v}_h) = \left\langle \left(\frac{1}{2} Id + C_\kappa \right) \boldsymbol{u}_h, \overline{\boldsymbol{v}_h} \right\rangle_{\times,\Gamma} = \langle \boldsymbol{f}, \overline{\boldsymbol{v}_h} \rangle_{\times,\Gamma}, \qquad \forall \boldsymbol{v}_h \in \mathbf{V}_h .$$

For any $\kappa' > 0$, it holds

$$I := \inf_{\boldsymbol{u}_h \in \mathbf{U}_h} \sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{\left| \left\langle \left(\frac{1}{2} Id + C_{i\kappa'} \right) \boldsymbol{u}_h, \overline{\boldsymbol{v}_h} \right\rangle_{\times,\Gamma} \right|}{\|\boldsymbol{u}_h\|_{\mathbf{X}_{\kappa'}} \left\| S_{i\kappa'}^{-1} \boldsymbol{v}_h \right\|_{\mathbf{X}_{\kappa'}}} \geq \frac{\alpha}{2} - \beta.$$

Proof:

$$I \geq \frac{1}{2} \inf_{\boldsymbol{u}_h \in \mathbf{U}_h} \sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{\left| \langle \boldsymbol{u}_h, \overline{\boldsymbol{v}_h} \rangle_{\times,\Gamma} \right|}{\|\boldsymbol{u}_h\|_{\mathbf{X}_{\kappa'}} \left\| S_{i\kappa'}^{-1} \boldsymbol{v}_h \right\|_{\mathbf{X}_{\kappa'}}} - \sup_{\boldsymbol{u}_h \in \mathbf{U}_h} \sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{\left| \langle C_{i\kappa'} \boldsymbol{u}_h, \overline{\boldsymbol{v}_h} \rangle_{\times,\Gamma} \right|}{\|\boldsymbol{u}_h\|_{\mathbf{X}_{\kappa'}} \left\| S_{i\kappa'}^{-1} \boldsymbol{v}_h \right\|_{\mathbf{X}_{\kappa'}}}.$$

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[1] K. Cools et. al., Accurate and conforming mixed discretization of the MFIE, *IEEE Antennas Wirel. Propag. Lett.*, pp. 528-531, 2011.

Unique solvability

Assumption 2: There is a $\kappa_0' > 0$ such that

$$\alpha(\kappa_0', \Gamma) > 2\beta(\kappa_0', \Gamma).$$

If Assumptions 1 and 2 are satisfied, then

$$\inf_{\boldsymbol{u}_h \in \mathbf{U}_h} \sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{|b(\boldsymbol{u}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{u}_h\|_{\mathbf{X}_{\kappa_0'}} \left\| S_{i\kappa_0'}^{-1} \boldsymbol{v}_h \right\|_{\mathbf{X}_{\kappa_0'}}} \geq \gamma > 0 \qquad \forall h < h_0.$$

Theorem (Unique solvability)

Let Assumptions 1 and 2 be satisfied. Then, there exists an $h_0 > 0$ such that for all $h < h_0$, the discrete problem has a unique solution $\boldsymbol{u}_h \in \mathbf{U}_h$ satisfying the quasi-optimal convergence

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma)} \leq C \inf_{\boldsymbol{w}_h \in \mathbf{U}_h} \|\boldsymbol{u} - \boldsymbol{w}_h\|_{\mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma)}.$$

Matrix representation

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\}$ be bases of $\mathbf{U}_h$ and $\mathbf{V}_h$, and

$$[\mathbf{B}]_{mn} := b(\boldsymbol{u}_n, \boldsymbol{v}_m), \qquad [\mathbf{D}]_{mn} := \langle \boldsymbol{u}_n, \overline{\boldsymbol{v}_m} \rangle_{\times,\Gamma} \,.$$

Let the solution

$$\boldsymbol{u} \approx \sum_{m=1}^{N} \widehat{u}_m \boldsymbol{u}_m.$$

The coefficient vector $\widehat{\boldsymbol{u}}_h := (\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_N)^\mathsf{T}$ is the solution to

$$\mathbf{B}\, \widehat{\boldsymbol{u}}_h = \boldsymbol{b}.$$

If Assumptions 1 and 2 are satisfied, then

$$\operatorname{cond}(\mathbf{D}^{-1}\mathbf{B}) \leq C, \qquad \forall h < h_0.$$

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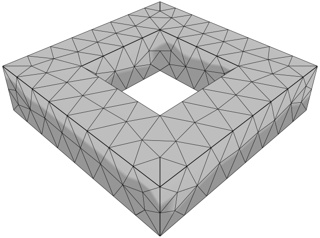
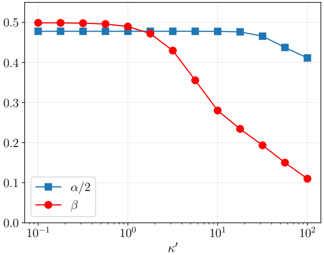
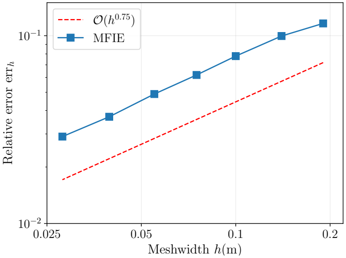
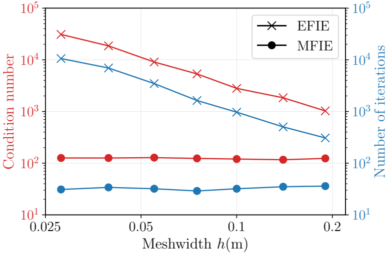
Numerical results[2]









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[2]K. Cools, BEAST.jl: Boundary Element Analysis and Simulation Toolkit, 2020.

Conclusions and future work

For the MFIE, we have shown that

- ✓ the continuous variational problem is uniquely solvable, under the uniqueness
- ✓ the Petrov-Galerkin discretization is uniquely solvable, under an additional assumption depending only on the geometry
- ✓ the numerical solutions satisfy the quasi-optimal convergence
- ✓ the Galerkin matrix system is well-conditioned.

Future work: to apply the proposed discretization scheme to

- combined field integral equations
- single source formulations.

References and acknowledgment

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How to approximately compute α and β?

$$\inf_{\boldsymbol{u}_h \in \mathbf{U}_h} \sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{\left| \langle \boldsymbol{u}_h, \overline{\boldsymbol{v}_h} \rangle_{\times,\Gamma} \right|}{\|\boldsymbol{u}_h\|_{\mathbf{X}_{\kappa'}} \left\| S_{i\kappa'}^{-1} \boldsymbol{v}_h \right\|_{\mathbf{X}_{\kappa'}}} \geq \alpha, \qquad \sup_{\boldsymbol{u}_h \in \mathbf{U}_h} \sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{\left| \langle C_{i\kappa'} \boldsymbol{u}_h, \overline{\boldsymbol{v}_h} \rangle_{\times,\Gamma} \right|}{\|\boldsymbol{u}_h\|_{\mathbf{X}_{\kappa'}} \left\| S_{i\kappa'}^{-1} \boldsymbol{v}_h \right\|_{\mathbf{X}_{\kappa'}}} \leq \beta.$$

Let $\widetilde{\mathbf{U}}_h$ be the RT space on the barycentric refinement $\widetilde{\Gamma}_h$ of Γ_h. We find $\widetilde{\boldsymbol{w}}_h \in \widetilde{\mathbf{U}}_h$ such that

$$\left\langle \widetilde{\boldsymbol{w}}_h, S_{i\kappa'} \overline{\widetilde{\boldsymbol{\varphi}}_h} \right\rangle_{\times,\Gamma} = - \left\langle \boldsymbol{v}_h, \overline{\widetilde{\boldsymbol{\varphi}}_h} \right\rangle_{\times,\Gamma}, \qquad \forall \widetilde{\boldsymbol{\varphi}}_h \in \widetilde{\mathbf{U}}_h.$$

Let $\mathbf{S}_{\kappa'}$ and $\widetilde{\mathbf{S}}_{\kappa'}$ be the Galerkin matrices of the $\mathbf{X}_{\kappa'}$-inner product, $\mathbf{G}$ be the Galerkin matrix of $\langle \cdot, \cdot \rangle_{\times,\Gamma}$ between $\mathbf{V}_h$ and $\widetilde{\mathbf{U}}_h$. Then,

$$\|\boldsymbol{u}_h\|_{\mathbf{X}_{\kappa'}} = \sqrt{\widehat{\boldsymbol{u}}_h^{\mathsf{T}} \mathbf{S}_{\kappa'} \widehat{\boldsymbol{u}}_h} = \left\| \mathbf{S}_{\kappa'}^{1/2} \widehat{\boldsymbol{u}}_h \right\|_{l^2},$$

$$\left\| S_{i\kappa'}^{-1} \boldsymbol{v}_h \right\|_{\mathbf{X}_{\kappa'}} \approx \sqrt{\widehat{\boldsymbol{w}}_h^{\mathsf{T}} \widetilde{\mathbf{S}}_{\kappa'} \widehat{\boldsymbol{w}}_h} = \left\| \left(\mathbf{G}^{\mathsf{T}} \widetilde{\mathbf{S}}_{\kappa'}^{-1} \mathbf{G} \right)^{1/2} \widehat{\boldsymbol{v}}_h \right\|_{l^2} =: \left\| \mathbf{M}^{1/2} \widehat{\boldsymbol{v}}_h \right\|_{l^2}.$$

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How to approximately compute α and β?

Substituting into the formulations

$$\alpha \leq \inf_{\widehat{\boldsymbol{u}}_h \in \mathbb{C}^N} \sup_{\widehat{\boldsymbol{v}}_h \in \mathbb{C}^N} \frac{\left| \widehat{\boldsymbol{v}}_h^{\mathsf{T}} \, \mathbf{D} \, \widehat{\boldsymbol{u}}_h \right|}{\left\| \mathbf{S}_{\kappa'}^{1/2} \, \widehat{\boldsymbol{u}}_h \right\|_{l^2} \left\| \mathbf{M}_{\kappa'}^{1/2} \, \widehat{\boldsymbol{v}}_h \right\|_{l^2}} = \inf_{\widehat{\boldsymbol{\xi}}_h \in \mathbb{C}^N} \sup_{\widehat{\boldsymbol{\psi}}_h \in \mathbb{C}^N} \frac{\left| \widehat{\boldsymbol{\psi}}_h^{\mathsf{T}} \, \mathbf{M}_{\kappa'}^{-\mathsf{T}/2} \, \mathbf{D} \, \mathbf{S}_{\kappa'}^{-1/2} \, \widehat{\boldsymbol{\xi}}_h \right|}{\left\| \widehat{\boldsymbol{\xi}}_h \right\|_{l^2} \left\| \widehat{\boldsymbol{\psi}}_h \right\|_{l^2}},$$

$$\beta \geq \sup_{\widehat{\boldsymbol{u}}_h \in \mathbb{C}^N} \sup_{\widehat{\boldsymbol{v}}_h \in \mathbb{C}^N} \frac{\left| \widehat{\boldsymbol{v}}_h^{\mathsf{T}} \, \mathbf{C}_{\kappa'} \, \widehat{\boldsymbol{u}}_h \right|}{\left\| \mathbf{S}_{\kappa'}^{1/2} \, \widehat{\boldsymbol{u}}_h \right\|_{l^2} \left\| \mathbf{M}_{\kappa'}^{1/2} \, \widehat{\boldsymbol{v}}_h \right\|_{l^2}} = \sup_{\widehat{\boldsymbol{\xi}}_h \in \mathbb{C}^N} \sup_{\widehat{\boldsymbol{\psi}}_h \in \mathbb{C}^N} \frac{\left| \widehat{\boldsymbol{\psi}}_h^{\mathsf{T}} \, \mathbf{M}_{\kappa'}^{-\mathsf{T}/2} \, \mathbf{C}_{\kappa'} \, \mathbf{S}_{\kappa'}^{-1/2} \, \widehat{\boldsymbol{\xi}}_h \right|}{\left\| \widehat{\boldsymbol{\xi}}_h \right\|_{l^2} \left\| \widehat{\boldsymbol{\psi}}_h \right\|_{l^2}}.$$

Thus, α and β can be approximated by

$$\alpha \approx \left| \lambda_{\min} \left(\mathbf{M}_{\kappa'}^{-\mathsf{T}/2} \, \mathbf{D} \, \mathbf{S}_{\kappa'}^{-1/2} \right) \right|, \qquad \beta \approx \left| \lambda_{\max} \left(\mathbf{M}_{\kappa'}^{-\mathsf{T}/2} \, \mathbf{K}_{\kappa'} \, \mathbf{S}_{\kappa'}^{-1/2} \right) \right|.$$

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