FACULTY OF ENGINEERING AND ARCHITECTURE

DEPARTMENT OF INFORMATION TECHNOLOGY
ELECTROMAGNETICS RESEARCH GROUP

2023 Söllerhaus Workshop on FastBEM

# A WELL-CONDITIONED COMBINED FIELD INTEGRAL EQUATION FOR ELECTROMAGNETIC SCATTERING IN LIPSCHITZ DOMAINS*

VAN CHIEN LE

October 3, 2023

GHENT UNIVERSITY

IDLab
INTERNET & DATA LAB

*A joint work with Kristof Cools

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# Outline



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# Electromagnetic scattering problem

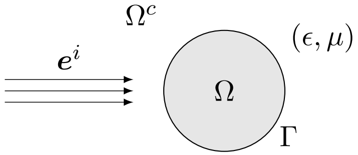
Exterior electric wave equation

$$\begin{aligned} \textbf{curl}\,\textbf{curl}\,\boldsymbol{e} - \kappa^2 \boldsymbol{e} &= 0 && \text{in} \quad \Omega^c, \\ \boldsymbol{e} \times \mathbf{n} &= -\boldsymbol{e}^i \times \mathbf{n} && \text{on} \quad \Gamma, \end{aligned}$$

with the Silver-Müller condition

$$\lim_{r\to\infty} \int_{\partial B_r} \left| \textbf{curl}\,\boldsymbol{e} \times \mathbf{n} + i\kappa(\mathbf{n} \times \boldsymbol{e}) \times \mathbf{n} \right|^2 \mathrm{d}s = 0.$$

The wave number $\kappa = \omega\sqrt{\varepsilon\mu} > 0$, with $\omega$ angular frequency.

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# Electromagnetic scattering problem

Exterior electric wave equation

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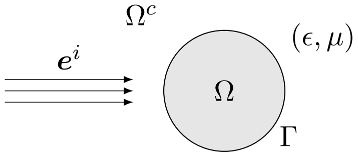


with the Silver-Müller condition

$$\lim_{r \to \infty} \int_{\partial B_r} \left| \mathbf{curl}\,\boldsymbol{e} \times \mathbf{n} + i\kappa(\mathbf{n} \times \boldsymbol{e}) \times \mathbf{n} \right|^2 \mathrm{d}s = 0.$$

The wave number $\kappa = \omega\sqrt{\varepsilon\mu} > 0$, with $\omega$ angular frequency.

**Well-posedness:**

- The exterior problem has a unique solution (Rellich's lemma).
- The solution to the interior problem is not unique at some *resonant frequencies*.

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# Trace operators and spaces

The space of tangential functions

$$\mathbf{L}^2_{\mathbf{t}}(\Gamma) := \left\{ \boldsymbol{u} \in \mathbf{L}^2(\Gamma) : \boldsymbol{u} \cdot \mathbf{n} = 0 \right\}.$$

Tangential (Dirichlet) trace $\gamma_D : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}_{\times}(\Gamma) \subset \mathbf{L}^2_{\mathbf{t}}(\Gamma)$ is defined by

$$\gamma_D : \boldsymbol{u} \mapsto \gamma(\boldsymbol{u}) \times \mathbf{n}.$$

The dual space $\mathbf{H}^{-1/2}_{\times}(\Gamma)$ of $\mathbf{H}^{1/2}_{\times}(\Gamma)$ with respect to

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\times,\Gamma} := \int_\Gamma (\boldsymbol{u} \times \mathbf{n}) \cdot \boldsymbol{v} \, \mathrm{d}s, \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbf{L}^2_{\mathbf{t}}(\Gamma).$$

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# Trace operators and spaces (cont.)

The surface operator $\mathbf{curl}_\Gamma : \mathrm{H}^{3/2}(\Gamma) \to \mathbf{H}_\times^{1/2}(\Gamma)$

$$\mathbf{curl}_\Gamma\, \gamma(\varphi) = \gamma_D(\mathbf{grad}\,\varphi), \qquad \forall \varphi \in \mathrm{H}^2(\Omega).$$

The surface divergence $\mathrm{div}_\Gamma : \mathbf{H}_\times^{-1/2}(\Gamma) \to \mathrm{H}^{-3/2}(\Gamma)$

$$\langle \mathrm{div}_\Gamma \boldsymbol{u}, \varphi \rangle_{3/2,\Gamma} = -\langle \boldsymbol{u}, \mathbf{curl}_\Gamma \varphi \rangle_{\times,\Gamma}, \qquad \forall \boldsymbol{u} \in \mathbf{H}_\times^{-1/2}(\Gamma),\ \varphi \in \mathrm{H}^{3/2}(\Gamma).$$

Let the trace space

$$\mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \left\{ \boldsymbol{u} \in \mathbf{H}_\times^{-1/2}(\Gamma) : \mathrm{div}_\Gamma \boldsymbol{u} \in \mathrm{H}^{-1/2}(\Gamma) \right\},$$

with the graph norm

$$\|\boldsymbol{u}\|^2_{\mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} := \|\boldsymbol{u}\|^2_{\mathbf{H}_\times^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma \boldsymbol{u}\|^2_{\mathrm{H}^{-1/2}(\Gamma)}.$$

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# Trace operators and spaces (cont.)

## Self-duality and integration by parts

- The Dirichlet trace $\gamma_D$ can be extended to the following continuous and surjective mapping

$$\gamma_D : \mathbf{H}(\textbf{curl}, \Omega) \to \mathbf{H}_\times^{-1/2}(\text{div}_\Gamma, \Gamma).$$

- The Neumann trace $\gamma_N : \mathbf{H}(\textbf{curl}^2, \Omega) \to \mathbf{H}_\times^{-1/2}(\text{div}_\Gamma, \Gamma)$ is defined by

$$\gamma_N = \gamma_D \circ \textbf{curl}.$$

- The space $\mathbf{H}_\times^{-1/2}(\text{div}_\Gamma, \Gamma)$ becomes its own dual with respect to $\langle \cdot, \cdot \rangle_{\times,\Gamma}$, and

$$\int_\Omega (\textbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \textbf{curl}\, \boldsymbol{v})\, \mathrm{d}\boldsymbol{x} = -\langle \gamma_D \boldsymbol{u}, \gamma_D \boldsymbol{v} \rangle_{\times,\Gamma}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}(\textbf{curl}, \Omega).$$

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# Potentials

Let $G_\sigma(\boldsymbol{x},\boldsymbol{y})$ be the fundamental solution associated with the operator $\Delta + \sigma^2$

$$G_\sigma(\boldsymbol{x},\boldsymbol{y}) := \frac{\exp(i\sigma\,|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi\,|\boldsymbol{x}-\boldsymbol{y}|}, \qquad \boldsymbol{x} \neq \boldsymbol{y},$$

where $\sigma = \kappa$, or $\sigma = i\kappa$ with $\kappa > 0$. The scalar and vectorial single layer potentials

$$\Psi_V^\sigma(\varphi)(\boldsymbol{x}) := \int_\Gamma \varphi(\boldsymbol{y}) G_\sigma(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}), \qquad \boldsymbol{\Psi}_{\boldsymbol{A}}^\sigma(\boldsymbol{u})(\boldsymbol{x}) := \int_\Gamma \boldsymbol{u}(\boldsymbol{y}) G_\sigma(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}).$$

Maxwell single and double layer potentials

$$\boldsymbol{\Psi}_{SL}^\sigma(\boldsymbol{u}) := \boldsymbol{\Psi}_{\boldsymbol{A}}^\sigma(\boldsymbol{u}) + \frac{1}{\sigma^2}\,\mathbf{grad}\,\Psi_V^\sigma(\mathrm{div}_\Gamma \boldsymbol{u}), \qquad \boldsymbol{\Psi}_{DL}^\sigma(\boldsymbol{u}) := \mathbf{curl}\,\boldsymbol{\Psi}_{\boldsymbol{A}}^\sigma(\boldsymbol{u}).$$

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# Potentials (cont.)

The potentials

$$\boldsymbol{\Psi}_{SL}^{\sigma}, \boldsymbol{\Psi}_{DL}^{\sigma} : \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega \cup \Omega^c)$$

are solutions to the "wave" equation and fulfill the radiation condition

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \sigma^2 \boldsymbol{u} = 0 \qquad \text{in} \quad \Omega \cup \Omega^c,$$

$$\lim_{r\to\infty} \int_{\partial B_r} \left| \mathbf{curl}\,\boldsymbol{u} \times \mathbf{n} + i\sigma(\mathbf{n} \times \boldsymbol{u}) \times \mathbf{n} \right|^2 \mathrm{d}s = 0.$$

Note: $\mathbf{curl} \circ \boldsymbol{\Psi}_{SL}^{\sigma} = \boldsymbol{\Psi}_{DL}^{\sigma}$ and $\mathbf{curl} \circ \boldsymbol{\Psi}_{DL}^{\sigma} = \sigma^2 \boldsymbol{\Psi}_{SL}^{\sigma}$.

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# Integral operators

For $\sigma = \kappa$ or $\sigma = i\kappa$ with $\kappa > 0$, the following integral operators are continuous

$$\begin{aligned}
V_\sigma &:= \{\gamma\}_\Gamma \circ \Psi_V^\sigma && : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Gamma),\\
A_\sigma &:= \{\gamma_D\}_\Gamma \circ \boldsymbol{\Psi}_{\boldsymbol{A}}^\sigma && : \mathbf{H}_\times^{-1/2}(\Gamma) \to \mathbf{H}_\times^{1/2}(\Gamma),\\
S_\sigma &:= \{\gamma_D\}_\Gamma \circ \boldsymbol{\Psi}_{SL}^\sigma && : \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma),\\
C_\sigma &:= \{\gamma_N\}_\Gamma \circ \boldsymbol{\Psi}_{SL}^\sigma && : \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma).
\end{aligned}$$

The jump relations

$$[\gamma]_\Gamma \circ \Psi_V^\sigma = 0, \qquad [\gamma_D]_\Gamma \circ \boldsymbol{\Psi}_{\boldsymbol{A}}^\sigma = 0, \qquad [\gamma_N]_\Gamma \circ \boldsymbol{\Psi}_{SL}^\sigma = -Id$$

imply that

$$\gamma_D^\pm \circ \boldsymbol{\Psi}_{SL}^\sigma = S_\sigma, \qquad \gamma_N^\pm \circ \boldsymbol{\Psi}_{SL}^\sigma = \mp\frac{1}{2} Id + C_\sigma.$$

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# Integral operators (cont.)

## Integral operators $V_{i\kappa}$ and $A_{i\kappa}$

For $\kappa > 0$, the operators $V_{i\kappa}$ and $A_{i\kappa}$ are self-adjoint

$$\langle \psi, V_{i\kappa}\varphi \rangle_{1/2,\Gamma} = \langle \varphi, V_{i\kappa}\psi \rangle_{1/2,\Gamma}\,, \qquad \forall \psi, \varphi \in \mathrm{H}^{-1/2}(\Gamma),$$

$$\langle \boldsymbol{v}, A_{i\kappa}\boldsymbol{u} \rangle_{\times,\Gamma} = \langle \boldsymbol{u}, A_{i\kappa}\boldsymbol{v} \rangle_{\times,\Gamma}\,, \qquad \forall \boldsymbol{v}, \boldsymbol{u} \in \mathbf{H}_{\times}^{-1/2}(\Gamma).$$

In addition, they are elliptic operators

$$\langle \varphi, V_{i\kappa}\varphi \rangle_{1/2,\Gamma} \geq C \left\| \varphi \right\|^2_{\mathrm{H}^{-1/2}(\Gamma)}\,, \qquad \forall \varphi \in \mathrm{H}^{-1/2}(\Gamma),$$

$$\langle \boldsymbol{u}, A_{i\kappa}\boldsymbol{u} \rangle_{\times,\Gamma} \geq C \left\| \boldsymbol{u} \right\|^2_{\mathbf{H}_{\times}^{-1/2}(\Gamma)}\,, \qquad \forall \boldsymbol{u} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma).$$

Consequence: The integral operator $S_{i\kappa}$ is self-adjoint and elliptic on $\mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

$$S_{i\kappa} = A_{i\kappa} - \frac{1}{\kappa^2}\mathbf{curl}_\Gamma \circ V_{i\kappa} \circ \mathrm{div}_\Gamma.$$

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# Representation formula

Any solution $\boldsymbol{e} \in \mathbf{H}_{\mathsf{loc}}(\mathbf{curl}^2, \Omega \cup \Omega^c)$ satisfies the Stratton-Chu representation formula

$$\boldsymbol{e}(\boldsymbol{x}) = -\boldsymbol{\Psi}^{\sigma}_{DL}([\gamma_D]_\Gamma\, \boldsymbol{e})(\boldsymbol{x}) - \boldsymbol{\Psi}^{\sigma}_{SL}([\gamma_N]_\Gamma\, \boldsymbol{e})(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \cup \Omega^c.$$

Taking the exterior traces $\gamma_D^+$ and $\gamma_N^+$ gives

$$\text{EFIE}: \qquad S_\kappa\left([\gamma_N]_\Gamma\, \boldsymbol{e}\right) = \gamma_D^+ \boldsymbol{e}^i,$$

$$\text{MFIE}: \qquad \left(\frac{1}{2} Id + C_\kappa\right)\left([\gamma_N]_\Gamma\, \boldsymbol{e}\right) = -\gamma_N^- \boldsymbol{e}^i.$$

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# Motivations

Numerical issues:

- EFIE yields ill-conditioned linear system when the surface mesh is fine
- EFIE and MFIE exhibit ill-conditioning when $\kappa^2$ is close to a resonant frequency
- $C_\sigma$ is compact when $\Gamma$ is smooth, but it does not hold when $\Gamma$ is non-smooth.

Literature:

- *A. Buffa and R. Hiptmair, 2005*: to introduce a "smooth operator" $M$

$$-i\eta S_\kappa(\boldsymbol{\xi}) + \left(\frac{1}{2}Id - C_\kappa\right)(M\boldsymbol{\xi}) = \gamma_D^+ \boldsymbol{e}^i,$$

- *O. Steinbach and M. Windisch, 2009*:

$$-i\eta S_\kappa(\boldsymbol{\xi}) + \left(\frac{1}{2}Id - C_\kappa\right) S_0^{*\,-1}\left(\frac{1}{2}Id + B_\kappa\right)(\boldsymbol{\xi}) = \gamma_D^+ \boldsymbol{e}^i.$$

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# Combined field integral equation

## Formulation

**Goal: to introduce a well-conditioned CFIE for Lipschitz domains.**

We consider the ansatz

$$\boldsymbol{e} = \left(i\eta\, \boldsymbol{\Psi}^{\kappa}_{SL} \circ \gamma_D^- \circ \boldsymbol{\Psi}^{i\kappa}_{SL} + \boldsymbol{\Psi}^{\kappa}_{DL} \circ \gamma_D^- \circ \boldsymbol{\Psi}^{i\kappa}_{DL}\right)(\boldsymbol{\xi}),$$

where $\boldsymbol{\xi} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\eta \in \mathbb{R} \setminus \{0\}$. Taking the exterior Dirichlet trace $\gamma_D^+$ gives

$$\mathcal{L}_\kappa(\boldsymbol{\xi}) = -\gamma_D^+ \boldsymbol{e}^i,$$

where

$$\mathcal{L}_\kappa = i\eta\, S_\kappa \circ S_{i\kappa} + \left(-\frac{1}{2} Id + C_\kappa\right) \circ \left(\frac{1}{2} Id + C_{i\kappa}\right).$$

**Variational formulation**: find $\boldsymbol{\xi} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ such that, for all $\boldsymbol{v} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

$$\langle \boldsymbol{v}, \mathcal{L}_\kappa(\boldsymbol{\xi}) \rangle_{\times,\Gamma} = -\left\langle \boldsymbol{v}, \gamma_D^+ \boldsymbol{e}^i \right\rangle_{\times,\Gamma}.$$

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# Combined field integral equation

## Uniqueness

**Theorem**

*For any* $\eta \in \mathbb{R} \setminus \{0\}$ *and any* $\kappa > 0$, *the CFIE has at most one solution* $\boldsymbol{\xi} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

*Sketch of the proof:*

Let $\boldsymbol{\xi}$ be the solution to the homogeneous problem. By means of the Stratton-Chu formula

$$\gamma_D^- \boldsymbol{e} = \left(\gamma_D^- \circ \boldsymbol{\Psi}_{DL}^{i\kappa}\right)(\boldsymbol{\xi}), \qquad \gamma_N^- \boldsymbol{e} = i\eta \left(\gamma_D^- \circ \boldsymbol{\Psi}_{SL}^{i\kappa}\right)(\boldsymbol{\xi}).$$

It holds that

$$\begin{aligned} \left\langle \gamma_D^- \boldsymbol{e}, \gamma_N^- \boldsymbol{e} \right\rangle_{\times,\Gamma} &= \int_\Omega \left( \kappa^2 \left| \boldsymbol{e}(\boldsymbol{x}) \right|^2 - \left| \mathbf{curl}\, \boldsymbol{e}(\boldsymbol{x}) \right|^2 \right) \mathrm{d}\boldsymbol{x} \in \mathbb{R} \\ &= i\eta \int_\Omega \left( \kappa^2 \left| \boldsymbol{\Psi}_{SL}^{i\kappa}(\boldsymbol{\xi})(\boldsymbol{x}) \right|^2 + \left| \mathbf{curl}\, \boldsymbol{\Psi}_{SL}^{i\kappa}(\boldsymbol{\xi})(\boldsymbol{x}) \right|^2 \right) \mathrm{d}\boldsymbol{x} \in i\mathbb{R}. \end{aligned}$$

Based on the ellipticity of $S_{i\kappa}$, we can conclude that $\boldsymbol{\xi} = 0$.

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# Combined field integral equation

## Coercivity

### Some comments:

- The ellipticity of $\mathcal{L}_\kappa$ is not available,
- If $\mathcal{L}_\kappa$ is a Fredholm operator of index 0, then the injectivity implies its surjectivity,
- The coercivity can be reached via a generalized Gårding inequality.

**Theorem**

*For any wave number $\kappa > 0$, there exist a positive constant $C$, an isomorphism mapping $\Theta : \mathbf{H}_\times^{-1/2}(\text{div}_\Gamma, \Gamma) \to \mathbf{H}_\times^{-1/2}(\text{div}_\Gamma, \Gamma)$, and a compact sesquilinear form $c : \mathbf{H}_\times^{-1/2}(\text{div}_\Gamma, \Gamma) \times \mathbf{H}_\times^{-1/2}(\text{div}_\Gamma, \Gamma) \to \mathbb{C}$ such that*

$$\left| \langle \Theta \boldsymbol{u}, \mathcal{L}_\kappa \boldsymbol{u} \rangle_{\times,\Gamma} + c(\boldsymbol{u}, \boldsymbol{u}) \right| \geq C \left\| \boldsymbol{u} \right\|^2_{\mathbf{H}_\times^{-1/2}(\text{div}_\Gamma,\Gamma)}, \qquad \forall \boldsymbol{u} \in \mathbf{H}_\times^{-1/2}(\text{div}_\Gamma, \Gamma).$$

# Combined field integral equation

## Coercivity

*Sketch of the proof:* Let us rewrite $\mathcal{L}_\kappa$ as follows

$$\begin{aligned}\mathcal{L}_\kappa &= i\eta\, S_\kappa \circ S_{i\kappa} + \left(-\frac{1}{2}Id + C_\kappa\right) \circ \left(\frac{1}{2}Id + C_{i\kappa}\right) \\ &= i\eta\, \widetilde{S}_{i\kappa} \circ S_{i\kappa} + \left(-\frac{1}{2}Id + C_{i\kappa}\right) \circ \left(\frac{1}{2}Id + C_{i\kappa}\right) + c_1 \\ &= \left(i\eta\, \widetilde{S}_{i\kappa} + \kappa^2 S_{i\kappa}\right) \circ S_{i\kappa} + c_1 = M_{i\kappa} \circ S_{i\kappa},\end{aligned}$$

where the operator

$$\begin{aligned}M_{i\kappa} &= i\eta\, \widetilde{S}_{i\kappa} + \kappa^2 S_{i\kappa} + c_1 \circ S_{i\kappa}^{-1} \\ &= \left(i\eta + \kappa^2\right) A_{i\kappa} - \left(1 - i\eta\kappa^{-2}\right) \mathbf{curl}_\Gamma \circ V_{i\kappa} \circ \mathrm{div}_\Gamma + c_1 \circ S_{i\kappa}^{-1}.\end{aligned}$$

By means of the Hodge decomposition, we get the generalized Gårding inequality.

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# Combined field integral equation

## Well-posedness

### Corollary

*The CFIE has a unique solution* $\boldsymbol{\xi} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. *In particular, the following inf-sup condition holds for all* $\boldsymbol{u} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

$$\sup_{\boldsymbol{v} \in \mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)} \frac{\left| \langle \boldsymbol{v}, \mathcal{L}_\kappa \boldsymbol{u} \rangle_{\times,\Gamma} \right|}{\|\boldsymbol{v}\|_{\mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)}} \geq C \, \|\boldsymbol{u}\|_{\mathbf{H}_{\times}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)},$$

*for some constant* $C > 0$ *independent of* $\boldsymbol{u}$.

*Proof:* by a Fredholm alternative argument.

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# Galerkin discretization

Let $(\Gamma_h)_{h>0}$ be a family of triangulations, and $\mathcal{T}_h$ and $\mathcal{E}_h$ the sets of triangles and edges of $\Gamma_h$.

On each triangle $T \in \mathcal{T}_h$, we equip the lowest-order triangular Raviart-Thomas space

$$\mathrm{RT}_0(T) := \left\{ \boldsymbol{x} \mapsto \boldsymbol{a} + b\boldsymbol{x} : \boldsymbol{a} \in \mathbb{C}^2, b \in \mathbb{C} \right\}.$$

This local space gives rise to the global $\mathsf{div}_\Gamma$-conforming boundary element space

$$\mathbf{V}_h := \left\{ \boldsymbol{u}_h \in \mathbf{H}_\times^{-1/2}(\mathsf{div}_\Gamma, \Gamma) : \boldsymbol{u}_h|_T \in \mathrm{RT}_0(T), \forall T \in \mathcal{T}_h \right\},$$

endowed with the edge degrees of freedom

$$\phi_e(\boldsymbol{u}_h) := \int_e (\boldsymbol{u}_h \times \mathbf{n}_j) \cdot \mathrm{d}s, \qquad \forall e \in \mathcal{E}_h.$$

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# Discrete inf-sup conditions

Let us recall that

$$\mathcal{L}_\kappa = M_{i\kappa} \circ S_{i\kappa}.$$

For any $\kappa > 0$ and any $h < h_0$, the following discrete inf-sup condition is satisfied

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{\left| \langle \boldsymbol{v}_h, M_{i\kappa} \boldsymbol{u}_h \rangle_{\times,\Gamma} \right|}{\| \boldsymbol{v}_h \|_{\mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}} \geq C \| \boldsymbol{u}_h \|_{\mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}, \qquad \forall \boldsymbol{u}_h \in \mathbf{V}_h .$$

We follow the approach of operator preconditioning in (*Hiptmair, 2006*).

Problem: search for a subspace $\mathbf{W}_h \subset \mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ such that $\dim \mathbf{W}_h = \dim \mathbf{V}_h$, and

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h} \frac{\left| \langle \boldsymbol{w}_h, \boldsymbol{v}_h \rangle_{\times,\Gamma} \right|}{\| \boldsymbol{v}_h \|_{\mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}} \geq C \| \boldsymbol{w}_h \|_{\mathbf{H}_\times^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}, \qquad \forall \boldsymbol{w}_h \in \mathbf{W}_h .$$

Solution: $\mathbf{W}_h$ = space of the Buffa-Christiansen (BC) basis functions.

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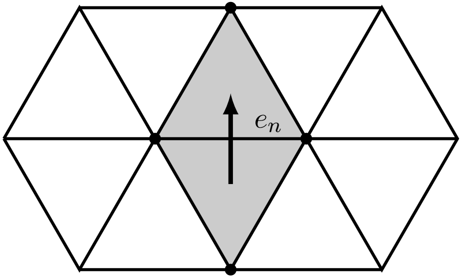
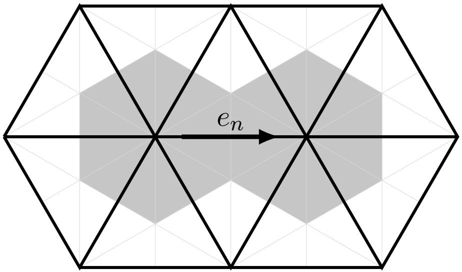
# Basis functions




Raviart-Thomas (RWG) function




Buffa-Christiansen function

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# Well-conditionedness

Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\}$ and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_N\}$ be bases of $\mathbf{V}_h$ and $\mathbf{W}_h$, respectively.

We introduce the Galerkin matrices

$$[\mathbf{M}]_{mn} := \langle \boldsymbol{v}_m, M_{i\kappa} \boldsymbol{v}_n \rangle_{\times,\Gamma}\,, \quad [\mathbf{S}_w]_{mn} := \langle \boldsymbol{w}_m, S_{i\kappa} \boldsymbol{w}_n \rangle_{\times,\Gamma}\,, \quad [\mathbf{G}]_{mn} := \langle \boldsymbol{w}_m, \boldsymbol{v}_n \rangle_{\times,\Gamma}\,.$$

Then, the condition number is uniformly bounded

$$\operatorname{cond}\left(\mathbf{G}^{-1}\,\mathbf{M}\,\mathbf{G}^{-\mathsf{T}}\,\mathbf{S}_w\right) \leq C.$$

Problem: $M_{i\kappa}$ consists of $c_1 \circ S_{i\kappa}^{-1}$.

Solution: rewrite the operator $\mathcal{L}_\kappa$ in the equivalent form

$$\mathcal{L}_\kappa = i\eta\, S_\kappa \circ S_{i\kappa} + \kappa^2 S_{i\kappa} \circ S_{i\kappa} + \delta C_\kappa \circ \left(\frac{1}{2} Id + C_{i\kappa}\right).$$

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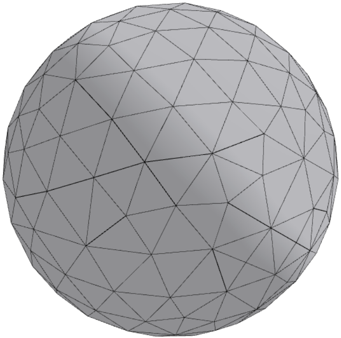
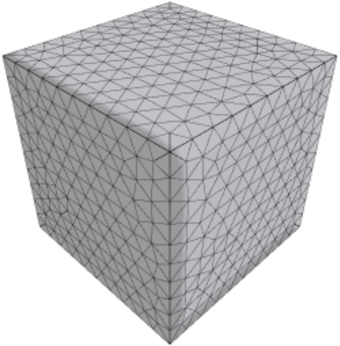
# Numerical results

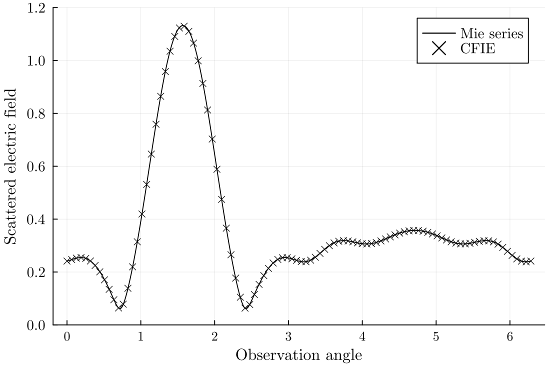
## Geometries

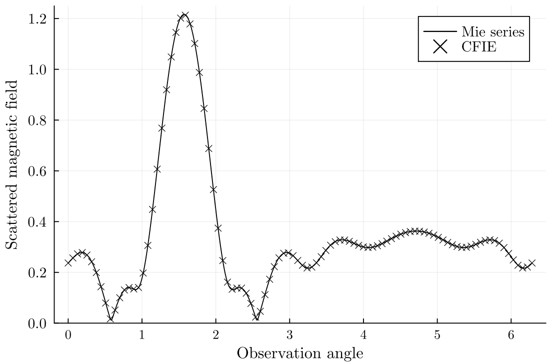




# Numerical results

## Sphere: scattered fields

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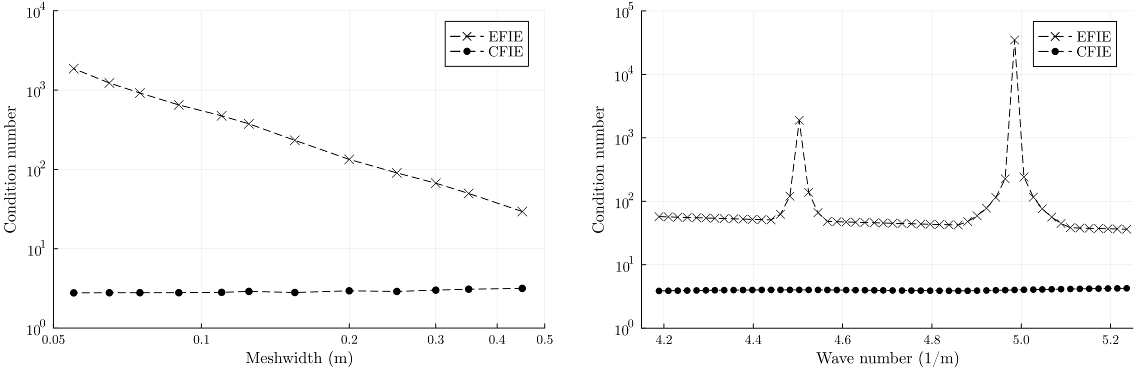
# Numerical results

## Sphere: condition numbers

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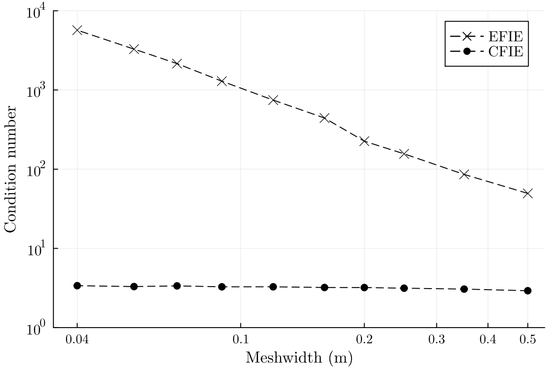
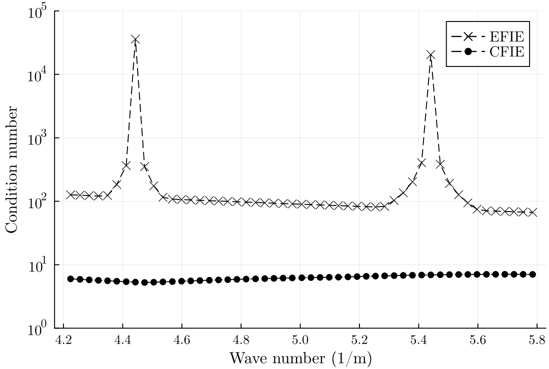
# Numerical results

## Cube: condition numbers



# Conclusions

- The proposed CFIE yields a unique solution for all wave numbers.
- The proposed Galerkin discretization scheme produces well-conditioned linear systems regardless the numerical resolution to the problem.

Open challenge: This discretization scheme differs from what practitioners typically use

Original: $$\mathcal{L}_\kappa = i\eta\, S_\kappa \circ S_{i\kappa} + \left(-\frac{1}{2}Id + C_\kappa\right) \circ \left(\frac{1}{2}Id + C_{i\kappa}\right)$$

Discretized: $$\mathcal{L}_\kappa = i\eta\, S_\kappa \circ S_{i\kappa} + \kappa^2 S_{i\kappa} \circ S_{i\kappa} + \delta C_\kappa \circ \left(\frac{1}{2}Id + C_{i\kappa}\right).$$

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