FACULTY OF ENGINEERING AND ARCHITECTURE

DEPARTMENT OF INFORMATION TECHNOLOGY
ELECTROMAGNETICS RESEARCH GROUP

IEEE AP-S/URSI 2023 Conference

A STABLE AND WELL-CONDITIONED TIME-DOMAIN COMBINED FIELD INTEGRAL EQUATION AT LOW FREQUENCIES*

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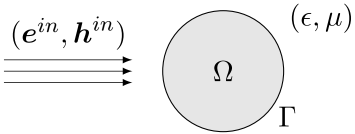
July 25, 2023

GHENT UNIVERSITY

IDLab
INTERNET & DATA LAB

*A joint work with P. Cordel, F. P. Andriulli and K. Cools.

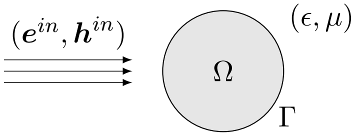
Time-domain integral equations



Time-domain integral equations

TD-EFIE: $\left(\partial_t \mathcal{T}^s + \partial_t^{-1} \mathcal{T}^h\right) \boldsymbol{j} = -\mathbf{n} \times \boldsymbol{e}^{in},$

TD-MFIE: $\left(\frac{1}{2}\mathcal{I} + \mathcal{K}\right) \boldsymbol{j} = \mathbf{n} \times \boldsymbol{h}^{in},$



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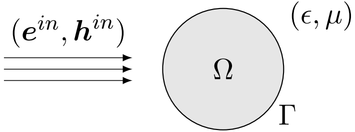

where $\eta = \sqrt{\mu/\epsilon}, c = 1/\sqrt{\mu\epsilon}, R = |\boldsymbol{r} - \boldsymbol{r}'|, \tau = t - R/c,$

$$(\mathcal{T}^s \boldsymbol{j})(\boldsymbol{r}, t) = -\frac{\eta}{c} \mathbf{n} \times \int_\Gamma \frac{\boldsymbol{j}(\boldsymbol{r}', \tau)}{4\pi R} \, \mathrm{d}s',$$

$$(\mathcal{T}^h \boldsymbol{j})(\boldsymbol{r}, t) = c\eta \; \mathbf{n} \times \mathbf{grad}_\mathbf{x} \int_\Gamma \frac{\mathrm{div}_\Gamma \, \boldsymbol{j}(\boldsymbol{r}', \tau)}{4\pi R} \, \mathrm{d}s',$$

$$(\mathcal{K} \boldsymbol{j})(\boldsymbol{r}, t) = -\mathbf{n} \times \mathbf{curl}_\mathbf{x} \int_\Gamma \frac{\boldsymbol{j}(\boldsymbol{r}', \tau)}{4\pi R} \, \mathrm{d}s'.$$

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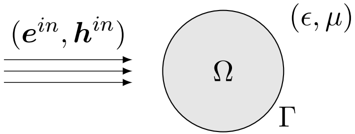

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TD vs FD: allows coupling to non-linear systems.

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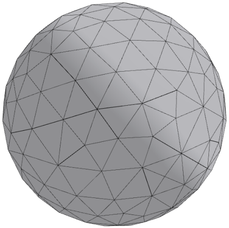
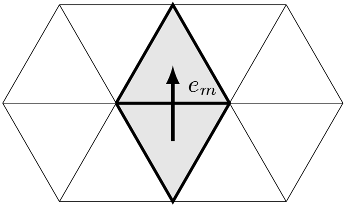
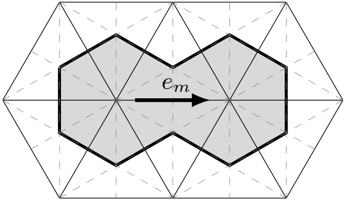
Spatial discretization



RWG function $\boldsymbol{f}_m(\boldsymbol{r})$



BC function $\boldsymbol{g}_m(\boldsymbol{r})$



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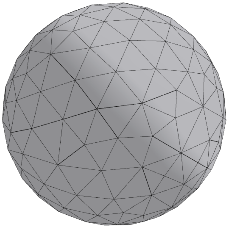

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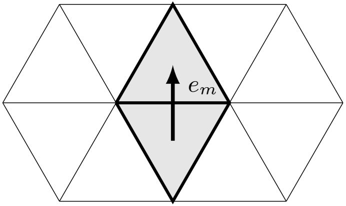

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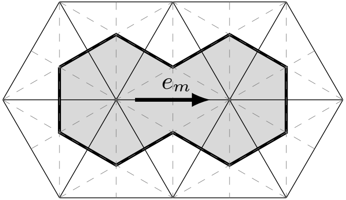

$$\boldsymbol{j}(\boldsymbol{r}, t) = \sum_{m}^{N_E} \mathrm{j}_m(t) \boldsymbol{f}_m(\boldsymbol{r}).$$

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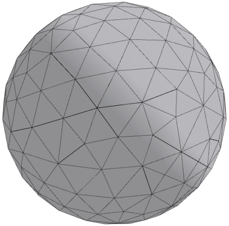
Spatial discretization



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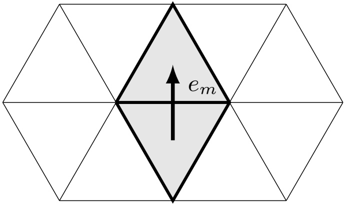

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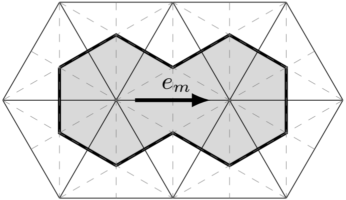

$$\boldsymbol{j}(\boldsymbol{r}, t) = \sum_{m}^{N_E} \mathrm{j}_m(t) \boldsymbol{f}_m(\boldsymbol{r}).$$

Testing schemes: $\int_\Gamma (\mathbf{n} \times \boldsymbol{f}_n) \cdot$ **TD-EFIE**, $\int_\Gamma (\mathbf{n} \times \boldsymbol{g}_n) \cdot$ **TD-MFIE**,

resulting in $\mathcal{Z}\mathrm{j}(t) = \mathrm{e}(t)$, $\mathcal{M}\mathrm{j}(t) = \mathrm{h}(t)$.

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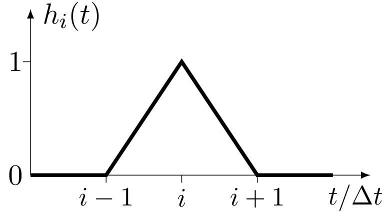
Temporal discretization

$$\mathrm{j}_m(t) = \sum_{i}^{N_T} [\mathrm{j}_i]_m \, h_i(t).$$

Testing scheme:

$$\int_{\mathbb{R}^+} \delta_j(t) \; . \; \textbf{TD-EFIE/TD-MFIE}.$$



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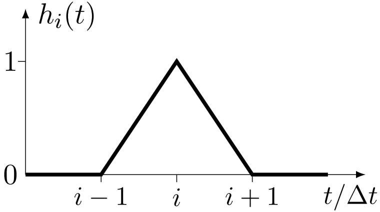

Testing scheme:

$$\int_{\mathbb{R}^+} \delta_j(t) \, . \, \textbf{TD-EFIE/TD-MFIE}.$$

Marching-on-in-time algorithm:

$$\begin{pmatrix} \mathbf{Z}_0 & & & \\ \mathbf{Z}_1 & \mathbf{Z}_0 & & \\ \vdots & \vdots & \ddots & \\ \mathbf{Z}_{N_T-1} & \mathbf{Z}_{N_T-2} & \dots & \mathbf{Z}_0 \end{pmatrix} \begin{pmatrix} \mathrm{j}_1 \\ \mathrm{j}_2 \\ \vdots \\ \mathrm{j}_{N_T} \end{pmatrix} = \begin{pmatrix} \mathrm{e}_1 \\ \mathrm{e}_2 \\ \vdots \\ \mathrm{e}_{N_T} \end{pmatrix} \implies \mathrm{j}_i = \mathbf{Z}_0^{-1} \left(\mathrm{e}_i - \sum_{k=1}^{i-1} \mathbf{Z}_k \, \mathrm{j}_{i-k} \right).$$

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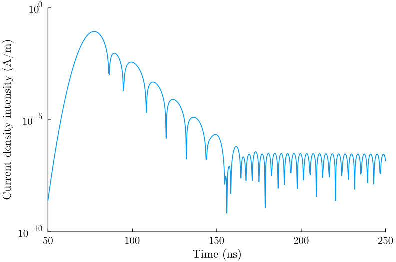
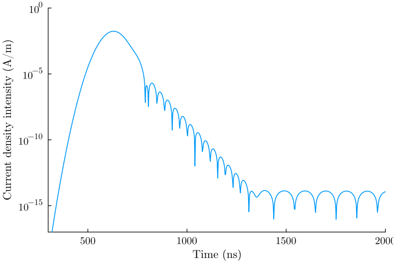
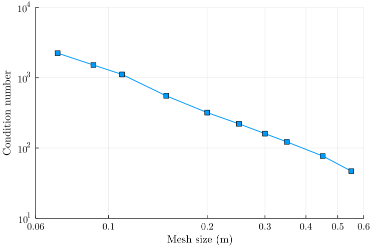
Numerical issues



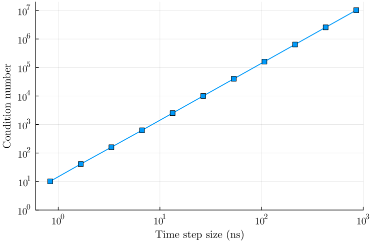
Resonant instability



Static kernel

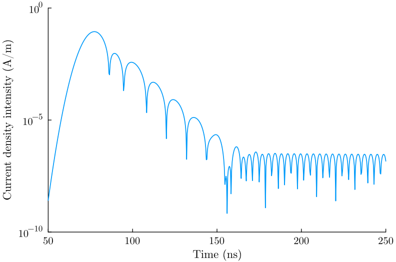


Dense discretization breakdown



Large time step breakdown

Numerical issues



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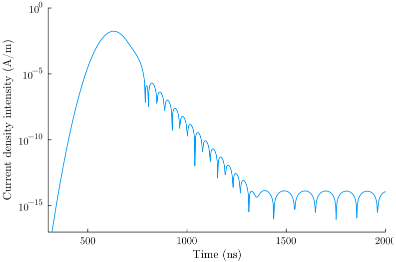

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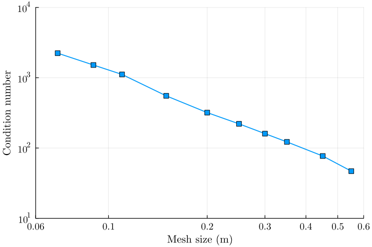

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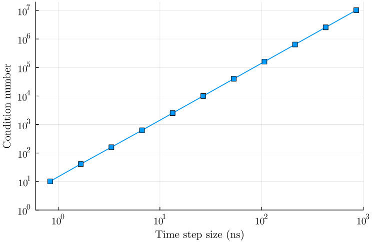

Large time step breakdown

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Yukawa-Calderón TD-CFIE

Yukawa-Calderón TD-CFIE:

$$\left[-\left(\kappa T^s_{-j\kappa} + \frac{1}{\kappa} T^h_{-j\kappa}\right)\left(\partial_t \mathcal{T}^s + \partial_t^{-1}\mathcal{T}^h\right) + \alpha\left(\frac{1}{2}I - K_{-j\kappa}\right)\left(\frac{1}{2}\mathcal{I} + \mathcal{K}\right)\right]\boldsymbol{j}$$
$$= \left(\kappa T^s_{-j\kappa} + \frac{1}{\kappa} T^h_{-j\kappa}\right)\left(\mathbf{n} \times \boldsymbol{e}^{in}\right) + \alpha\left(\frac{1}{2}I - K_{-j\kappa}\right)\left(\mathbf{n} \times \boldsymbol{h}^{in}\right),$$

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$$= \left(\kappa T^s_{-j\kappa} + \frac{1}{\kappa}T^h_{-j\kappa}\right)\left(\mathbf{n}\times\boldsymbol{e}^{in}\right) + \alpha\left(\frac{1}{2}I - K_{-j\kappa}\right)\left(\mathbf{n}\times\boldsymbol{h}^{in}\right),$$

where $\alpha > 0$ and

$$(T^s_{-j\kappa}\boldsymbol{j})(\boldsymbol{r}) = \eta\,\mathbf{n}\times\int_\Gamma \frac{e^{-\kappa R}}{4\pi R}\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{d}s',$$

$$(T^h_{-j\kappa}\boldsymbol{j})(\boldsymbol{r}) = -\eta\,\mathbf{n}\times\mathbf{grad}_{\mathbf{x}}\int_\Gamma \frac{e^{-\kappa R}}{4\pi R}\mathrm{div}_\Gamma\,\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{d}s',$$

$$(K_{-j\kappa}\boldsymbol{j})(\boldsymbol{r}) = -\mathbf{n}\times\mathbf{curl}_{\mathbf{x}}\int_\Gamma \frac{e^{-\kappa R}}{4\pi R}\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{d}s'.$$

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Generalization: see the presentation of Pierrick Cordel on Thursday, July 27th.

Spatial discretization

Testing schemes:

$$\mathbf{Z}_{mn} = \int_\Gamma (\mathbf{n} \times \boldsymbol{g}_m) \cdot \left(\kappa\, T^s_{-j\kappa} + \frac{1}{\kappa} T^h_{-j\kappa} \right) (\boldsymbol{g}_n)\, \mathrm{d}s,$$

$$\mathbf{M}_{mn} = \int_\Gamma (\mathbf{n} \times \boldsymbol{g}_m) \cdot \left(\frac{1}{2} I - K_{-j\kappa} \right) (\boldsymbol{f}_n)\, \mathrm{d}s,$$

$$\mathbf{G}_{mn} = \int_\Gamma (\mathbf{n} \times \boldsymbol{g}_m) \cdot \boldsymbol{f}_n\, \mathrm{d}s,$$

yield consistent spatial discretization

$$\left(-\mathbf{Z}\, \mathbf{G}^{-\mathsf{T}}\, \mathcal{Z} + \mathbf{M}\, \mathbf{G}^{-1}\, \mathcal{M} \right) \mathrm{j}(t) = -\mathbf{Z}\, \mathbf{G}^{-\mathsf{T}}\, \mathrm{e}(t) + \mathbf{M}\, \mathbf{G}^{-1}\, \mathrm{h}(t).$$

Parameters: $\kappa = (c\Delta t)^{-1}, \alpha = \eta^2.$

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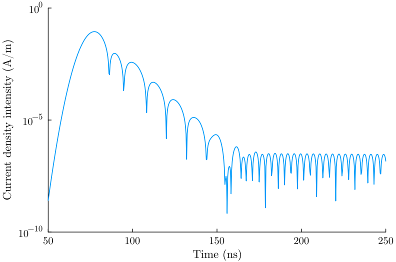
Numerical issues



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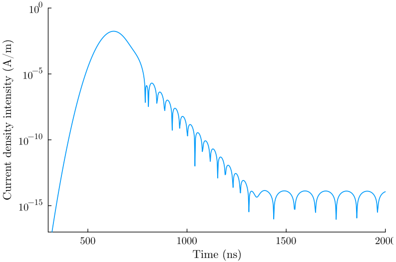

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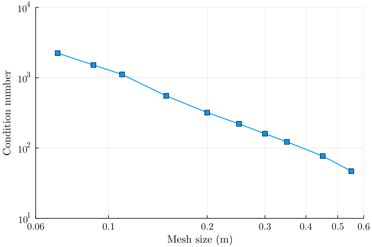
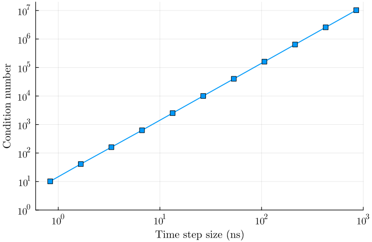


Dense discretization breakdown



Large time step breakdown

Loop-star decomposition

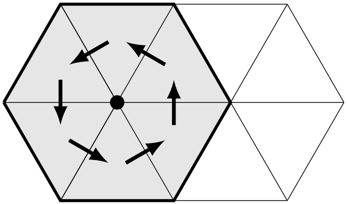
Goal: to separate solenoidal ($\mathrm{div}_\Gamma \boldsymbol{j} = 0$) and non-solenoidal currents.

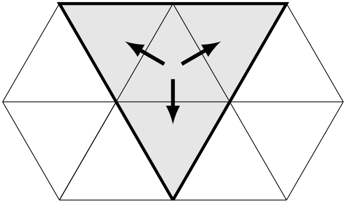
Loop $\Lambda : N_E \times N_V$

$$\Lambda_{m,i} = \begin{cases} \pm 1 & \text{if edge m contains vertex i,} \\ 0 & \text{otherwise.} \end{cases}$$

Star $\Sigma : N_E \times N_F$

$$\Sigma_{m,i} = \begin{cases} \pm 1 & \text{if edge m is contained in face i,} \\ 0 & \text{otherwise.} \end{cases}$$





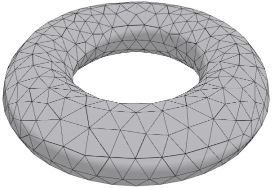
Note: $\mathrm{div}_\Gamma \, \Lambda \boldsymbol{j} = 0.$

Quasi-Helmholtz decomposition

Problem: For multiply-connected geometries:

$$N_V + N_F - N_E = 2 - 2g.$$

$\Longrightarrow \Lambda$ and Σ do not span the entire space.



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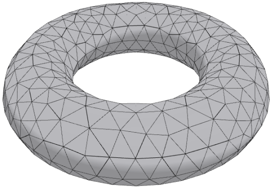

Solution: RWG-based Helmholtz decomposition:

$$\boldsymbol{j}_\Sigma = \mathbf{P}^\Sigma \boldsymbol{j} = \Sigma(\Sigma^\mathsf{T}\Sigma)^{-1}\Sigma^\mathsf{T}\boldsymbol{j},$$
$$\boldsymbol{j}_{\Lambda H} = \mathbf{P}^{\Lambda H} \boldsymbol{j} = (\boldsymbol{I} - \mathbf{P}^\Sigma)\boldsymbol{j}.$$

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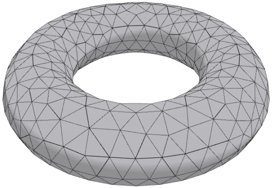

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BC-based Helmholtz decomposition:

$$\boldsymbol{j}_\Lambda = \mathbb{P}^\Lambda \boldsymbol{j} = \Lambda(\Lambda^\mathsf{T}\Lambda)^{-1}\Lambda^\mathsf{T}\boldsymbol{j},$$
$$\boldsymbol{j}_{\Sigma H} = \mathbb{P}^{\Sigma H}\boldsymbol{j} = (\boldsymbol{I} - \mathbb{P}^\Lambda)\boldsymbol{j}.$$

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Large-time step stabilization

Goal: to combat large-time step breakdown.

In a Helmholtz decomposed basis $[\Lambda H \Sigma]$, the scaling of **TD-EFIE** reads

$$\begin{pmatrix} \Delta t^{-1} & \Delta t^{-1} & \Delta t^{-1} \\ \Delta t^{-1} & \Delta t^{-1} & \Delta t^{-1} \\ \Delta t^{-1} & \Delta t^{-1} & \color{red}{\Delta t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \Delta t^{-1} \end{pmatrix} = \begin{pmatrix} \Delta t^{-1} \\ \Delta t^{-1} \\ 1 \end{pmatrix}.$$

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Stabilization: Rescaling the unknown and the right-hand side

$$\left(\mathbf{P}^{\Sigma} + \Delta t \, \mathbf{P}^{\Lambda H}\right) \mathcal{Z} \left(\partial_t \mathbf{P}^{\Sigma} + \mathbf{P}^{\Lambda H}\right) \mathrm{y}(t) = \left(\mathbf{P}^{\Sigma} + \Delta t \, \mathbf{P}^{\Lambda H}\right) \mathrm{e}(t),$$

gives

$$\begin{pmatrix} 1 & 1 & \Delta t^{-1} \\ 1 & 1 & \Delta t^{-1} \\ \Delta t^{-1} & \Delta t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

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Large-time step stabilization

Starting from the **symmetrized TD-MFIE**

$$\mathbf{M}\,\mathbf{G}^{-1}\,\mathcal{M}\mathrm{j}(t) = \mathbf{M}\,\mathbf{G}^{-1}\,\mathrm{h}(t),$$

the scaling reads

$$\begin{pmatrix} 1 & 1 & 1 \\ \Delta t^{-2} & 1 & 1 \\ \Delta t^{-2} & \Delta t^{-2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \Delta t^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ \Delta t^{-1} \\ \Delta t^{-1} \end{pmatrix}.$$

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Applying the rescaling procedure

$$\left(\mathbb{P}^{\Lambda} + \Delta t\,\mathbb{P}^{\Sigma H}\right)\mathbf{M}\,\mathbf{G}^{-1}\,\mathcal{M}\left(\partial_t\,\mathbf{P}^{\Sigma} + \mathbf{P}^{\Lambda H}\right)\mathrm{y}(t) = \left(\mathbb{P}^{\Lambda} + \Delta t\,\mathbb{P}^{\Sigma H}\right)\mathbf{M}\,\mathbf{G}^{-1}\,\mathrm{h}(t),$$

gives

$$\begin{pmatrix} 1 & 1 & \Delta t^{-1} \\ \Delta t^{-1} & {\color{red}\Delta t} & 1 \\ \Delta t^{-1} & \Delta t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

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Large-time step stabilization

Note: the special property

$$\mathbb{P}^{\Sigma H} \left(\frac{1}{2} \boldsymbol{I} - \mathbf{K}^s \right) \mathbf{G}^{-1} \left(\frac{1}{2} \boldsymbol{I} + \mathbf{K}^s \right) \mathbf{P}^{\Lambda H} = 0,$$

where

$$\mathbf{K}^s_{mn} = \int_\Gamma (\mathbf{n} \times \boldsymbol{g}_m) \cdot (K_0 \boldsymbol{f}_n) \, \mathrm{d}s,$$

with

$$(K_0 \boldsymbol{j})(\boldsymbol{r}) = -\mathbf{n} \times \mathbf{curl}_{\mathbf{x}} \int_\Gamma \frac{\boldsymbol{j}(\boldsymbol{r}')}{4\pi R} \, \mathrm{d}s'.$$

This property leads to

$$\begin{pmatrix} 1 & 1 & \Delta t^{-1} \\ \Delta t^{-1} & \Delta t^{-1} & 1 \\ \Delta t^{-1} & \Delta t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

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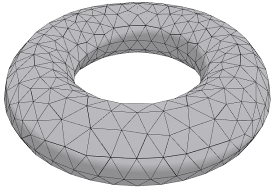
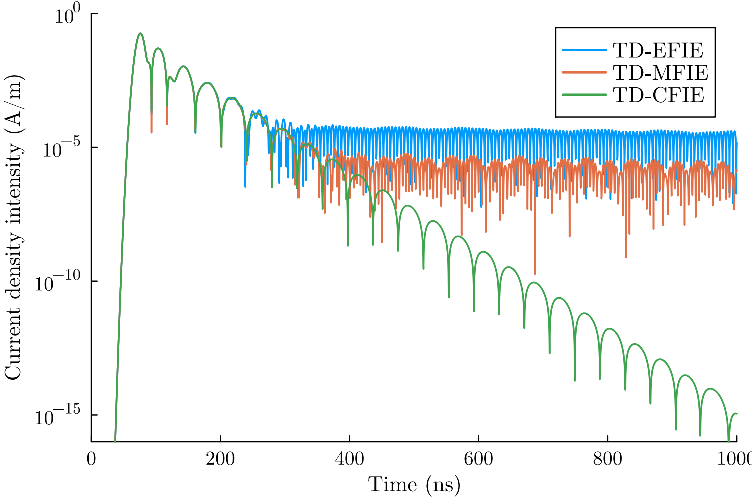
Numerical results

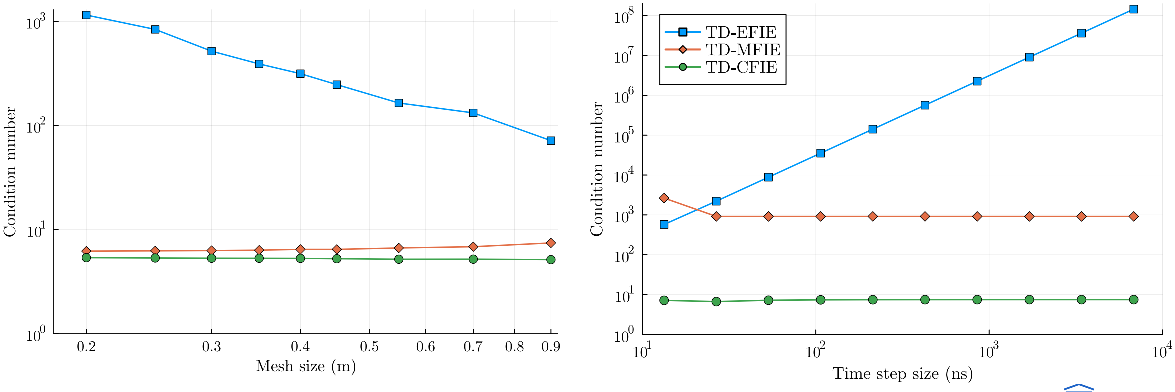
Torus: surface current





Numerical results

Torus: condition numbers



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Conclusions

- Resonant instability, dense discretization breakdown and static kernel can be eliminated by the Yukawa-Calderón TD-CFIE.
- Loop-star decomposition fails when applied to multiply-connected geometries.
- The projector-based Helmholtz rescaling gets rid of large-time step breakdown.
- Setting "loop-loop" static component of the symmetrized TD-MFIE to zero is required.

References and acknowledgment

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