FACULTY OF ENGINEERING AND ARCHITECTURE

DEPARTMENT OF INFORMATION TECHNOLOGY
IDLAB - INTERNET TECHNOLOGY AND DATA SCIENCE LAB

The East and Southeast Asia Workshop on Inverse Problems and Optimal Control (ESEAW: IPOC)

EXISTENCE OF A WEAK SOLUTION TO A NONLINEAR INDUCTION HARDENING PROBLEM FOR A STEEL WORKPIECE

VAN CHIEN LE*

August 2, 2022

GHENT UNIVERSITY

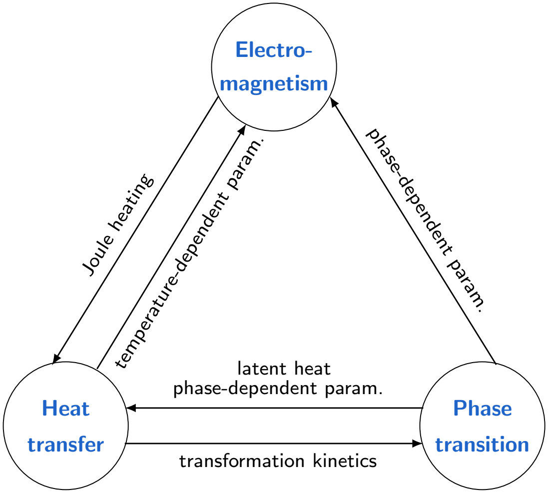
imec

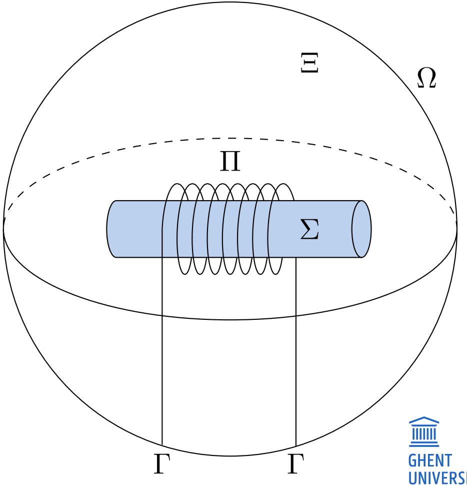
*Van Chien Le is supported by ERC Consolidator grant BET3D.

Outline

GHENT UNIVERSITY

Induction hardening





GHENT UNIVERSITY

Jumping material coefficients

The subscripts Σ, Π and Ξ are used to distinguish the value of material coefficients on subdomains. For instances,

$$\mu = \begin{cases} \mu_\Pi & \text{in} \quad \Pi, \\ \mu_\Sigma & \text{in} \quad \Sigma, \\ \mu_\Xi & \text{in} \quad \Xi. \end{cases} \qquad \text{and} \qquad \sigma = \begin{cases} \sigma_\Pi & \text{in} \quad \Pi, \\ \sigma_\Sigma & \text{in} \quad \Sigma, \\ 0 & \text{in} \quad \Xi. \end{cases}$$

In general, material coefficients have jumps at the interfaces of different subdomains.

$$[\![\mu]\!]_{\partial\Sigma} := \mu_\Xi - \mu_\Sigma \neq 0,$$

$$[\![\mu]\!]_{\partial\Pi\setminus\Gamma} := \mu_\Xi - \mu_\Pi \neq 0.$$

GHENT UNIVERSITY

Electromagnetic subproblem

Maxwell's equations

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \textit{(Gauss's law)}$$
$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \qquad \textit{(Faraday's law)}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}, \qquad \textit{(Ampère's law)}$$

with the constitutive relations

$$\boldsymbol{B} = \mu \boldsymbol{H},$$
$$\boldsymbol{J} = \sigma \boldsymbol{E}. \qquad \textit{(Ohm's law)}$$

Boundary condition

$$\boldsymbol{B} \cdot \mathbf{n} = 0.$$

Interface conditions

$$[\![\boldsymbol{B} \cdot \mathbf{n}]\!] = 0, \qquad [\![\boldsymbol{H} \times \mathbf{n}]\!] = \mathbf{0}, \qquad [\![\boldsymbol{E} \times \mathbf{n}]\!] = \mathbf{0}.$$

GHENT UNIVERSITY

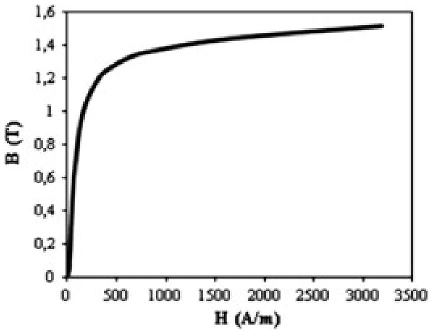
Nonlinear $\boldsymbol{B} - \boldsymbol{H}$ relation

Suppose that

$$\mu_\Sigma(\boldsymbol{B}) = \hat{\mu}_\Sigma \, m^{-1}(|\boldsymbol{B}|),$$

then the magnetization curve

$$\boldsymbol{H} = \hat{\mu}^{-1} \boldsymbol{M}(\boldsymbol{B}),$$



where

$$\hat{\mu} = \begin{cases} \hat{\mu}_\Sigma & \text{in} \quad \Sigma, \\ \mu & \text{in} \quad \Pi \cup \Xi, \end{cases} \quad \text{and} \quad \boldsymbol{M}(\boldsymbol{b}) = \begin{cases} m(|\boldsymbol{b}|)\, \boldsymbol{b} & \text{in} \quad \Sigma, \\ \boldsymbol{b} & \text{in} \quad \Pi \cup \Xi. \end{cases}$$

GHENT UNIVERSITY

Assumptions on the function m

The function $m : \mathbb{R}^+ \to \mathbb{R}^+$ is supposed to be differentiable and satisfy

$$m(s) + \left| m'(s) \right| s \le C_M \qquad \forall s \in \mathbb{R}^+.$$

In addition, for each $s \in \mathbb{R}^+$, one of the following conditions is satisfied

$$m(s) - \left| m'(s) \right| s \ge c_M > 0,$$

or

$$m(s) \ge c_M \quad \text{and} \quad m'(s) \ge 0.$$

Lemma

Under these assumptions, the nonlinear operator $\boldsymbol{M}$ is strongly monotone, coercive and Lipschitz continuous.

$\boldsymbol{A} - \phi$ potential formulation

The BVP for the electric scalar potential ϕ

$$\begin{cases} \nabla \cdot (-\sigma_\Pi \nabla \phi) = 0 & \text{in} \quad \Pi \times (0,T), \\ -\sigma_\Pi \nabla \phi \cdot \mathbf{n} = 0 & \text{on} \quad (\partial\Pi \setminus \Gamma) \times (0,T), \\ -\sigma_\Pi \nabla \phi \cdot \mathbf{n} = j & \text{on} \quad \Gamma \times (0,T). \end{cases}$$

The IBVP for the magnetic vector potential $\boldsymbol{A}$ ($\boldsymbol{B} := \nabla \times \boldsymbol{A}$)

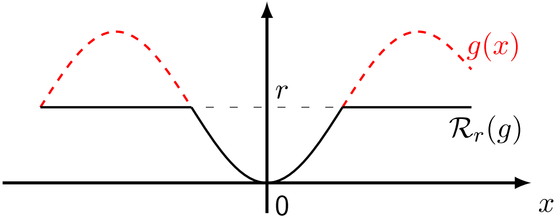
$$\begin{cases} \sigma \partial_t \boldsymbol{A} + \nabla \times \hat{\mu}^{-1} \boldsymbol{M} (\nabla \times \boldsymbol{A}) + \chi_\Pi \sigma \nabla \phi = \mathbf{0} & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{A} = 0 & \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{A} \times \mathbf{n} = \mathbf{0} & \text{on} \quad \partial\Omega \times (0,T), \\ [\![\hat{\mu}^{-1} \boldsymbol{M} (\nabla \times \boldsymbol{A}) \times \mathbf{n}]\!] = \mathbf{0} & \text{on} \quad (\partial\Theta \setminus \Gamma) \times (0,T), \\ \boldsymbol{A}(\cdot, 0) = \boldsymbol{A}_0 & \text{in} \quad \Theta := \Sigma \cup \Pi. \end{cases}$$

GHENT UNIVERSITY

Energy balance in the steel workpiece

The Joule heat caused by the eddy current

$$\boldsymbol{J} = \sigma_\Sigma |\partial_t \boldsymbol{A}|^2 .$$



The IBVP for the temperature $\mathcal{T}$

$$\begin{cases} \rho c_p(\mathcal{T})\partial_t \mathcal{T} - \nabla \cdot (\kappa(\mathcal{T})\nabla \mathcal{T}) = \mathcal{R}_r\left(\sigma_\Sigma |\partial_t \boldsymbol{A}|^2\right) - \rho f(z, \mathcal{T})\partial_t z & \text{in} \quad \Sigma \times (0,T), \\ -\kappa(\mathcal{T})\nabla\mathcal{T} \cdot \mathbf{n} = \alpha(\mathcal{T} - u^c) & \text{on} \quad \partial\Sigma \times (0,T), \\ \mathcal{T}(\cdot, 0) = \mathcal{T}_0 & \text{in} \quad \Sigma, \end{cases}$$

where

$$f(z, \mathcal{T}) = -L\left(z_{\text{eq}}(\mathcal{T}) - z - \mathcal{T} z'_{\text{eq}}(\mathcal{T})\right).$$

GHENT UNIVERSITY

Kirchhoff transformation

We introduce the primitive function $\hat{\kappa}$ from κ

$$\hat{\kappa}(x) = \int_0^x \kappa(s)\,\mathrm{d}s.$$

Let $u := \hat{\kappa}(\mathcal{T})$ and $u_0 := \hat{\kappa}(\mathcal{T}_0)$. The IBVP for the transformed temperature u

$$\begin{cases} \beta(\hat{\kappa}^{-1}(u))\partial_t u - \Delta u = \mathcal{R}_r\left(\sigma_\Sigma |\partial_t \boldsymbol{A}|^2\right) - \rho f(z, \hat{\kappa}^{-1}(u))\partial_t z & \text{in} \quad \Sigma \times (0,T), \\ -\nabla u \cdot \mathbf{n} = \alpha(\hat{\kappa}^{-1}(u) - u^c) & \text{on} \quad \partial\Sigma \times (0,T), \\ u(\cdot, 0) = u_0 & \text{in} \quad \Sigma. \end{cases}$$

GHENT UNIVERSITY

Leblond-Devaux model

Leblond-Devaux model for the austenite proportion z

$$\begin{cases} \partial_t z = \dfrac{z_{\mathsf{eq}}(\mathcal{T}) - z}{\vartheta(\mathcal{T})} & \text{in} \quad \Sigma \times (0,T), \\ z(\cdot,0) = z_0 = 0 & \text{in} \quad \Sigma. \end{cases}$$

or with the transformed temperature u

$$\begin{cases} \partial_t z = \dfrac{z_{\mathsf{eq}}(\hat{\kappa}^{-1}(u)) - z}{\vartheta(\hat{\kappa}^{-1}(u))} & \text{in} \quad \Sigma \times (0,T), \\ z(\cdot,0) = z_0 = 0 & \text{in} \quad \Sigma. \end{cases}$$

GHENT UNIVERSITY

Induction hardening problem

Electric potential ϕ

$$\begin{cases} \nabla \cdot (-\sigma_\Pi \nabla \phi) = 0, \\ -\sigma_\Pi \nabla \phi \cdot \mathbf{n} = 0, \\ -\sigma_\Pi \nabla \phi \cdot \mathbf{n} = j. \end{cases}$$

Magnetic potential $\boldsymbol{A}$

$$\begin{cases} \sigma(z)\partial_t \boldsymbol{A} + \nabla \times \hat{\mu}^{-1}\boldsymbol{M}(\nabla \times \boldsymbol{A}) \\ \qquad\qquad + \chi_\Pi \sigma \nabla \phi = \mathbf{0}, \\ \nabla \cdot \boldsymbol{A} = 0, \\ \boldsymbol{A} \times \mathbf{n} = \mathbf{0}, \\ [\![\hat{\mu}^{-1}\boldsymbol{M}(\nabla \times \boldsymbol{A}) \times \mathbf{n}]\!] = \mathbf{0}, \\ \boldsymbol{A}(\cdot, 0) = \boldsymbol{A}_0. \end{cases}$$

Temperature u

$$\begin{cases} \beta(z, \hat{\kappa}^{-1}(u))\partial_t u - \Delta u \\ \qquad = \mathcal{R}_r \left(\sigma_\Sigma(z)|\partial_t \boldsymbol{A}|^2\right) - \rho(z) f(z, \hat{\kappa}^{-1}(u))\partial_t z, \\ -\nabla u \cdot \mathbf{n} = \alpha(\hat{\kappa}^{-1}(u) - u^c), \\ u(\cdot, 0) = u_0. \end{cases}$$

Austenite proportion z

$$\begin{cases} \partial_t z = \dfrac{z_{\text{eq}}(\hat{\kappa}^{-1}(u)) - z}{\vartheta(\hat{\kappa}^{-1}(u))}, \\ z(\cdot, 0) = z_0 = 0. \end{cases}$$

GHENT UNIVERSITY

Function spaces

Let us introduce the following Hilbert spaces

$$\mathrm{Z} := \mathrm{H}^1(\Pi)/\mathbb{R},$$
$$\boldsymbol{W}_0 := \left\{\boldsymbol{f} \in \mathbf{H}(\mathrm{div}, \Omega) \cap \mathbf{H}_0(\mathrm{curl}, \Omega) : \nabla \cdot \boldsymbol{f} = 0\right\},$$

equipped with the norms

$$\|f\|_{\mathrm{Z}} := \|\nabla f\|_{\mathbf{L}^2(\Pi)},$$
$$\|\boldsymbol{f}\|_{\boldsymbol{W}_0} := \|\nabla \times \boldsymbol{f}\|_{\mathbf{L}^2(\Omega)}.$$

Remark: The space $\boldsymbol{W}_0$ is continuously embedded into $\mathbf{H}^1(\Omega)$ if the domain Ω is bounded and either the boundary $\partial\Omega$ is of class $\mathrm{C}^{1,1}$ or Ω is a convex polyhedron.

GHENT UNIVERSITY

Variational formulation

Find $\phi \in \mathrm{Z}, z \in \mathrm{L}^2(\Sigma), \boldsymbol{A} \in \boldsymbol{W}_0$ and $u \in \mathrm{H}^1(\Sigma)$ such that

$$
\begin{aligned}
&\sigma_\Pi \left(\nabla\phi, \nabla\psi\right)_\Pi + (j, \psi)_\Gamma = 0,\\
&(\partial_t z, \varsigma)_\Sigma = \left(\frac{z_{\mathsf{eq}}(\hat{\kappa}^{-1}(u)) - z}{\vartheta(\hat{\kappa}^{-1}(u))}, \varsigma\right)_\Sigma,\\
&(\sigma(z)\partial_t \boldsymbol{A}, \boldsymbol{\varphi})_\Theta + \left(\hat{\mu}^{-1}\boldsymbol{M}(\nabla \times \boldsymbol{A}), \nabla \times \boldsymbol{\varphi}\right)_\Omega + \sigma_\Pi \left(\nabla\phi, \boldsymbol{\varphi}\right)_\Pi = 0,\\
&\left(\beta(z, \hat{\kappa}^{-1}(u))\partial_t u, w\right)_\Sigma + (\nabla u, \nabla w)_\Sigma + \alpha\left(\hat{\kappa}^{-1}(u) - u^c, w\right)_{\partial\Sigma}\\
&\qquad = \left(\mathcal{R}_r\left(\sigma_\Sigma(z)|\partial_t \boldsymbol{A}|^2\right), w\right)_\Sigma - \left(\rho(z) f(z, \hat{\kappa}^{-1}(u))\partial_t z, w\right)_\Sigma
\end{aligned}
$$

are valid for any $\psi \in \mathrm{Z}, \varsigma \in \mathrm{L}^2(\Sigma), \boldsymbol{\varphi} \in \boldsymbol{W}_0$ and $w \in \mathrm{H}^1(\Sigma)$.

GHENT UNIVERSITY

Time discretization

Let $[0, T]$ be partitioned into n equidistant subintervals with time step

$$\tau = \frac{T}{n}.$$

For any function v, we introduce the notations

$$v_i = v(t_i), \qquad \delta v_i = \frac{v_i - v_{i-1}}{\tau},$$

which stand for the value of function v and the approximation of its time derivative at the time-point $t_i := i\tau, i = 1, 2, ..., n$.

GHENT UNIVERSITY

Time discretization (cont.)

For every $i = 1, 2, \ldots, n$, find $\phi_i \in \mathrm{Z}, z_i \in \mathrm{L}^2(\Sigma), \boldsymbol{A}_i \in \boldsymbol{W}_0$ and $u_i \in \mathrm{H}^1(\Sigma)$ such that

$$
\begin{aligned}
&\sigma_\Pi \left(\nabla\phi_i, \nabla\psi\right)_\Pi + (j_i, \psi)_\Gamma = 0, \\
&(\delta z_i, \varsigma)_\Sigma = \left(\frac{z_{\text{eq}}(\hat{\kappa}^{-1}(u_{i-1})) - z_i}{\vartheta(\hat{\kappa}^{-1}(u_{i-1}))}, \varsigma\right)_\Sigma, \\
&\left(\sigma(z_i)\delta \boldsymbol{A}_i, \boldsymbol{\varphi}\right)_\Theta + \left(\hat{\mu}^{-1}\boldsymbol{M}(\nabla\times\boldsymbol{A}_i), \nabla\times\boldsymbol{\varphi}\right)_\Omega + \sigma_\Pi\left(\nabla\phi_i, \boldsymbol{\varphi}\right)_\Pi = 0, \\
&\left(\beta(z_i, \hat{\kappa}^{-1}(u_{i-1}))\delta u_i, w\right)_\Sigma + (\nabla u_i, \nabla w)_\Sigma + \alpha\left(\hat{\kappa}^{-1}(u_{i-1}) - u_i^c, w\right)_{\partial\Sigma} \\
&\qquad = \left(\mathcal{R}_r\left(\sigma_\Sigma(z_i)|\delta\boldsymbol{A}_i|^2\right), w\right)_\Sigma - \left(\rho(z_i) f(z_i, \hat{\kappa}^{-1}(u_{i-1}))\ \delta z_i, w\right)_\Sigma
\end{aligned}
$$

are valid for any $\psi \in \mathrm{Z}, \varsigma \in \mathrm{L}^2(\Sigma), \boldsymbol{\varphi} \in \boldsymbol{W}_0$ and $w \in \mathrm{H}^1(\Sigma)$.

GHENT UNIVERSITY

Solvability of the discretized problem

Lemma

There exists a unique solution $(\phi_i, z_i, \boldsymbol{A}_i, u_i) \in \mathrm{Z} \times \mathrm{L}^2(\Sigma) \times \boldsymbol{W}_0 \times \mathrm{H}^1(\Sigma)$ *to the discretized variational problem for any* $i = 1, 2, \ldots, n$.

Proof:

- The existence of a unique solution (ϕ_i, z_i, u_i) is guaranteed by using the Lax-Milgram lemma.
- Thanks to the properties of the nonlinear operator $\boldsymbol{M}$, we invoke the Browder-Minty theorem to prove the existence of a unique solution $\boldsymbol{A}_i$.

GHENT UNIVERSITY

A priori estimates

Lemma

There exists a constant $C > 0$ such that

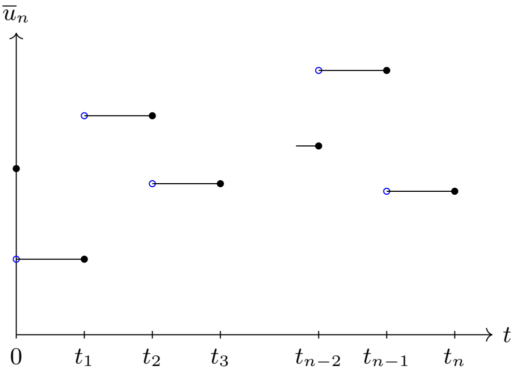
$$(i)\quad \max_{1\le l\le n} \|\nabla \delta\phi_l\|_{\mathbf{L}^2(\Pi)} \le C,$$

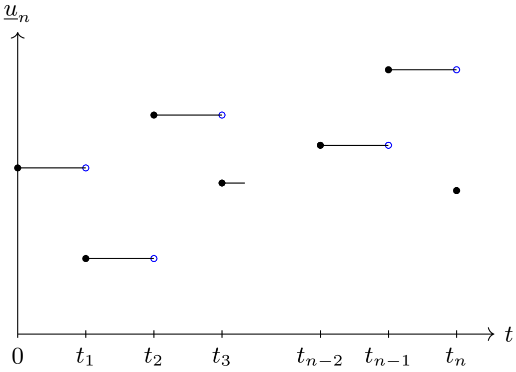
$$(ii)\quad 0 \le z_i(\boldsymbol{x}) \le 1, \qquad \left|\delta z_i(\boldsymbol{x})\right| \le C, \qquad \max_{1\le l\le n} \|\nabla z_l\|_{\mathbf{L}^2(\Sigma)} \le C,$$

$$(iii)\quad \max_{1\le l\le n} \|\delta \boldsymbol{A}_l\|^2_{\mathbf{L}^2(\Theta)} + \sum_{i=1}^{n} \|\delta \boldsymbol{A}_i - \delta \boldsymbol{A}_{i-1}\|^2_{\mathbf{L}^2(\Theta)} + \sum_{i=1}^{n} \|\nabla \times \delta \boldsymbol{A}_i\|^2_{\mathbf{L}^2(\Omega)}\, \tau \le C,$$

$$(iv)\quad \sum_{i=1}^{n} \|\delta u_i\|^2_{\mathrm{L}^2(\Sigma)}\, \tau + \max_{1\le l\le n} \|\nabla u_l\|^2_{\mathbf{L}^2(\Sigma)} + \sum_{i=1}^{n} \|\nabla u_i - \nabla u_{i-1}\|^2_{\mathbf{L}^2(\Sigma)} + \max_{1\le l\le n} \|u_l\|^2_{\mathrm{L}^2(\partial\Sigma)} \le C.$$

Rothe's piecewise constant in time functions



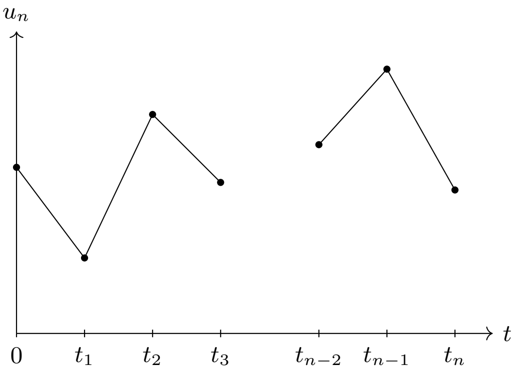


$$\overline{u}_n(t) = u_i, \qquad t \in (t_{i-1}, t_i].$$

$$\underline{u}_n(t) = u_{i-1}, \qquad t \in [t_{i-1}, t_i).$$

GHENT UNIVERSITY

Rothe's piecewise affine in time function



$$u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i, \qquad t \in [t_{i-1}, t_i].$$

GHENT UNIVERSITY

Discretized problem in the continous sense

The following identities are valid for any $\psi \in \mathrm{Z}, \varsigma \in \mathrm{L}^2(\Sigma), \boldsymbol{\varphi} \in \boldsymbol{W}_0$ and $w \in \mathrm{H}^1(\Sigma)$

$$\sigma_\Pi \left(\nabla \overline{\phi}_n, \nabla \psi\right)_\Pi + \left(\overline{j}_n, \psi\right)_\Gamma = 0,$$

$$(\partial_t z_n, \varsigma)_\Sigma = \left(\frac{z_{\text{eq}}(\hat{\kappa}^{-1}(\underline{u}_n)) - \overline{z}_n}{\vartheta(\hat{\kappa}^{-1}(\underline{u}_n))}, \varsigma \right)_\Sigma,$$

$$(\sigma(\overline{z}_n)\partial_t \boldsymbol{A}_n, \boldsymbol{\varphi})_\Theta + \left(\hat{\mu}^{-1}\boldsymbol{M}(\nabla \times \overline{\boldsymbol{A}}_n), \nabla \times \boldsymbol{\varphi}\right)_\Omega + \sigma_\Pi \left(\nabla \overline{\phi}_n, \boldsymbol{\varphi}\right)_\Pi = 0,$$

$$\left(\beta(\overline{z}_n, \hat{\kappa}^{-1}(\underline{u}_n))\partial_t u_n, w\right)_\Sigma + (\nabla \overline{u}_n, \nabla w)_\Sigma + \alpha \left(\hat{\kappa}^{-1}(\underline{u}_n) - \overline{u}_n^c, w\right)_{\partial\Sigma}$$

$$= \left(\mathcal{R}_r \left(\sigma(\overline{z}_n)|\partial_t \boldsymbol{A}_n|^2\right), w\right)_\Sigma - \left(\rho(\overline{z}_n) f(\overline{z}_n, \hat{\kappa}^{-1}(\underline{u}_n)) \, \partial_t z_n, w\right)_\Sigma.$$

GHENT UNIVERSITY

Existence of a weak solution

Theorem

There exists a weak solution

$$(\phi, z, u, \boldsymbol{A}) \in \mathrm{Lip}([0,T],\mathrm{Z}) \times \left[\mathrm{C}([0,T],\mathrm{L}^2(\Sigma)) \cap \mathrm{L}^\infty((0,T),\mathrm{H}^1(\Sigma))\right]^2 \times \mathrm{C}([0,T],\boldsymbol{W}_0)$$

to the variational system. Moreover, the following convergences hold for subsequences

$$\begin{aligned}
&\overline{\phi}_{n_k} \to \phi && \text{in} \quad \mathrm{C}([0,T],\mathrm{Z}),\\
&z_{n_k} \to z, \quad u_{n_k} \to u && \text{in} \quad \mathrm{C}([0,T],\mathrm{L}^2(\Sigma)),\\
&\partial_t z_{n_k} \rightharpoonup \partial_t z, \quad \partial_t u_{n_k} \rightharpoonup \partial_t u && \text{in} \quad \mathrm{L}^2((0,T),\mathrm{L}^2(\Sigma)),\\
&\overline{z}_{n_k} \to z, \quad \overline{u}_{n_k} \to u, \quad \underline{u}_{n_k} \to u && \text{in} \quad \mathrm{L}^2((0,T),\mathrm{L}^2(\Sigma)),\\
&\boldsymbol{A}_{n_k} \to \boldsymbol{A} && \text{in} \quad \mathrm{C}([0,T],\mathbf{L}^2(\Omega)),\\
&\overline{\boldsymbol{A}}_{n_k} \to \boldsymbol{A}, \quad \partial_t \boldsymbol{A}_{n_k} \rightharpoonup \partial_t \boldsymbol{A} && \text{in} \quad \mathrm{L}^2((0,T),\boldsymbol{W}_0).
\end{aligned}$$

Reference

V. C. Le, M. Slodička, and K. Van Bockstal.
Existence of a weak solution to a nonlinear induction hardening problem with Leblond-Devaux model for a steel workpiece.
Communications in Nonlinear Science and Numerical Simulation, 107:106156, 2022.

GHENT UNIVERSITY

Conclusions

- A mathematical model has been introduced for the induction hardening process.
- A time discretization scheme has been proposed.
- Numerical analysis have been performed, and the existence of a weak solution has been proved.
- The uniqueness of a solution to the nonlinear induction hardening problem is still a challenge.

GHENT UNIVERSITY

THANK YOU FOR YOUR ATTENTION!