International Conference on Differential Equations and Applications
Dedicated to Professor Dinh Nho Hao on the Occasion of his 60th Birthday

SPACE-TIME DISCRETIZATION OF A DEGENERATE PARABOLIC PROBLEM INVOLVING A MOVING BODY*

VAN CHIEN LE

August 17, 2022

GHENT UNIVERSITY

imec

*A joint work with M. Slodička and K. Van Bockstal

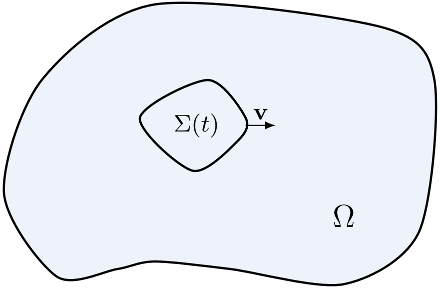
Outline

GHENT UNIVERSITY

Mathematical problem

We consider the following degenerate parabolic problem

$$\begin{cases} \alpha \partial_t \boldsymbol{u} - \beta \Delta \boldsymbol{u} + \chi_\Sigma \boldsymbol{A} \boldsymbol{u} = \boldsymbol{f} & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} = \boldsymbol{0} & \text{on} \quad \partial\Omega \times (0,T), \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0 & \text{in} \quad \Sigma_0, \end{cases}$$



where

$$\alpha = \begin{cases} \alpha_\Sigma & \text{in} \quad \Sigma(t) \\ 0 & \text{in} \quad \Omega \setminus \overline{\Sigma(t)} \end{cases}, \qquad \boldsymbol{A}\boldsymbol{u} = \left[\sum_{j,k=1}^{d} a^{ijk} \frac{\partial u^j}{\partial x^k} + \sum_{j=1}^{d} b^{ij} u^j \right]_{i=1}^{d}$$

GHENT UNIVERSITY

Mixed variational formulation

Let us introduce the following Hilbert spaces

$$\mathrm{L}_0^2(\Omega) := \mathrm{L}^2(\Omega)/\mathbb{R},$$
$$\mathbf{H}_0^1(\mathrm{div},\Omega) := \left\{\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\boldsymbol{\varphi} = 0\right\}.$$

Find $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$ with $\partial_t\boldsymbol{u} \in \mathbf{L}^2(\Sigma(t))$ and $p \in \mathrm{L}_0^2(\Omega)$ such that

$$\alpha_\Sigma\,(\partial_t\boldsymbol{u},\boldsymbol{\varphi})_{\Sigma(t)} + \beta\,(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi})_\Omega + (\boldsymbol{A}\boldsymbol{u},\boldsymbol{\varphi})_{\Sigma(t)} + (p,\nabla\cdot\boldsymbol{\varphi})_\Omega = (\boldsymbol{f},\boldsymbol{\varphi})_\Omega\,,$$
$$(\nabla\cdot\boldsymbol{u},q)_\Omega = 0.$$

are valid for any $\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega)$ and any $q \in \mathrm{L}_0^2(\Omega)$.

GHENT UNIVERSITY

Well-posedness

Theorem (Well-posedness)

Let us assume

$$\boldsymbol{u}_0 \in \mathbf{H}_0^1(\mathsf{div}, \Omega) \cap \mathbf{H}^2(\Omega), \qquad \Delta \boldsymbol{u}_0 = \mathbf{0} \quad \textit{on} \quad \Omega \setminus \overline{\Sigma_0}, \qquad \mathbf{v} \in \mathbf{C}^1(\overline{\Omega} \times [0,T]),$$
$$\boldsymbol{f} \in \mathrm{Lip}([0,T], \mathbf{L}^2(\Omega)), \qquad a^{ijk}, b^{ij} \in \mathrm{Lip}([0,T], \mathrm{L}^\infty(\Omega)), \qquad i,j,k = 1,\dots,d.$$

Then, the mixed variational system admits exactly one solution $(\boldsymbol{u}, p)$ *satisfying*

$$p \in \mathrm{L}^2((0,T), \mathrm{L}_0^2(\Omega)), \qquad \boldsymbol{u} \in \mathrm{C}([0,T], \mathbf{H}_0^1(\mathsf{div}, \Omega)), \qquad \partial_t \boldsymbol{u} \in \mathrm{L}^2((0,T), \mathbf{H}_0^1(\mathsf{div}, \Omega)).$$

Proof: Using Brezzi's theorem combined with the Reynolds transport theorem

$$\frac{\mathsf{d}}{\mathrm{d}t} \int_{\omega(t)} f \, \mathrm{d}\boldsymbol{x} = \int_{\omega(t)} \partial_t f \, \mathrm{d}\boldsymbol{x} + \int_{\partial\omega(t)} f \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}s.$$

GHENT UNIVERSITY

Space-time discretization

The time interval $[0, T]$ is partitioned into n equidistant subintervals with time step

$$\tau = \frac{T}{n}.$$

Let $\mathbf{V}_0^h \subset \mathbf{H}_0^1(\Omega)$ and $\mathrm{V}^h \subset \mathrm{L}_0^2(\Omega)$ be FE spaces with orthogonal projection operators

$$\mathbf{P}^h \in \mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{V}_0^h), \qquad \mathrm{P}^h \in \mathcal{L}(\mathrm{L}_0^2(\Omega), \mathrm{V}^h).$$

According to Céa's lemma, there exists a constant $C > 0$ such that

$$\left\| q - \mathrm{P}^h q \right\|_{\mathrm{L}^2(\Omega)} \le C \inf_{q^h \in \mathrm{V}^h} \left\| q - q^h \right\|_{\mathrm{L}^2(\Omega)},$$

$$\left\| \boldsymbol{\varphi} - \mathbf{P}^h \boldsymbol{\varphi} \right\|_{\mathbf{H}_0^1(\Omega)} \le C \inf_{\boldsymbol{\varphi}^h \in \mathbf{V}_0^h} \left\| \boldsymbol{\varphi} - \boldsymbol{\varphi}^h \right\|_{\mathbf{H}_0^1(\Omega)}.$$

GHENT UNIVERSITY

Space-time discretization scheme

Let $\boldsymbol{u}_i^h$ and p_i^h be approximations of $\boldsymbol{u}$ and p at time $t_i := i\tau$. Moreover, we introduce the following notations

$$\delta \boldsymbol{u}_i^h = \frac{\boldsymbol{u}_i^h - \boldsymbol{u}_{i-1}^h}{\tau}, \qquad \boldsymbol{u}_0^h = \mathbf{P}^h \boldsymbol{u}_0, \qquad \boldsymbol{A}_i = \boldsymbol{A}(t_i), \qquad \Sigma_i = \Sigma(t_i).$$

Find $\boldsymbol{u}_i^h \in \mathbf{V}_0^h$ and $p_i^h \in \mathrm{V}^h$ such that

$$\begin{aligned} \alpha_\Sigma \left(\delta \boldsymbol{u}_i^h, \boldsymbol{\varphi}^h\right)_{\Sigma_i} + \beta \left(\nabla \boldsymbol{u}_i^h, \nabla \boldsymbol{\varphi}^h\right)_\Omega + \left(\boldsymbol{A}_i \boldsymbol{u}_i^h, \boldsymbol{\varphi}^h\right)_{\Sigma_i} & \\ + \left(p_i^h, \nabla \cdot \boldsymbol{\varphi}^h\right)_\Omega &= \left(\boldsymbol{f}_i, \boldsymbol{\varphi}^h\right)_\Omega, \\ \left(\nabla \cdot \boldsymbol{u}_i^h, q^h\right)_\Omega &= 0 \end{aligned}$$

are valid for any $\boldsymbol{\varphi}^h \in \mathbf{V}_0^h, q^h \in \mathrm{V}^h$ and for any $i = 1, 2, \ldots, n$.

Solvability of the discretized problem

Lemma (Solvability)

We assume that the discrete inf-sup condition is satisfied, i.e.

$$\sup_{\boldsymbol{\varphi}^h \in \mathbf{V}_0^h,\ \boldsymbol{\varphi}^h \neq \mathbf{0}} \frac{\left(\nabla \cdot \boldsymbol{\varphi}^h, q^h\right)_\Omega}{\left\|\boldsymbol{\varphi}^h\right\|_{\mathbf{H}_0^1(\Omega)}} \geq C \left\|q^h\right\|_{\mathrm{L}^2(\Omega)} \qquad \forall q^h \in \mathrm{V}^h .$$

Then, for any $i = 1, 2, \ldots, n$, *there exists a unique couple* $(\boldsymbol{u}_i^h, p_i^h) \in \mathbf{V}_0^h \times \mathrm{V}^h$ *solving the discretized variational problem. Moreover, there exists a constant* $C > 0$ *such that*

$$\max_{1 \leq l \leq n} \left\|\delta \boldsymbol{u}_l^h\right\|^2_{\mathbf{L}^2(\Sigma_l)} + \sum_{i=1}^n \left\|\nabla \delta \boldsymbol{u}_i^h\right\|^2_{\mathbf{L}^2(\Omega)} \tau + \sum_{i=1}^n \left\|\delta \boldsymbol{u}_i^h - \delta \boldsymbol{u}_{i-1}^h\right\|^2_{\mathbf{L}^2(\Sigma_{i-1})} + \max_{1 \leq l \leq n} \left\|p_l^h\right\|^2_{\mathrm{L}^2(\Omega)} \leq C.$$

Solvability of the discretized problem (cont.)

Proof

- The solvability of the mixed variational problem is guaranteed by Brezzi's theorem.
- We subtract the discretized problem for $i = i - 1$ from itself, then set $\boldsymbol{\varphi}^h = \delta \boldsymbol{u}_i^h, q^h = \delta p_i^h$ to get the a priori estimate.

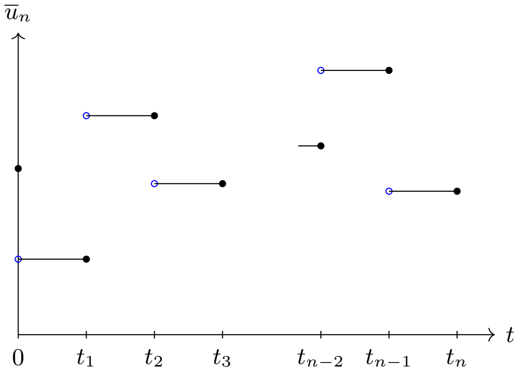
Remarks:

- The discretized inf-sup condition depends on the choice of FE spaces.
- The Reynolds transport theorem and the following estimate are crucial

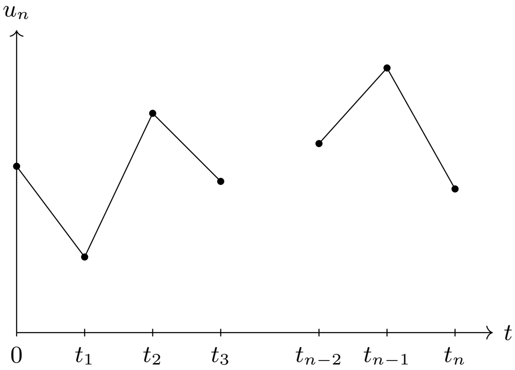
$$\sum_{i=1}^{n} \left\| \nabla \boldsymbol{u}_i^h \right\|_{\mathbf{H}^1(\Omega')}^2 \leq C(\Omega') \qquad \forall\ \Omega' \subset\subset \Omega\ (\overline{\Omega'} \subset \Omega).$$

GHENT UNIVERSITY

Rothe's functions



$$\overline{u}_n(t) = u_i, \qquad t \in (t_{i-1}, t_i].$$



$$u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i.$$

GHENT UNIVERSITY

Discretized problem in the continuous sense

The following identities are valid for any $\boldsymbol{\varphi}^h \in \mathbf{V}_0^h$ and $q^h \in \mathrm{V}^h$

$$\alpha_\Sigma \left(\partial_t \boldsymbol{u}_n^h, \boldsymbol{\varphi}^h\right)_{\overline{\Sigma}_n} + \beta \left(\nabla \overline{\boldsymbol{u}}_n^h, \nabla \boldsymbol{\varphi}^h\right)_\Omega + \left(\overline{\boldsymbol{A}}_n \overline{\boldsymbol{u}}_n^h, \boldsymbol{\varphi}^h\right)_{\overline{\Sigma}_n} + \left(\overline{p}_n^h, \nabla \cdot \boldsymbol{\varphi}^h\right)_\Omega = \left(\overline{\boldsymbol{f}}_n, \boldsymbol{\varphi}^h\right)_\Omega,$$

$$\left(\nabla \cdot \overline{\boldsymbol{u}}_n^h, q^h\right)_\Omega = 0.$$

Error estimate

Theorem

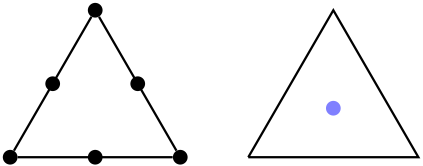
The following relation holds true for any $\xi \in [0, T]$

$$\int_0^\xi \left\| \partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t) \right\|^2_{\mathbf{L}^2(\Sigma(t))} \, \mathrm{d}t + \int_0^\xi \left\| \overline{p}_n^h(t) - p(t) \right\|^2_{\mathrm{L}^2(\Omega)} \, \mathrm{d}t$$

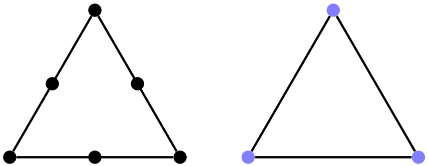
$$+ \left\| \nabla \boldsymbol{u}_n^h(\xi) - \nabla \boldsymbol{u}(\xi) \right\|^2_{\mathbf{L}^2(\Omega)} \leq C \left(\tau + \left\| \nabla \boldsymbol{u}_0 - \nabla \mathbf{P}^h \boldsymbol{u}_0 \right\|^2_{\mathbf{L}^2(\Omega)} \right.$$

$$\left. + \sqrt{\int_0^\xi \left\| p(t) - \mathrm{P}^h p(t) \right\|^2_{\mathrm{L}^2(\Omega)} \, \mathrm{d}t} + \int_0^\xi \left\| \nabla \partial_t \boldsymbol{u}(t) - \nabla \mathbf{P}^h \partial_t \boldsymbol{u}(t) \right\|^2_{\mathbf{L}^2(\Omega)} \, \mathrm{d}t \right).$$

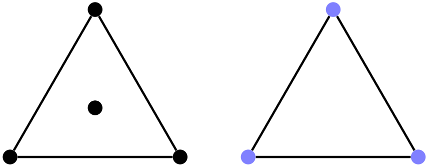
Some finite element pairs



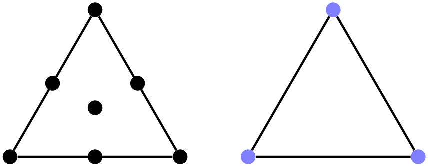
(a) $\mathcal{P}_2 - \mathcal{P}_0$ *elements*



(b) *Taylor-Hood elements*



(c) *Mini elements*



(d) $\mathcal{P}_{2b} - \mathcal{P}_1$ *elements*

$\mathcal{P}_2 - \mathcal{P}_0$ elements

Let

$$\mathbf{V}_0^h = (\mathcal{P}_2)^d \cap \mathbf{H}_0^1(\Omega) \qquad \text{and} \qquad \mathrm{V}^h = \mathcal{P}_0 \cap \mathrm{L}_0^2(\Omega).$$

In addition, we assume that

$$\boldsymbol{u} \in \mathrm{C}([0,T], \mathbf{H}_0^1(\mathsf{div}, \Omega)) \cap \mathrm{L}^2((0,T), \mathbf{H}^2(\Omega)), \qquad p \in \mathrm{L}^2((0,T), \mathrm{H}^1(\Omega) \cap \mathrm{L}_0^2(\Omega)).$$

Then, there exists a constant $C > 0$ such that

$$\int_0^\xi \left\| \partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t) \right\|_{\mathbf{L}^2(\Sigma(t))}^2 \mathrm{d}t + \left\| \nabla \boldsymbol{u}_n^h(\xi) - \nabla \boldsymbol{u}(\xi) \right\|_{\mathbf{L}^2(\Omega)}^2$$
$$+ \int_0^\xi \left\| \overline{p}_n^h(t) - p(t) \right\|_{\mathrm{L}^2(\Omega)}^2 \mathrm{d}t \le C(\tau + h).$$

GHENT UNIVERSITY

The convergence rate: $\mathcal{O}(\sqrt{\tau} + \sqrt{h})$.

Taylor-Hood $\mathcal{P}_2 - \mathcal{P}_1$ elements

Let

$$\mathbf{V}_0^h = (\mathcal{P}_2)^d \cap \mathbf{H}_0^1(\Omega) \qquad \text{and} \qquad \mathrm{V}^h = \mathcal{P}_1 \cap \mathrm{L}_0^2(\Omega).$$

In addition, we assume that

$$\boldsymbol{u} \in \mathrm{C}([0,T], \mathbf{H}_0^1(\mathrm{div}, \Omega)) \cap \mathrm{L}^2((0,T), \mathbf{H}^2(\Omega)), \qquad p \in \mathrm{L}^2((0,T), \mathrm{H}^2(\Omega) \ \cap \mathrm{L}_0^2(\Omega)).$$

Then, there exists a constant $C > 0$ such that

$$\int_0^\xi \left\| \partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t) \right\|_{\mathbf{L}^2(\Sigma(t))}^2 \mathrm{d}t + \left\| \nabla \boldsymbol{u}_n^h(\xi) - \nabla \boldsymbol{u}(\xi) \right\|_{\mathbf{L}^2(\Omega)}^2$$

$$+ \int_0^\xi \left\| \overline{p}_n^h(t) - p(t) \right\|_{\mathrm{L}^2(\Omega)}^2 \mathrm{d}t \le C(\tau + h^2).$$

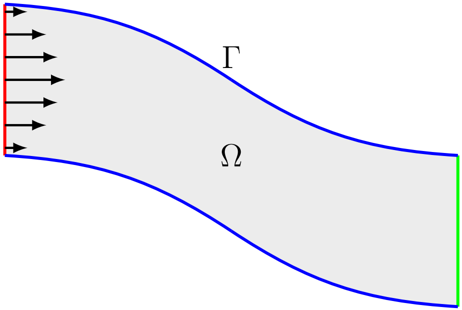
GHENT UNIVERSITY

The convergence rate: $\mathcal{O}(\sqrt{\tau} + h)$.

Stokes equations

Let us consider the following Stokes equations

$$\begin{cases} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} = \boldsymbol{g} & \text{on} \quad \Gamma \times (0,T), \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0 & \text{in} \quad \Omega. \end{cases}$$



GHENT UNIVERSITY

Error estimates

- The present estimate for mini elements

$$\int_0^{\xi} \left\| \partial_t \boldsymbol{u}_n^h(t) - \partial_t \boldsymbol{u}(t) \right\|_{\mathbf{L}^2(\Omega)}^2 \, \mathrm{d}t + \left\| \nabla \boldsymbol{u}_n^h(\xi) - \nabla \boldsymbol{u}(\xi) \right\|_{\mathbf{L}^2(\Omega)}^2 \leq C(\tau + h^2).$$

Convergence rate: $\mathcal{O}(\sqrt{\tau} + h)$.

- The existing error estimate for mini elements (*Acevedo et al. 2021*)

$$\left\| \overline{\boldsymbol{u}}_n^h(\xi) - \boldsymbol{u}(\xi) \right\|_{\mathbf{L}^2(\Omega)}^2 + \tau \int_0^{\xi} \left\| \nabla \boldsymbol{u}_n^h(t) - \nabla \boldsymbol{u}(t) \right\|_{\mathbf{L}^2(\Omega)}^2 \, \mathrm{d}t \leq C(\tau^2 + h^2).$$

Convergence rate: $\mathcal{O}(\sqrt{\tau} + h/\sqrt{\tau})$.

GHENT UNIVERSITY

References and acknowledgement

R. Acevedo, C. Gómez, and B. López-Rodríguez.
Fully discrete finite element approximation for a family of degenerate parabolic mixed equations.
Computers and Mathematics with Applications, 96:155–177, 2021.

V. C. Le, M. Slodička, and K. Van Bockstal.
A full discretization for the saddle-point approach of a degenerate parabolic problem involving a moving body.
Applied Mathematics Letters, 124:107660, 2022.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 101001847).

GHENT UNIVERSITY

THANK YOU FOR YOUR ATTENTION!