FACULTY OF ENGINEERING AND ARCHITECTURE

DEPARTMENT OF INFORMATION TECHNOLOGY
IDLAB - INTERNET TECHNOLOGY AND DATA SCIENCE LAB

The 8th international conference on Advanced Computational Methods in Engineering (ACOMEN2022)

A SPACE-TIME DISCRETIZATION FOR THE MIXED VARIATIONAL FORMULATION OF AN ELECTROMAGNETIC PROBLEM WITH A MOVING NON-MAGNETIC CONDUCTOR

VAN CHIEN LE

September 1, 2022

GHENT UNIVERSITY

imec

Outline

GHENT UNIVERSITY

Mathematical model

Eddy current equations

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \textit{(Gauss's law)}$$
$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \qquad \textit{(Faraday's law)}$$
$$\nabla \times \boldsymbol{H} = \boldsymbol{J}, \qquad \textit{(Ampère's law)}$$

with the constitutive relations

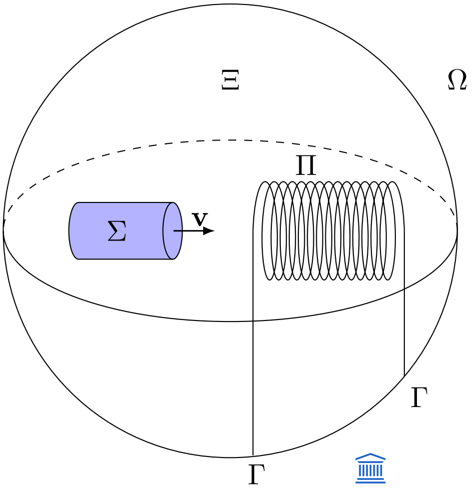
$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{J} = \sigma(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}).$$

Boundary condition

$$\boldsymbol{B} \cdot \mathbf{n} = 0.$$

Interface conditions

$$[\![\boldsymbol{B} \cdot \mathbf{n}]\!] = 0, \qquad [\![\boldsymbol{H} \times \mathbf{n}]\!] = \mathbf{0}, \qquad [\![(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}) \times \mathbf{n}]\!] = \mathbf{0}.$$



GHENT UNIVERSITY

Material coefficients

The subscripts Σ, Π and Ξ are used to distinguish the value of material coefficients on subdomains. For instances,

$$\mu = \begin{cases} \mu_\Pi & \text{in} \quad \Pi, \\ \mu_\Sigma & \text{in} \quad \Sigma, \\ \mu_\Xi & \text{in} \quad \Xi, \end{cases} \qquad \text{and} \qquad \sigma = \begin{cases} \sigma_\Pi & \text{in} \quad \Pi, \\ \sigma_\Sigma & \text{in} \quad \Sigma, \\ \sigma_\Xi & \text{in} \quad \Xi. \end{cases}$$

In general, material coefficients have jumps at the interfaces of different subdomains

$$[\![\sigma]\!]_{\partial\Sigma} := \sigma_\Xi - \sigma_\Sigma \neq 0,$$

$$[\![\sigma]\!]_{\partial\Pi\setminus\Gamma} := \sigma_\Xi - \sigma_\Pi \neq 0.$$

Assumptions: $\sigma_\Xi = 0$ and $\mu = \mu_0$ in Ω.

GHENT UNIVERSITY

$\boldsymbol{A} - \phi$ potential formulation

The BVP for the electric scalar potential ϕ

$$\begin{cases} \nabla \cdot (-\sigma_\Pi \nabla \phi) = 0 & \text{in} \quad \Pi \times (0,T), \\ -\sigma_\Pi \nabla \phi \cdot \mathbf{n} = 0 & \text{on} \quad (\partial \Pi \setminus \Gamma) \times (0,T), \\ -\sigma_\Pi \nabla \phi \cdot \mathbf{n} = j & \text{on} \quad \Gamma \times (0,T). \end{cases}$$

The IBVP for the magnetic vector potential $\boldsymbol{A}$ ($\boldsymbol{B} := \nabla \times \boldsymbol{A}$)

$$\begin{cases} \sigma \partial_t \boldsymbol{A} + \mu_0^{-1} \nabla \times \nabla \times \boldsymbol{A} & \\ \qquad + \chi_\Pi \sigma \nabla \phi - \sigma \mathbf{v} \times (\nabla \times \boldsymbol{A}) = \mathbf{0} & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{A} = 0 & \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{A} \times \mathbf{n} = \mathbf{0} & \text{on} \quad \partial\Omega \times (0,T), \\ [\![(\nabla \times \boldsymbol{A}) \times \mathbf{n}]\!] = \mathbf{0} & \text{on} \quad (\partial\Theta \setminus \Gamma) \times (0,T), \\ \boldsymbol{A}(\cdot, 0) = \boldsymbol{A}_0 & \text{in} \quad \Theta(0) := \Sigma(0) \cup \Pi. \end{cases}$$

Function spaces

Goal: proposing a finite element (FE) space-time discretization for the variational problem.

Let us introduce the following Hilbert spaces

$$\boldsymbol{W} = \mathbf{H}(\mathrm{div}, \Omega) \cap \mathbf{H}_0(\mathrm{curl}, \Omega),$$
$$\boldsymbol{W}_0 = \{\boldsymbol{f} \in \boldsymbol{W} : \nabla \cdot \boldsymbol{f} = 0\},$$

equipped with the norms

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}} = \|\boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \times \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \cdot \boldsymbol{\varphi}\|_{\mathrm{L}^2(\Omega)},$$
$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_0} = \|\nabla \times \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)}.$$

In addition, we introduce the spaces

$$\mathrm{Z} = \mathrm{H}^1(\Pi)/\mathbb{R},$$
$$\mathrm{L}^2_0(\Omega) = \mathrm{L}^2(\Omega)/\mathbb{R},$$
$$\mathbf{H}^1_0(\mathrm{div}, \Omega) = \left\{\boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega) : \nabla \cdot \boldsymbol{\varphi} = 0\right\}.$$

GHENT UNIVERSITY

Motivation

The mixed variational formulation for $\boldsymbol{A}$ reads as: find $\boldsymbol{A} \in \boldsymbol{W}$ and $p \in \mathrm{L}^2(\Omega)$ such that

$$\begin{aligned}(\sigma\partial_t \boldsymbol{A}, \boldsymbol{\varphi})_{\Theta(t)} + \mu_0^{-1} (\nabla \times \boldsymbol{A}, \nabla \times \boldsymbol{\varphi})_{\Omega} + \sigma_{\Pi} (\nabla \phi, \boldsymbol{\varphi})_{\Pi} & \\ - \sigma_{\Sigma} \left(\mathbf{v} \times (\nabla \times \boldsymbol{A}), \boldsymbol{\varphi}\right)_{\Sigma(t)} + (p, \nabla \cdot \boldsymbol{\varphi})_{\Omega} = 0 & \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{W}, \\ (\nabla \cdot \boldsymbol{A}, q)_{\Omega} = 0 & \qquad \forall q \in \mathrm{L}^2(\Omega).\end{aligned}$$

Let V^h be a FE subspace of $\mathrm{L}^2(\Omega)$. Since

$$\left(\nabla \cdot \boldsymbol{A}^h, q^h\right)_{\Omega} = 0, \quad \forall q^h \in \mathrm{V}^h \qquad \not\Rightarrow \qquad \nabla \cdot \boldsymbol{A}^h = 0 \quad \text{in } \mathrm{L}^2(\Omega),$$

in general, $\boldsymbol{A}^h \notin \boldsymbol{W}_0$. The solvability of the discretized system is not guaranteed.

Therefore, this approach is impractical.

GHENT UNIVERSITY

Modified problem for $\boldsymbol{A}$

Based on the vector identity

$$\Delta \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla \times \nabla \times \boldsymbol{A},$$

we propose the following modified IBVP for the magnetic potential $\boldsymbol{A}$

$$\begin{cases} \sigma \partial_t \boldsymbol{A} - \mu_0^{-1} \Delta \boldsymbol{A} + \chi_\Pi \sigma \nabla \phi - \sigma \mathbf{v} \times (\nabla \times \boldsymbol{A}) = \mathbf{0} & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{A} = 0 & \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{A} = \mathbf{0} & \text{on} \quad \partial\Omega \times (0,T), \\ [\![(\nabla \boldsymbol{A}) \mathbf{n}]\!] = \mathbf{0} & \text{on} \quad (\partial\Theta \setminus \Gamma) \times (0,T), \\ \boldsymbol{A}(\cdot, 0) = \boldsymbol{A}_0 & \text{in} \quad \Theta(0). \end{cases}$$

GHENT UNIVERSITY

Mixed variational formulation

Find $\phi \in \mathrm{Z}, \boldsymbol{A} \in \mathbf{H}_0^1(\Omega)$ and $p \in \mathrm{L}_0^2(\Omega)$ such that

$$
\begin{aligned}
&\sigma_\Pi \left(\nabla\phi, \nabla\psi\right)_\Pi + \left(j, \psi\right)_\Gamma = 0, \\
&\left(\sigma\partial_t \boldsymbol{A}, \boldsymbol{\varphi}\right)_{\Theta(t)} + \mu_0^{-1}\left(\nabla\boldsymbol{A}, \nabla\boldsymbol{\varphi}\right)_\Omega + \sigma_\Pi \left(\nabla\phi, \boldsymbol{\varphi}\right)_\Pi \\
&\qquad\qquad - \sigma_\Sigma \big(\mathbf{v} \times (\nabla \times \boldsymbol{A}), \boldsymbol{\varphi}\big)_{\Sigma(t)} + \left(p, \nabla\cdot\boldsymbol{\varphi}\right)_\Omega = 0, \\
&\left(\nabla\cdot\boldsymbol{A}, q\right)_\Omega = 0.
\end{aligned}
$$

for any $\psi \in \mathrm{Z}, \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega)$ and any $q \in \mathrm{L}_0^2(\Omega)$.

GHENT UNIVERSITY

Well-posedness

Theorem (Well-posedness)

Let us assume

$$\mathbf{v} \in \mathbf{C}^1(\overline{\Omega} \times [0,T]), \qquad j \in \mathrm{Lip}([0,T], \mathrm{H}^{-1/2}(\Gamma)),$$

$$\boldsymbol{A}_0 \in \mathbf{H}_0^1(\mathsf{div}, \Omega) \cap \mathbf{H}^2(\Omega), \qquad \Delta \boldsymbol{A}_0 = \mathbf{0} \quad \textit{on} \quad \Omega \setminus \overline{\Theta_0}.$$

Then, the mixed variational system admits exactly one solution $(\phi, \boldsymbol{A}, p)$ *satisfying*

$$\phi \in \mathrm{Lip}([0,T], \mathrm{Z}), \qquad \boldsymbol{A} \in \mathrm{C}([0,T], \mathbf{H}_0^1(\mathsf{div}, \Omega)), \qquad p \in \mathrm{L}^2((0,T), \mathrm{L}_0^2(\Omega)).$$

Proof: Using Brezzi's theorem combined with the Reynolds transport theorem

$$\frac{\mathsf{d}}{\mathrm{d}t} \int_{\omega(t)} f \, \mathrm{d}\boldsymbol{x} = \int_{\omega(t)} \partial_t f \, \mathrm{d}\boldsymbol{x} + \int_{\partial\omega(t)} f \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}s.$$

GHENT UNIVERSITY

Space-time discretization

The time interval $[0, T]$ is partitioned into n equidistant subintervals with time step

$$\tau = \frac{T}{n}.$$

Let $\mathrm{U}^h \subset \mathrm{Z}, \mathbf{V}_0^h \subset \mathbf{H}_0^1(\Omega)$ and $\mathrm{V}^h \subset \mathrm{L}_0^2(\Omega)$ be FE spaces with projection operators

$$\mathrm{Q}^h \in \mathcal{L}(\mathrm{Z}, \mathrm{U}^h), \qquad \mathbf{P}^h \in \mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{V}_0^h), \qquad \mathrm{P}^h \in \mathcal{L}(\mathrm{L}_0^2(\Omega), \mathrm{V}^h).$$

According to Céa's lemma, there exists a constant $C > 0$ such that

$$\begin{aligned}
\left\|\psi - \mathrm{Q}^h \psi\right\|_{\mathrm{Z}} &\le C \inf_{\psi^h \in \mathrm{U}^h} \left\|\psi - \psi^h\right\|_{\mathrm{Z}}, \\
\left\|q - \mathrm{P}^h q\right\|_{\mathrm{L}^2(\Omega)} &\le C \inf_{q^h \in \mathrm{V}^h} \left\|q - q^h\right\|_{\mathrm{L}^2(\Omega)}, \\
\left\|\boldsymbol{\varphi} - \mathbf{P}^h \boldsymbol{\varphi}\right\|_{\mathbf{H}_0^1(\Omega)} &\le C \inf_{\boldsymbol{\varphi}^h \in \mathbf{V}_0^h} \left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}^h\right\|_{\mathbf{H}_0^1(\Omega)}.
\end{aligned}$$

GHENT UNIVERSITY

Space-time discretization scheme

Let $\phi_i^h, \boldsymbol{A}_i^h$ and p_i^h be approximations of $\phi, \boldsymbol{A}$ and p at time $t_i := i\tau$. Moreover, we introduce

$$\delta \boldsymbol{A}_i^h = \frac{\boldsymbol{A}_i^h - \boldsymbol{A}_{i-1}^h}{\tau}, \qquad \boldsymbol{A}_0^h = \mathbf{P}^h \boldsymbol{A}_0, \qquad \Sigma_i = \Sigma(t_i), \qquad \Theta_i = \Theta(t_i).$$

Find $\phi_i^h \in \mathrm{U}^h, \boldsymbol{A}_i^h \in \mathbf{V}_0^h$ and $p_i^h \in \mathrm{V}^h$ such that

$$\begin{aligned}
&\sigma_\Pi \left(\nabla \phi_i^h, \nabla \psi^h\right)_\Pi + \left(j_i, \psi^h\right)_\Gamma = 0, \\
&\left(\sigma_i \delta \boldsymbol{A}_i^h, \boldsymbol{\varphi}^h\right)_{\Theta_i} + \mu_0^{-1} \left(\nabla \boldsymbol{A}_i^h, \nabla \boldsymbol{\varphi}^h\right)_\Omega + \sigma_\Pi \left(\nabla \phi_i^h, \boldsymbol{\varphi}^h\right)_\Pi \\
&\qquad - \sigma_\Sigma \left(\mathbf{v}_i \times (\nabla \times \boldsymbol{A}_i^h), \boldsymbol{\varphi}^h\right)_{\Sigma_i} + \left(p_i^h, \nabla \cdot \boldsymbol{\varphi}^h\right)_\Omega = 0, \\
&\left(\nabla \cdot \boldsymbol{A}_i^h, q^h\right)_\Omega = 0
\end{aligned}$$

for any $\psi^h \in \mathrm{U}^h, \boldsymbol{\varphi}^h \in \mathbf{V}_0^h, q^h \in \mathrm{V}^h$ and for any $i = 1, 2, \ldots, n$.

GHENT UNIVERSITY

Solvability of the discretized problem

Lemma (Solvability)

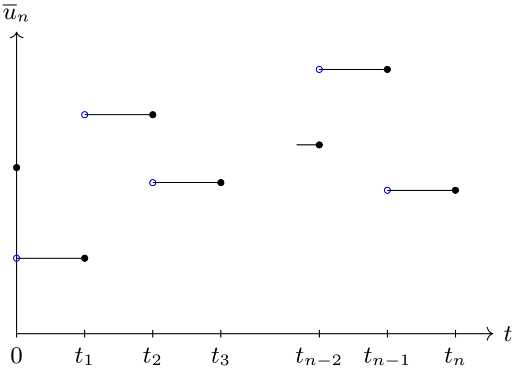
We assume that the discrete inf-sup condition is satisfied, i.e.

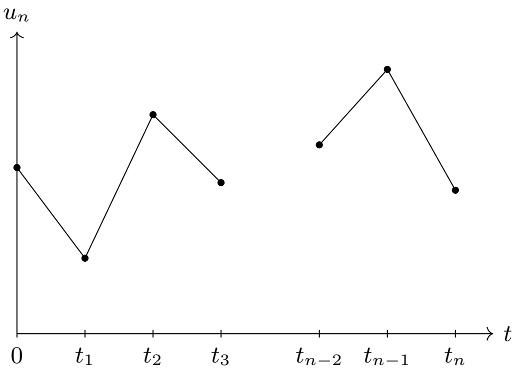
$$\sup_{\boldsymbol{\varphi}^h \in \mathbf{V}_0^h,\ \boldsymbol{\varphi}^h \neq \mathbf{0}} \frac{\left(\nabla \cdot \boldsymbol{\varphi}^h, q^h\right)_\Omega}{\left\|\boldsymbol{\varphi}^h\right\|_{\mathbf{H}_0^1(\Omega)}} \geq C \left\|q^h\right\|_{\mathrm{L}^2(\Omega)} \qquad \forall q^h \in \mathrm{V}^h .$$

Then, for any $i = 1, 2, \ldots, n$, *there exists a solution* $(\phi_i^h, \boldsymbol{A}_i^h, p_i^h) \in \mathrm{U}^h \times \mathbf{V}_0^h \times \mathrm{V}^h$ *to the discretized variational problem. Moreover, there exists a constant* $C > 0$ *such that*

$$\max_{1 \leq l \leq n} \left\|\nabla \delta \phi_l^h\right\|^2_{\mathbf{L}^2(\Pi)} + \max_{1 \leq l \leq n} \left\|p_l^h\right\|^2_{\mathrm{L}^2(\Omega)} + \max_{1 \leq l \leq n} \left\|\delta \boldsymbol{A}_l^h\right\|^2_{\mathbf{L}^2(\Theta_l)} + \sum_{i=1}^{n} \left\|\nabla \delta \boldsymbol{A}_i^h\right\|^2_{\mathbf{L}^2(\Omega)} \tau + \sum_{i=1}^{n} \left\|\delta \boldsymbol{A}_i^h - \delta \boldsymbol{A}_{i-1}^h\right\|^2_{\mathbf{L}^2(\Theta_{i-1})} \leq C.$$

Rothe's functions





$$\overline{u}_n(t) = u_i, \qquad t \in (t_{i-1}, t_i].$$

$$u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i.$$

GHENT UNIVERSITY

Discretized problem in the continuous sense

The following identities are valid for any $\psi^h \in \mathrm{U}^h, \boldsymbol{\varphi}^h \in \mathbf{V}_0^h$ and $q^h \in \mathrm{V}^h$

$$
\begin{aligned}
&\sigma_\Pi \left(\nabla \overline{\phi}_n^h, \nabla \psi^h\right)_\Pi + \left(\overline{j}_n, \psi^h\right)_\Gamma = 0,\\
&\left(\overline{\sigma}_n \partial_t \boldsymbol{A}_n^h, \boldsymbol{\varphi}^h\right)_{\overline{\Sigma}_n} + \mu_0^{-1} \left(\nabla \overline{\boldsymbol{A}}_n^h, \nabla \boldsymbol{\varphi}^h\right)_\Omega + \sigma_\Pi \left(\nabla \overline{\phi}_n^h, \boldsymbol{\varphi}^h\right)_\Pi \\
&\qquad - \sigma_\Sigma \left(\overline{\mathbf{v}}_n \times (\nabla \times \overline{\boldsymbol{A}}_n^h), \boldsymbol{\varphi}^h\right)_{\overline{\Sigma}_n} + \left(\overline{p}_n^h, \nabla \cdot \boldsymbol{\varphi}^h\right)_\Omega = 0,\\
&\left(\nabla \cdot \overline{\boldsymbol{A}}_n^h, q^h\right)_\Omega = 0.
\end{aligned}
$$

GHENT UNIVERSITY

Error estimates

Theorem (Error estimates)

The following relations hold true for any $\xi \in [0, T]$

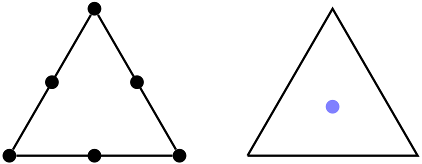
$$i.\ \left\| \nabla \overline{\phi}_n^h(\xi) - \nabla \phi(\xi) \right\|^2_{\mathbf{L}^2(\Pi)} \leq C \left(\tau^2 + \left\| \nabla \phi(\xi) - \nabla \mathrm{Q}^h \phi(\xi) \right\|^2_{\mathbf{L}^2(\Pi)} \right),$$

$$ii.\ \int_0^\xi \left\| \partial_t \boldsymbol{A}_n^h(t) - \partial_t \boldsymbol{A}(t) \right\|^2_{\mathbf{L}^2(\Theta(t))} \mathrm{d}t + \left\| \nabla \boldsymbol{A}_n^h(\xi) - \nabla \boldsymbol{A}(\xi) \right\|^2_{\mathbf{L}^2(\Omega)} + \int_0^\xi \left\| \overline{p}_n^h(t) - p(t) \right\|^2_{\mathrm{L}^2(\Omega)} \mathrm{d}t$$

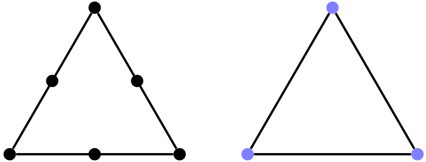
$$\leq C \left(\tau + \left\| \nabla \boldsymbol{A}_0 - \nabla \mathbf{P}^h \boldsymbol{A}_0 \right\|^2_{\mathbf{L}^2(\Omega)} + \left\| \nabla \phi(\xi) - \nabla \mathrm{Q}^h \phi(\xi) \right\|^2_{\mathbf{L}^2(\Pi)} \right.$$

$$\left. + \sqrt{\int_0^\xi \left\| p(t) - \mathrm{P}^h p(t) \right\|^2_{\mathrm{L}^2(\Omega)} \mathrm{d}t} + \int_0^\xi \left\| \nabla \partial_t \boldsymbol{A}(t) - \nabla \mathbf{P}^h \partial_t \boldsymbol{A}(t) \right\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}t \right).$$

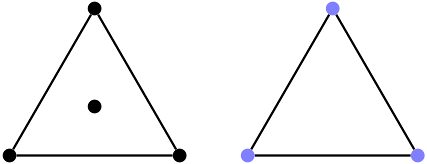
Some finite element pairs



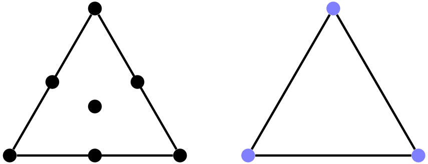
(a) $\mathcal{P}_2 - \mathcal{P}_0$ *elements*



(b) *Taylor-Hood elements*



(c) *Mini elements*



(d) $\mathcal{P}_{2b} - \mathcal{P}_1$ *elements*

Convergence rate for Taylor-Hood elements

Let

$$\mathrm{U}^h = \mathcal{P}_1 \cap \mathrm{Z}, \qquad \mathbf{V}_0^h = (\mathcal{P}_2)^d \cap \mathbf{H}_0^1(\Omega) \qquad \text{and} \qquad \mathrm{V}^h = \mathcal{P}_1 \cap \mathrm{L}_0^2(\Omega).$$

In addition, we assume that

$$\phi \in \mathrm{Lip}([0,T],\mathrm{Z}) \cap \mathrm{C}([0,T],\mathrm{H}^2(\Pi)), \qquad \boldsymbol{A} \in \mathrm{C}([0,T],\mathbf{H}_0^1(\mathrm{div},\Omega)) \cap \mathrm{L}^2((0,T),\mathbf{H}^2(\Omega)).$$

Then, there exists a constant $C > 0$ such that

$$\left\| \nabla \overline{\phi}_n^h(\xi) - \nabla \phi(\xi) \right\|_{\mathbf{L}^2(\Pi)}^2 \le C(\tau^2 + h^2),$$

$$\int_0^\xi \left\| \partial_t \boldsymbol{A}_n^h(t) - \partial_t \boldsymbol{A}(t) \right\|_{\mathbf{L}^2(\Theta(t))}^2 \,\mathrm{d}t + \left\| \nabla \boldsymbol{A}_n^h(\xi) - \nabla \boldsymbol{A}(\xi) \right\|_{\mathbf{L}^2(\Omega)}^2 \le C(\tau + h^2).$$

The convergence rate: $\mathcal{O}(\sqrt{\tau} + h)$.

GHENT UNIVERSITY

References and acknowledgement

V. C. Le, M. Slodička, and K. Van Bockstal.
A full discretization for the saddle-point approach of a degenerate parabolic problem involving a moving body.
Applied Mathematics Letters, 124:107660, 2022.

V. C. Le, M. Slodička, and K. Van Bockstal.
A space-time discretization for an electromagnetic problem with moving non-magnetic conductor.
Applied Numerical Mathematics, 173:345–364, 2022.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 101001847).

GHENT UNIVERSITY

THANK YOU FOR YOUR ATTENTION!