A time discretization for solving an electromagnetic contact problem with moving conductor[1]

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[1] This is a joint work with Marián Slodička and Karel Van Bockstal (Ghent University)

Outline

Electromagnetic contact problem

Let us consider the eddy current equations

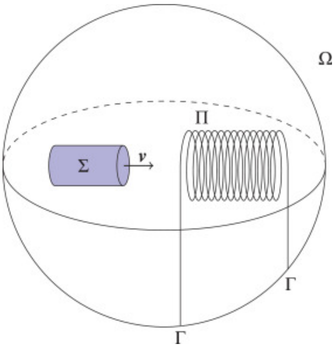
$$\begin{aligned} \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \times \boldsymbol{E} &= -\partial_t \boldsymbol{B}, \\ \nabla \times \boldsymbol{H} &= \boldsymbol{J}, \end{aligned}$$

with the constitutive relations

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{J} = \sigma \left(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B} \right).$$

We acquire the following interface conditions

$$\begin{aligned} [\![\boldsymbol{B} \cdot \mathbf{n}]\!] &= 0, \\ [\![\boldsymbol{H} \times \mathbf{n}]\!] &= \mathbf{0}, \\ [\![(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}) \times \mathbf{n}]\!] &= \mathbf{0}. \end{aligned}$$



$$\Theta = \Sigma \cup \Pi \subset \Omega.$$

The jumping material coefficients

The material coefficients σ and μ are defined by

$$\sigma(\boldsymbol{x},t) = \begin{cases} \sigma_\Sigma(\boldsymbol{x},t) & \text{in } \Sigma(t), \\ \sigma_\Pi(\boldsymbol{x},t) & \text{in } \Pi, \\ 0 & \text{in } \Omega \setminus \overline{\Theta(t)}, \end{cases}$$

and

$$\mu(\boldsymbol{x},t) = \begin{cases} \mu_\Sigma(\boldsymbol{x},t) & \text{in } \Sigma(t), \\ \mu_\Pi(\boldsymbol{x},t) & \text{in } \Pi, \\ \mu_{air}(\boldsymbol{x},t) & \text{in } \Omega \setminus \overline{\Theta(t)}. \end{cases}$$

At the interfaces of the different materials, σ and μ have jumps, i.e.

$$[\![\sigma]\!] \neq 0, \qquad [\![\mu]\!] \neq 0.$$

$\boldsymbol{A} - \phi$ formulation

Since $\nabla \cdot \boldsymbol{B} = 0$, there exists a unique vector potential $\boldsymbol{A} \in \mathbf{H}^1(\Omega)$ of $\boldsymbol{B}$ such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \nabla \cdot \boldsymbol{A} = 0, \qquad \boldsymbol{A} \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega.$$

Therefore, from $\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$, we get that

$$\nabla \times (\boldsymbol{E} + \partial_t \boldsymbol{A}) = \mathbf{0}$$

or

$$\boldsymbol{E} + \partial_t \boldsymbol{A} = -\nabla \phi.$$

Thanks to $\boldsymbol{J} = \sigma(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B})$, the total current density $\boldsymbol{J}$ can be split into

$$\boldsymbol{J} = \boldsymbol{J}_{source} + \boldsymbol{J}_{eddy},$$

where

$$\boldsymbol{J}_{source} = -\sigma \nabla \phi, \qquad \boldsymbol{J}_{eddy} = -\sigma \partial_t \boldsymbol{A} + \sigma \mathbf{v} \times (\nabla \times \boldsymbol{A}).$$

$\boldsymbol{A} - \phi$ formulation (cont.)

The boundary value problem for the scalar potential ϕ

$$\begin{cases} \nabla \cdot (-\sigma \nabla \phi) = 0 & \text{in} \quad \Pi \times (0,T), \\ -\sigma \nabla \phi \cdot \mathbf{n} = 0 & \text{on} \quad \left(\partial \Pi \setminus \Gamma\right) \times (0,T), \\ -\sigma \nabla \phi \cdot \mathbf{n} = j & \text{on} \quad \Gamma \times (0,T)\,, \end{cases}$$

where

$$\int\limits_{\Gamma} j(s,t)\,\mathrm{d}s = 0.$$

From $\nabla \times \boldsymbol{H} = \boldsymbol{J}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$, we derive the initial-boundary value problem for the vector potential $\boldsymbol{A}$

$$\begin{cases} \sigma \partial_t \boldsymbol{A} + \nabla \times (\mu^{-1} \nabla \times \boldsymbol{A}) + \chi_\Pi \sigma \nabla \phi & \\ \qquad\qquad -\sigma \mathbf{v} \times (\nabla \times \boldsymbol{A}) = \mathbf{0} & \text{in } \Omega \times (0,T), \\ \boldsymbol{A} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{A}(\boldsymbol{x},0) = \boldsymbol{A}_0 & \text{in } \Theta(0). \end{cases}$$

Functional setting

Let us consider the following Hilbert spaces

$$\mathrm{Z} = \left\{ \psi \in \mathrm{H}^1(\Pi) : (\psi, 1)_\Pi = 0 \right\},$$

$$\boldsymbol{W}_0 = \left\{ \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) : \nabla \times \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega), \nabla \cdot \boldsymbol{\varphi} = 0, \boldsymbol{\varphi} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \right\},$$

with the equipped norms

$$\|\psi\|_{\mathrm{Z}} = \|\nabla \psi\|_{\mathbf{L}^2(\Pi)},$$
$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_0} = \|\nabla \times \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)}.$$

We recall the well-known Reynolds transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega(t)} f \,\mathrm{d}\boldsymbol{x} = \int_{\omega(t)} \partial_t f \,\mathrm{d}\boldsymbol{x} + \int_{\partial\omega(t)} f \mathbf{v} \cdot \mathbf{n} \,\mathrm{d}s.$$

Weak formulation

The weak problem for ϕ reads as:

$$\big(\sigma(t)\nabla\phi(t), \nabla\psi\big)_{\Pi} + \big(j(t), \psi\big)_{\Gamma} = 0 \qquad \forall\psi \in \mathrm{Z}\,.$$

For any $\boldsymbol{\varphi} \in \boldsymbol{W}_0$, it holds that

$$\big(\sigma(t)\partial_t \boldsymbol{A}(t), \boldsymbol{\varphi}\big)_{\Theta(t)} + \Big(\mu^{-1}(t)\nabla\times\boldsymbol{A}(t), \nabla\times\boldsymbol{\varphi}\Big)_{\Omega} + \big(\sigma(t)\nabla\phi(t), \boldsymbol{\varphi}\big)_{\Pi} - \big(\sigma(t)\mathbf{v}(t)\times(\nabla\times\boldsymbol{A}(t)), \boldsymbol{\varphi}\big)_{\Theta(t)} = 0.$$

Using the Reynolds transport theorem, the variational formulation for $\boldsymbol{A}$ is defined as: Find $\boldsymbol{A}(t) \in \boldsymbol{W}_0$ such that the following identity holds true for any $\boldsymbol{\varphi} \in \boldsymbol{W}_0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\sigma(t)\boldsymbol{A}(t), \boldsymbol{\varphi}\big)_{\Theta(t)} - \big(\sigma(t)\boldsymbol{A}(t), \boldsymbol{\varphi}(\mathbf{v}\cdot\mathbf{n})(t)\big)_{\partial\Theta(t)} - \big(\gamma(t)\boldsymbol{A}(t), \boldsymbol{\varphi}\big)_{\Theta(t)} + \Big(\mu^{-1}(t)\nabla\times\boldsymbol{A}(t), \nabla\times\boldsymbol{\varphi}\Big)_{\Omega} + \big(\sigma(t)\nabla\phi(t), \boldsymbol{\varphi}\big)_{\Pi} - \big(\sigma(t)\mathbf{v}(t)\times(\nabla\times\boldsymbol{A}(t)), \boldsymbol{\varphi}\big)_{\Theta(t)} = 0,$$

where $\gamma = \partial_t\sigma$.

Uniqueness

Theorem

Let the material functions $\mu \in \mathrm{C}^1$ *and* $\sigma \in \mathrm{C}^2$ *on each component (in all variables). Moreover, assume that* $\boldsymbol{A}_0 \in \mathbf{L}^2\left(\Theta_0\right), j \in \mathrm{Lip}\left([0,T], \mathrm{H}^{-1/2}(\Gamma)\right)$, $\mathbf{v} \in \mathbf{C}^1(\overline{\Omega} \times [0,T])$ *and*

$$0 < \sigma_* \leq \sigma \leq \sigma^* \quad \text{in} \;\; \Theta; \qquad \sigma = 0 \quad \text{in air}$$

and

$$0 < \mu_* \leq \mu \leq \mu^* \qquad \text{in} \quad \Omega.$$

The variational system admits at most one solution $(\phi, \boldsymbol{A})$ *satisfying* $\phi \in \mathrm{L}^2\left((0,T), \mathrm{Z}\right)$, $\boldsymbol{A} \in \mathrm{L}^2\left((0,T), \boldsymbol{W}_0\right)$ *and* $\left\|\sqrt{\sigma}\boldsymbol{A}\right\|_{\mathbf{L}^2(\Omega)} \in \mathrm{C}([0,T])$.

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Time discretization

We partition the time range $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals with the time step

$$\tau = \frac{T}{n}.$$

The time discretization scheme is defined as: Find $\phi_i \in \mathrm{Z}$ and $\boldsymbol{A}_i \in \boldsymbol{W}_0$ such that the following identities are valid for any $\psi \in \mathrm{Z}$ and $\boldsymbol{\varphi} \in \boldsymbol{W}_0$

$$\begin{aligned}
&(\sigma_i \nabla \phi_i, \nabla \psi)_\Pi + (j_i, \psi)_\Gamma = 0, \\
&\delta\left(\sigma_i \boldsymbol{A}_i, \boldsymbol{\varphi}\right)_{\Theta_i} - \left(\sigma_i \boldsymbol{A}_i, \boldsymbol{\varphi}(\mathbf{v}_i \cdot \mathbf{n})\right)_{\partial\Theta_i} - (\gamma_i \boldsymbol{A}_i, \boldsymbol{\varphi})_{\Theta_i} \\
&\quad + \left(\mu_i^{-1} \nabla \times \boldsymbol{A}_i, \nabla \times \boldsymbol{\varphi}\right)_\Omega + (\sigma_i \nabla \phi_i, \boldsymbol{\varphi})_\Pi - \left(\sigma_i \mathbf{v}_i \times (\nabla \times \boldsymbol{A}_i), \boldsymbol{\varphi}\right)_{\Theta_i} = 0,
\end{aligned}$$

where

$$\delta\left(\sigma_i \boldsymbol{A}_i, \boldsymbol{\varphi}\right)_{\Theta_i} = \frac{1}{\tau}\left((\sigma_i \boldsymbol{A}_i, \boldsymbol{\varphi})_{\Theta_i} - (\sigma_{i-1} \boldsymbol{A}_{i-1}, \boldsymbol{\varphi})_{\Theta_{i-1}}\right).$$

Time discretization (cont.)

Lemma

For any $\tau < \tau_0$ *and for any* $i = 1, 2, \ldots, n$, *there exists a unique couple* $(\phi_i, \boldsymbol{A}_i) \in \mathrm{Z} \times \boldsymbol{W}_0$ *solving the time discrete system.*

Lemma

There exist positive constants C *and* τ_0 *such that the following estimates hold true for any* $\tau < \tau_0$

$$(i) \quad \max_{1 \le i \le n} \|\nabla \phi_i\|^2_{\mathbf{L}^2(\Pi)} \le C,$$

$$(ii) \quad \max_{1 \le l \le n} \|\boldsymbol{A}_l\|^2_{\mathbf{L}^2(\Theta_l)} + \sum_{i=1}^{n} \|\nabla \times \boldsymbol{A}_i\|^2_{\mathbf{L}^2(\Omega)} \tau + \sum_{i=1}^{n} \|\boldsymbol{A}_i - \boldsymbol{A}_{i-1}\|^2_{\mathbf{L}^2(\Theta_{i-1})} \le C.$$

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Rothe's method

We introduce the following Rothe's functions and domain

$$\overline{\phi}_n(t) = \phi_i, \quad \overline{\boldsymbol{A}}_n(t) = \boldsymbol{A}_i,$$
$$g_n(t) = (\sigma_{i-1}\boldsymbol{A}_{i-1}, \boldsymbol{\varphi})_{\Theta_{i-1}} + (t - t_{i-1})\,\delta\,(\sigma_i \boldsymbol{A}_i, \boldsymbol{\varphi})_{\Theta_i},$$
$$\overline{j}_n(t) = j_i, \quad \overline{\sigma}_n(t) = \sigma_i, \quad \overline{\gamma}_n(t) = \gamma_i, \quad \overline{\mu}_n(t) = \mu_i, \quad \overline{\mathbf{v}}_n(t) = \mathbf{v}_i, \quad \overline{\Theta}_n(t) = \Theta_i,$$

with the initial data $\overline{\phi}_n(0) = \phi_0, \overline{\boldsymbol{A}}_n(0) = \boldsymbol{A}_0, g_n(0) = (\sigma_0 \boldsymbol{A}_0, \boldsymbol{\varphi})_{\Theta_0}$.
The discrete problems can be rewritten in the continuous sense: for any $\psi \in \mathrm{Z}$ and $\boldsymbol{\varphi} \in \boldsymbol{W}_0$, it holds that

$$\left(\overline{\sigma}_n(t)\nabla\overline{\phi}_n(t), \nabla\psi\right)_{\Pi} + \left(\overline{j}_n(t), \psi\right)_{\Gamma} = 0,$$
$$g_n'(t) - \left(\overline{\sigma}_n(t)\overline{\boldsymbol{A}}_n(t), \boldsymbol{\varphi}(\overline{\mathbf{v}}_n \cdot \mathbf{n})(t)\right)_{\partial\overline{\Theta}_n(t)} - \left(\overline{\gamma}_n(t)\overline{\boldsymbol{A}}_n(t), \boldsymbol{\varphi}\right)_{\overline{\Theta}_n(t)}$$
$$+ \left(\overline{\mu}_n^{-1}(t)(\nabla \times \overline{\boldsymbol{A}}_n(t)), \nabla \times \boldsymbol{\varphi}\right)_{\Omega} + \left(\overline{\sigma}_n(t)\nabla\overline{\phi}_n(t), \boldsymbol{\varphi}\right)_{\Pi}$$
$$- \left(\overline{\sigma}_n(t)\overline{\mathbf{v}}_n(t) \times (\nabla \times \overline{\boldsymbol{A}}_n(t)), \boldsymbol{\varphi}\right)_{\overline{\Theta}_n(t)} = 0.$$

Convergence on given data

Lemma

There exists a constant $C > 0$ such that the following relations hold true

$$
\begin{aligned}
\max_{t\in[0,T]} \left\| \overline{j}_n(t) - j(t) \right\|_{\mathrm{H}^{-1/2}(\Gamma)} &\leq C\tau, \\
\max_{t\in[0,T]} \left\| \overline{\mathbf{v}}_n(t) - \mathbf{v}(t) \right\|_{\mathbf{H}(\mathsf{div},\Omega)} &\leq C\tau, \\
\max_{t\in[0,T]} \left\| \overline{\mu}_n(t) - \mu(t) \right\|_{\mathrm{L}^2(\Omega)} &\leq C\sqrt{\tau}, \\
\max_{t\in[0,T]} \left\| \overline{\gamma}_n(t) - \gamma(t) \right\|_{\mathrm{L}^2(\Omega)} &\leq C\sqrt{\tau}, \\
\max_{t\in[0,T]} \left\| \overline{\sigma}_n(t) - \sigma(t) \right\|_{\mathrm{L}^2(\Omega)} &\leq C\sqrt{\tau}, \\
\max_{t\in[0,T]} \left\| \nabla\overline{\sigma}_n(t) - \nabla\sigma(t) \right\|_{\mathbf{L}^2(\Omega)} &\leq C\sqrt{\tau}.
\end{aligned}
$$

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Existence

Theorem

There exists a unique weak solution $(\phi, \boldsymbol{A})$ *to the variational system satisfying* $\phi \in \mathrm{L}^2((0,T),\mathrm{Z})$ *and* $\boldsymbol{A} \in \mathrm{L}^2((0,T),\boldsymbol{W}_0)$ *with* $\left\|\sqrt{\sigma}\boldsymbol{A}\right\|_{\mathbf{L}^2(\Omega)} \in \mathrm{C}([0,T])$. *Moreover, the following convergences hold true*

$$
\begin{aligned}
(i) \quad & \overline{\phi}_n \rightharpoonup \phi && \textit{in} \quad \mathrm{L}^2\left((0,T),\mathrm{Z}\right), \\
(ii) \quad & \overline{\boldsymbol{A}}_n \rightharpoonup \boldsymbol{A} && \textit{in} \quad \mathrm{L}^2\left((0,T),\boldsymbol{W}_0\right), \\
& g_n \to (\sigma \boldsymbol{A}, \boldsymbol{\varphi})_\Theta && \textit{in} \quad \mathrm{L}^1\left((0,T)\right).
\end{aligned}
$$

Proof of the existence

Proof. The boundedness of $\left\{\overline{\phi}_n\right\}$ and $\left\{\overline{\boldsymbol{A}}_n\right\}$ implies the existence of $\phi \in \mathrm{L}^2((0,T),\mathrm{Z})$ and $\boldsymbol{A} \in \mathrm{L}^2((0,T),\boldsymbol{W}_0)$ such that

$$\begin{aligned} \overline{\phi}_n &\rightharpoonup \phi && \text{in} \quad \mathrm{L}^2\left((0,T),\mathrm{Z}\right), \\ \overline{\boldsymbol{A}}_n &\rightharpoonup \boldsymbol{A} && \text{in} \quad \mathrm{L}^2\left((0,T),\boldsymbol{W}_0\right). \end{aligned}$$

Thanks to the stability of $\overline{\boldsymbol{A}}_n$, the following strong convergence holds true

$$g_n - \left(\overline{\sigma}_n \overline{\boldsymbol{A}}_n, \boldsymbol{\varphi}\right)_{\overline{\Theta}_n} \to 0 \qquad \text{in} \quad \mathrm{L}^1((0,T)).$$

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Proof of the existence (cont.)

Moreover, we can also deduce the following convergences in $\mathrm{L}^1\left((0,T)\right)$

$$
\begin{aligned}
&\left(\overline{\gamma}_n \overline{\boldsymbol{A}}_n, \boldsymbol{\varphi}\right)_{\overline{\Theta}_n} \to (\gamma \boldsymbol{A}, \boldsymbol{\varphi})_{\Theta}\,,\\
&\left(\overline{\mu}_n^{-1} \nabla \times \overline{\boldsymbol{A}}_n, \nabla \times \boldsymbol{\varphi}\right)_{\Omega} \to \left(\mu^{-1} \nabla \times \boldsymbol{A}, \nabla \times \varphi\right)_{\Omega}\,,\\
&\left((\nabla \overline{\sigma}_n \cdot \overline{\mathbf{v}}_n) \overline{\boldsymbol{A}}_n, \boldsymbol{\varphi}\right)_{\overline{\Theta}_n} \to \left((\nabla \sigma \cdot \mathbf{v}) \boldsymbol{A}, \boldsymbol{\varphi}\right)_{\Theta}\,,\\
&\left(\overline{\sigma}_n \left(\nabla \cdot \overline{\mathbf{v}}_n\right) \overline{\boldsymbol{A}}_n, \boldsymbol{\varphi}\right)_{\overline{\Theta}_n} \to \left(\sigma \left(\nabla \cdot \mathbf{v}\right) \boldsymbol{A}, \boldsymbol{\varphi}\right)_{\Theta}\,,\\
&\left(\overline{\sigma}_n \overline{\mathbf{v}}_n, \nabla \left(\overline{\boldsymbol{A}}_n \cdot \boldsymbol{\varphi}\right)\right)_{\overline{\Theta}_n} \to \left(\sigma \mathbf{v}, \nabla \left(\boldsymbol{A} \cdot \boldsymbol{\varphi}\right)\right)_{\Theta}\,,\\
&\left(\overline{\sigma}_n \overline{\mathbf{v}}_n \times (\nabla \times \overline{\boldsymbol{A}}_n), \boldsymbol{\varphi}\right)_{\overline{\Theta}_n} \to \left(\sigma \mathbf{v} \times (\nabla \times \boldsymbol{A}), \boldsymbol{\varphi}\right)_{\Theta}\,.
\end{aligned}
$$

Therefore, we can pass to the limit and use the Reynolds transport theorem to conclude that $(\phi, \boldsymbol{A})$ solves the variational system.

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Proof of the existence (cont.)

Finally, the following relation can be obtained

$$\left|(\sigma \boldsymbol{A}, \boldsymbol{A})_{\Theta}(\eta) - (\sigma \boldsymbol{A}, \boldsymbol{A})_{\Theta}(\xi)\right| \lesssim |\eta - \xi| + \int_{\xi}^{\eta} \|\nabla \times \boldsymbol{A}\|^2_{\mathbf{L}^2(\Omega)}\, \mathrm{d}t$$

which implies the continuity in time, i.e.

$$\lim_{\eta \to \xi} (\sigma \boldsymbol{A}, \boldsymbol{A})_{\Theta}(\eta) = (\sigma \boldsymbol{A}, \boldsymbol{A})_{\Theta}(\xi)$$

and thus $\left\|\sqrt{\sigma} \boldsymbol{A}\right\|_{\mathbf{L}^2(\Omega)} \in \mathrm{C}([0, T])$.

Conclusion

- A mathematical model for an electromagnetic contact problem with moving conductor is investigated.
- A time discretization for solving the variational problem is introduced.
- The convergence of the numerical scheme is shown.
- The existence of a unique weak solution is present.

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Reference

V. C. Le, M. Slodička, and K. Van Bockstal.
A time discrete scheme for an electromagnetic contact problem with moving conductor.
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Thank you for your attention!