Induction heating with moving conductor

A time discretization scheme & numerical analysis

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Introduction

Induction heating is a commonly industrial manufacturing process. Some scientific papers deal with the well-posedness and provide theoretical results, while others propose some numerical schemes. A problem has imposed in the real industrial processes, where the conductors are moving in the electromagnetic system. We investigate a model of multi-physics in the domain consisting out of multiple components: moving workpiece, static coil and air. To our best knowledge, there aren't any similar works that have been done before.

Mathematical model

Let Ω be an open, bounded and simply-connected domain in $\mathbb{R}^3$ containing the moving workpiece Σ, the static coil Π and the rest air. We denote the velocity vector $\boldsymbol{v}$, the outward unit normal vector $\mathbf{n}$ and the conductors by $\Theta = \Sigma \cup \Pi$ (see Figure 1).

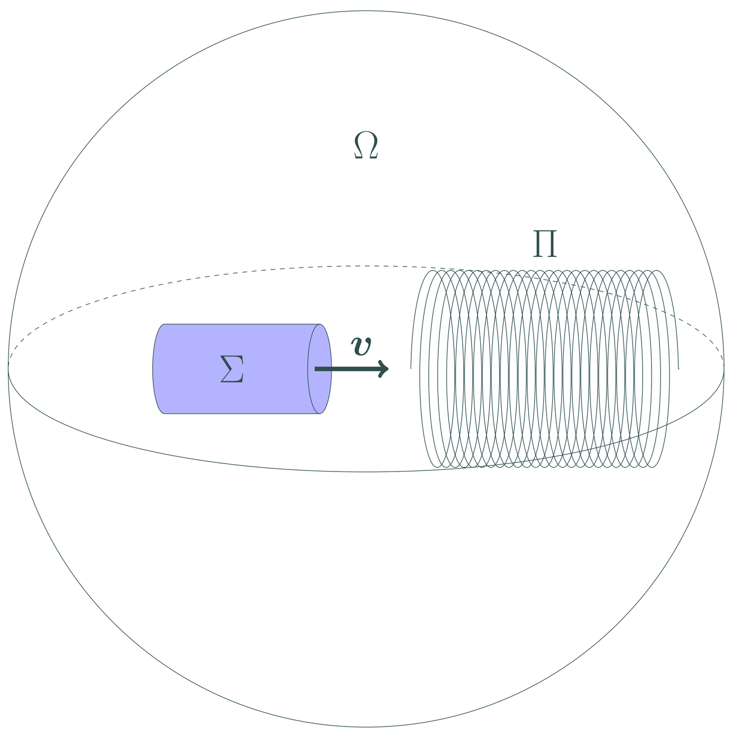

Figure 1: The domain

Each of subdomains is characterized by different material coefficients:

- the magnetic permeability μ,
- the electrical conductivity σ,
- the thermal conductivity k,
- volumetric heat capacity β.

The time frame is denoted by $[0, T]$.

The Maxwell equations for electromagnetics in each of continuous media are:

$$\nabla \cdot \boldsymbol{B} = 0,\quad \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B},\quad \nabla \times \boldsymbol{H} = \boldsymbol{J},$$

with the interface conditions:

$$[\boldsymbol{B}\cdot\mathbf{n}]_{\partial\Theta(t)} = 0, \qquad [\boldsymbol{H}\times\mathbf{n}]_{\partial\Theta(t)} = \mathbf{0}, \qquad \text{and} \qquad [(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\times\mathbf{n}]_{\partial\Theta(t)} = 0, \qquad \forall t \in [0, T].$$

We consider the constitutive relation between $\boldsymbol{H}$ and $\boldsymbol{B}$ in the linear form:

$$\boldsymbol{H} = \mu^{-1}\boldsymbol{B}.$$

The Ohm law for moving problem with the velocity $\boldsymbol{v}$ is given as:

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + v \times \boldsymbol{B}\right).$$

Since Ω is a simply-connected domain, we have a unique vector potential $\boldsymbol{A} \in \mathbf{H}^1(\Omega)$:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \nabla\cdot\boldsymbol{A} = 0, \qquad \boldsymbol{A}\times\mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega,$$

and a unique scalar potential $\phi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ such that:

$$\boldsymbol{E} + \partial_t\boldsymbol{A} = -\nabla\phi.$$

Therefore, we have the following initial boundary value problem for the vector potential $\boldsymbol{A}$:

$$\begin{cases} \sigma\partial_t\boldsymbol{A} + \nabla\times\left(\mu^{-1}\nabla\times\boldsymbol{A}\right) + \chi_\Pi\sigma\nabla\phi - \sigma\boldsymbol{v}\times(\nabla\times\boldsymbol{A}) = \mathbf{0} & \text{in } \Omega\times(0,T),\\ \boldsymbol{A}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\times(0,T),\\ \boldsymbol{A}(\boldsymbol{x},0) = \boldsymbol{A}_0 & \text{in } \Omega. \end{cases}$$

The elliptic problem for the scalar potential ϕ is formed by:

$$\begin{cases} -\nabla\cdot(\sigma\nabla\phi) = 0 & \text{in } \Pi\times(0,T),\\ -\sigma\nabla\phi\cdot\mathbf{n} = 0 & \text{on } (\partial\Pi\setminus\Gamma)\times(0,T),\\ -\sigma\nabla\phi\cdot\mathbf{n} = j & \text{on } \Gamma\times(0,T). \end{cases}$$

The heat transfer problem is as follows:

$$\begin{cases} \beta\partial_t\mathcal{T} + \nabla\cdot(-k\nabla\mathcal{T} + \beta\mathcal{T}\boldsymbol{v}) = \mathcal{R}_r(Q) & \text{in } \Omega\times(0,T),\\ (-k\nabla\mathcal{T} + \beta\mathcal{T}\boldsymbol{v})\cdot\mathbf{n} = \alpha(\mathcal{T} - \mathcal{T}_c), & \text{on } \partial\Omega\times(0,T),\\ \mathcal{T}(\boldsymbol{x},0) = \mathcal{T}_0 & \text{in } \Omega, \end{cases}$$

where R_r is the cut function and the Joule heating source Q is the following:

$$Q = \frac{1}{\sigma}|\boldsymbol{J}|^2 = \sigma\left|\partial_t\boldsymbol{A} + \nabla\phi - \boldsymbol{v}\times\nabla\times\boldsymbol{A}\right|^2.$$

Finally, we need some transmission conditions on the interface $\partial\Theta(t)$:

$$[\![\mathcal{T}]\!] = 0, \qquad \text{and} \qquad [\![-k\nabla\mathcal{T} + \beta\mathcal{T}\boldsymbol{v}]\!] = \mathbf{0}.$$

Variational problems

Define the Banach space $\boldsymbol{X}_{N,0} = \left\{\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)\,|\,\nabla\cdot\boldsymbol{u} = 0, \boldsymbol{u}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\right\}$ with the norm:

$$\|\boldsymbol{u}\|_{\boldsymbol{X}_{N,0}} = \|\nabla\times\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}.$$

For any $\boldsymbol{\varphi}\in\boldsymbol{X}_{N,0}$, $\xi\in\mathrm{H}^1(\Pi)/\mathbb{R}$ and $\psi\in\mathrm{H}^1(\Omega)$, it holds that

$$\begin{aligned} &(\sigma\partial_t\boldsymbol{A},\boldsymbol{\varphi})_{\Theta(t)} + \left(\mu^{-1}\nabla\times\boldsymbol{A},\nabla\times\boldsymbol{\varphi}\right)_\Omega + (\sigma\nabla\phi,\boldsymbol{\varphi})_\Pi + (\sigma\nabla\times\boldsymbol{A},\boldsymbol{v}\times\boldsymbol{\varphi})_{\Theta(t)} = 0,\\ &(\sigma\nabla\phi,\nabla\xi)_\Pi + (j,\xi)_\Gamma = 0,\\ &(\beta\partial_t\mathcal{T},\psi)_\Omega + (k\nabla\mathcal{T} - \beta\mathcal{T}\boldsymbol{v},\nabla\psi)_\Omega + (\alpha(\mathcal{T}-\mathcal{T}_c),\psi)_{\partial\Omega} = (\mathcal{R}_r(Q),\psi)_\Omega. \end{aligned}$$

Time discretization scheme

The time range $[0,T]$ is divided into n subintervals. For any $i = 1, 2, \cdots, n$, find $\phi_{c_i}\in\mathrm{H}^1(\Pi)/\mathbb{R}$, $\boldsymbol{A}_i\in\boldsymbol{X}_{N,0}$ and $\mathcal{T}_i\in\mathrm{H}^1(\Omega)$ such that: for any $\xi\in\mathrm{H}^1(\Pi)/\mathbb{R}$, $\boldsymbol{\varphi}\in\boldsymbol{X}_{N,0}$ and $\psi\in\mathrm{H}^1(\Omega)$

$$\begin{aligned} &(\sigma_i(\mathcal{T}_{i-1})\nabla\phi_{c_i},\nabla\xi)_\Pi + (j_i,\xi)_\Gamma = 0,\\ &(\sigma_i(\mathcal{T}_{i-1})\delta\boldsymbol{A}_i,\boldsymbol{\varphi})_{\Theta_i} + \left(\mu_i^{-1}\nabla\times\boldsymbol{A}_i,\nabla\times\boldsymbol{\varphi}\right)_\Omega + (\sigma_i(\mathcal{T}_{i-1})\nabla\times\boldsymbol{A}_i,\boldsymbol{v}_i\times\boldsymbol{\varphi})_{\Theta_i} = -(\sigma_i(\mathcal{T}_{i-1})\nabla\phi_{c_i},\boldsymbol{\varphi})_\Pi,\\ &(\beta_i\delta\mathcal{T}_i,\psi)_\Omega + (k_i\nabla\mathcal{T}_i - \beta_i\mathcal{T}_i\boldsymbol{v}_i,\nabla\psi)_\Omega + (\alpha(\mathcal{T}_i-\mathcal{T}_c),\psi)_{\partial\Omega} = (\mathcal{R}_r(Q_i),\psi)_\Omega, \end{aligned}$$

where

$$Q_i = \sigma_i(\mathcal{T}_{i-1})\left|\delta\boldsymbol{A}_i + \nabla\phi_{c_i} - \boldsymbol{v}_i\times\nabla\times\boldsymbol{A}_i\right|^2.$$

Theorem 1. *For any $i = 1, 2, \cdots, n$, there exists a unique triplet $\phi_{c_i}\in\mathrm{H}^1(\Pi)/\mathbb{R}, \boldsymbol{A}_i\in\boldsymbol{X}_{N,0}$ and $\mathcal{T}_i\in\mathrm{H}^1(\Omega)$ solving the time discretization problem.*

Convergence

Firstly, we assume that there exists a transportation mapping γ which belongs to C^2 class in both of time and space such that

$$\gamma(t):\Theta_0\to\Theta(t), \qquad \gamma'(t) = \boldsymbol{v}.$$

Define the following Rothe's functions:

$$\begin{aligned} &\overline{\phi}_n(t) = \phi_{c_i} && \forall t\in(t_{i-1},t_i],\\ &\overline{\boldsymbol{A}}_n(t) = \boldsymbol{A}_i, \quad \boldsymbol{A}_n(t) = \boldsymbol{A}_{i-1} + (t - t_{i-1})\,\delta\boldsymbol{A}_i, && \forall t\in(t_{i-1},t_i],\\ &\overline{\mathcal{T}}_n(t) = \mathcal{T}_i, \quad \underline{\mathcal{T}}_n(t) = \mathcal{T}_{i-1}, \quad \mathcal{T}_n(t) = \mathcal{T}_{i-1} + (t-t_{i-1})\,\delta\mathcal{T}_i, && \forall t\in(t_{i-1},t_i],\\ &\overline{\gamma}_n(t) = \gamma_i, && \forall t\in(t_{i-1},t_i],\\ &\overline{\boldsymbol{A}}_n(0) = \boldsymbol{A}_n(0) = \boldsymbol{A}_0, \quad \overline{\mathcal{T}}_n(0) = \underline{\mathcal{T}}_n(0) = \mathcal{T}_n(0) = \mathcal{T}_0, && \overline{\gamma}_n(0) = \gamma_0. \end{aligned}$$

Theorem 2. *There exists a solution-triplet $\{\phi,\boldsymbol{A},\mathcal{T}\}$ where $\phi\in\mathrm{L}^2\left((0,T),\mathrm{H}^1(\Pi)/\mathbb{R}\right)$, $\boldsymbol{A}\in\mathrm{L}^2\left((0,T),\boldsymbol{X}_{N,0}\right)$ with $\partial_t\boldsymbol{A}\circ\gamma\in\mathrm{L}^2\left((0,T),\mathbf{L}^2(\Theta_0)\right)$ and $\mathcal{T}\in\mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right)\cap\mathrm{L}^\infty\left((0,T),\mathrm{H}^1(\Omega)\right)$ with $\partial_t\mathcal{T}\in\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$ such that*

$$\begin{aligned} (i)\quad & \mathcal{T}_n\to\mathcal{T} && \text{in } \mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right),\\ & \overline{\mathcal{T}}_n\to\mathcal{T}, \quad \underline{\mathcal{T}}_n\to\mathcal{T} && \text{in } \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right),\\ & \mathcal{T}_n(t)\rightharpoonup\mathcal{T}(t), \quad \overline{\mathcal{T}}_n(t)\rightharpoonup\mathcal{T}(t) && \text{in } \mathrm{H}^1(\Omega) \quad \text{for all} \quad t\in[0,T],\\ & \partial_t\mathcal{T}_n\rightharpoonup\partial_t\mathcal{T} && \text{in } \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right),\\ (ii)\quad & \overline{\boldsymbol{A}}_n\rightharpoonup\boldsymbol{A}, \quad \nabla\times\overline{\boldsymbol{A}}_n\rightharpoonup\nabla\times\boldsymbol{A} && \text{in } \mathrm{L}^2\left((0,T),\mathbf{L}^2(\Omega)\right),\\ & \partial_t\boldsymbol{A}_n\circ\overline{\gamma}_n\rightharpoonup\partial_t\boldsymbol{A}\circ\gamma && \text{in } \mathrm{L}^2\left((0,T),\mathbf{L}^2(\Theta_0)\right),\\ (iii)\quad & \nabla\overline{\phi}_n\to\nabla\phi && \text{in } \mathrm{L}^2\left((0,T),\mathbf{L}^2(\Pi)\right),\\ (iv)\quad & \phi,\boldsymbol{A} \text{ and } \mathcal{T} \text{ solve the variational system.} \end{aligned}$$

Conclusions

In this study, we have

- investigated a mathematical model for induction heating with the moving workpiece problem,
- proposed an effective discretization algorithm to solve the variational problems,
- proved the stability and the convergence of the approximation scheme.

Future research

In the future, we should

- investigate the nonlinear model for the induction heating problem,
- deal the induction hardening procedure by considering additional the phase transition kinetic of workpiece.

References

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