Towards a theory of MCDM; Stepping away from social choice theory.

Many axiomatic results concerning aggregation procedures in multi-criteria decision aiding have been obtained in the framework of voting theory. We show that voting theory, although helpful for a better understanding of some aggregation procedures, is not totally appropriate for multi-criteria decision aiding. We propose a slight modification of this framework making it more relevant for the axiomatization of aggregation procedures in the context of multi-criteria decision aiding. As an illustration, we present two characterizations of the simple weighted majority: one in our new framework and one in voting theory.

Keywords : multicriteria decision-making, voting, weighted majority

1. INTRODUCTION

The similarity between aggregation procedures in social choice theory and multi-criteria decision aiding is well-known : the alternatives play the role of the candidates, the criteria play the role of the voters and the decision-maker plays the role of the society. Many authors, aware of this similarity, have used the formalism of social choice theory in multi-criteria decision aiding[1]; see for example (Arrow & Raynaud, 1986; Bouyssou, 1992; Durand & Trentesaux, 2000; Fishburn, 1978; Lansdowne, 1996; Marchant, 1996; Nurmi & Meskanen, 1968; Pérez & Barba-Romero, 1995; Perny, 1992; Vansnick, 1986; Vincke, 1982). They transposed, adapted or generalized some results of social choice theory into the more recent context of multi-criteria decision aiding. When the required result was not available in social choice theory, they created it in multi-criteria decision aiding while still using the formalism of social choice theory.

The above-mentioned results are mostly characterizations of aggregation procedures. They tell us what the fundamental characteristics of an aggregation procedure are. This approach to the analysis and characterization of aggregation procedures has often been used for so-called outranking methods. It has considerably improved our understanding of these methods but, nevertheless, it suffers some limitations due to the differences between an election and a decision aiding process. We list four of these differences.

- In all papers inspired by social choice theory, it is assumed that the preferences along each criterion are very simple structures: total orders (e.g. Pérez & Barba-Romero, 1995) or weak orders. Even when more general structures are considered, it is assumed that the information provided by each voter is of the same nature (e.g. Marchant, 1996). But in decision aiding, most of the time, the information along each criterion is very different. For one criterion, we have a weak order, for another, a real valued function on the set of alternatives and for a third, linguistic assessments. The information may be uncertain on some criteria while it is certain for others.
- In most of the social choice literature, all voters are treated equally. This is called anonymity. In decision aiding, some criteria are more important or

[1] Utility theory has also been used by many authors. We concentrate in this paper only on the social choice approach.

more relevant than others. The information for some criteria is more reliable than the information for others. For this reason, some authors have introduced weights in the social choice theory framework. For example, (Marchant, 1996) shows that using weights in PROMETHEE is equivalent to considering a population of voters in which the proportion of voters having some particular preferences is equal to the weight of the corresponding criterion.

- In some axiomatic results (e.g. Marchant, 1996), the number of voters is allowed to vary without limits (and varies dramatically in the proofs). In decison aiding, such variations in the number of criteria are not possible. If a decision-maker starts a decision process based upon 10 criteria, he might, at some further stage, use 15 or 20 criteria but definitely not several thousand. Conversely, a decision-maker starting with 1000 criteria will not, in the same decision problem, later abandon 990 of them and use only the remaining 10. Note that he could summarize the 1000 criteria in 10 broader criteria, but this is another problem.

- When a decision-maker resorts to a decision aiding process, its is because he doesn't know how to compare some alternatives. He is of course not completely ignorant. He knows, for example, that x is definitely better than y or that w is probably a good alternative or ... even if he doesn't know how to compare x and z. Let us call these preferences *initial preferences*.

  When we question a decision-maker in order to determine the parameters of a model (utility functions, subjective probabilities, indifference thresholds, weights, concordance thresholds, ...), his answers are mainly based on his knowledge of the problem, his values, his experience, ..., all the things that determine what we call initial preferences. Of course, his answers are also influenced by the questioning process and by the decision aiding process, but his initial preferences are essential. Otherwise all decision-makers using the same method would end up with the same global preferences. Therefore, trying to describe a decision aiding method without speaking of the a priori preferences seems inadequate.

  In social choice theory, initial preferences are eventually used in the framing of the choice problem (some options are not on the table) or later in the process (alternative x can ultimately be chosen even if the aggregation procedure yielded $y \succ x$). But initial preferences play no role in the aggregation procedure itself. No parameter of the aggregation procedure is fixed by asking the society if it prefers x to y.

  This is probably the most fundamental difference between social choice theory and multi-criteria decision aiding. We can think of voting theorems that would avoid the three above-mentioned problems, but a voting theory with a priori preferences is hardly conceivable. Yet, the existence of a priori preferences is essential in decision aiding.

In this paper, we modify the framework inherited from social choice theory by adding one primitive : initial preferences. Then, in order to illustrate the impact of this change, we characterize a single method–simple weighted majority–in the two frameworks : with and without initial preferences. Finally, we comment the differences between the two characterizations.

2. NOTATION

2.1. Alternatives, criteria and assessment structures

The set of criteria is denoted by $C = \{1, 2, \ldots, i, \ldots, k\}$ and the set of alternatives by $X = \{x, y, \ldots\}$. For each criterion, some information about the alternatives is available: we call this an *assessment structure*. It can be a ranking of the alternatives, a real-valued function on X, a linguistic assessment of the alternatives (e.g. a mapping from X into the set { "very good", "good", "average", "bad" }) and so on. Note that an assessment structure can even contain parameters, utility curves, indifference thresholds, probability distributions, ... For each criterion i, the set of possible assessment structures is denoted by E_i.

Up to now, our definition of an assessment structure is extremely broad: it can be anything. In order to avoid a vacuous definition, we impose four conditions on the assessment structures.

1. Let e be an element of E_i. Considering only criterion i, there is a function δ^i that maps e on a reflexive[2] binary preference relation $\delta^i(e)$ on X. The relation $\delta^i(e)$ is interpreted as a preference relation on X, when criterion i is taken into account alone. Therefore, $\delta^i(e)$ will be called a *single-criterion preference relation*. If e is a weak order representing the preferences of the decision-maker along criterion i, then $\delta^i(e) = e$. If e is a mapping from X into $\mathbb{R}$, then $\delta^i(e)$ could be defined as follows:

   $$x\delta^i(e)y \Leftrightarrow \begin{cases} e(x) \geq e(y) \text{ and } i \text{ is to be maximized} \\ \text{or} \\ e(x) \leq e(y) \text{ and } i \text{ is to be minimized.} \end{cases}$$

   A single-criterion preference relation does not need to be complete. For example, if e is a mapping from X into $\mathbb{R}$, then $\delta^i(e)$ could be defined as follows:

   $$x\delta^i(e)y \Leftrightarrow \begin{cases} e(x) \geq e(y) + \epsilon \text{ and } i \text{ is to be maximized} \\ \text{or} \\ e(x) \leq e(y) - \epsilon \text{ and } i \text{ is to be minimized,} \end{cases}$$

   where ϵ is a positive constant. In this case, $\delta^i(e)$ is the asymmetric part of a semiorder. There are cases where the single-criterion preference relation is almost empty. Suppose that an assessment structure maps each alternative on a set of real numbers and a probability distribution over this set. It is very likely that, even when he considers only that criterion, for most pairs, the decision-maker cannot tell if he prefers x to y.

2. Let π be a permutation on X and e an assessment structure. Then $\pi(e)$ represents an assessment structure such that the roles of the alternatives are exchanged according to the permutation π. For example, if e is a binary relation, then $\pi(e)$ is another binary relation defined by

   $$xey \Leftrightarrow \pi(x)\pi(e)\pi(y).$$

[2] A binary relation R on X is reflexive iff, for any x in X, xRx.

If e is a mapping from X into some set, then $\pi(e)$ is another mapping from X into the same set, defined by

$$e(x) = \pi(e)(\pi(x)).$$

3. Let Y be a subset of X such that Y contains at least two alternatives. An assessment structure e must be such that $e|_Y$, its restriction to Y, is defined. For example, if e is a binary relation, then $e|_Y$ is the binary relation defined by $e|_Y = \{(x, y) : x \in Y,\ y \in Y \text{ and } (x, y) \in e\}$. If e is a quaternary relation, then $e|_Y$ is the quaternary relation defined by $e|_Y = \{(x, y, z, w) : x \in Y,\ y \in Y,\ z \in Y,\ w \in Y, \text{ and } (x, y, z, w) \in e\}$. When Y contains only two elements, say x and y, we write $e|_{xy}$ instead of $e|_{\{x,y\}}$

4. Let e and f be two assessment structures belonging to E_i such that $\delta^i(e)|_Y = \delta^i(f)|_Y$. Then, there are e' and f' in E_i such that
   - $\delta^i(e') = \delta^i(e)$ and $\delta^i(f') = \delta^i(f)$
   - $e'|_Y = f'|_Y$.

An assessment structure can be anything, provided it satisfies the following four conditions. The first one tells us that the structure actually contains at least some minimal preferential information. The three others have a more practical interest. They allow us to be sure that some manipulations (needed in the axioms) will be possible. We do not know of any aggregation procedure using assessment structures that violates one of these conditions. Therefore, we include them in the definition of an assessment structure.

A profile is defined as a point $p = (p_1, p_2, \ldots, p_k)$ in $E_1 \times E_2 \times \ldots \times E_k$. When all sets E_i contain only mappings from X into some set, the profile is usually called a performance matrix. Given X and $E_1, E_2, \ldots, E_k$, the set of all possible profiles is denoted by $\mathcal{P}(X, E_1, \ldots, E_k)$, often abbreviated as $\mathcal{P}$.

2.2. Weights, initial preferences and aggregation procedures

A weight vector is a point $w = (w_1, \ldots, w_k)$ in $(\mathbb{R}^+)^C \setminus \{\mathbf{0}\}$, where $\mathbb{R}^+$ is the set of the non-negative real numbers and $\mathbf{0} = (0, \ldots, 0)$. The set $(\mathbb{R}^+)^C \setminus \{\mathbf{0}\}$ is denoted by $\mathcal{W}$. The weight vector u^j such that $u^j_j = 1, u^j_i = 0$, for all $i \neq j$ will play a special role.

In this paper, we define the initial preference as a reflexive binary relation. It is denoted by $\trianglerighteq$. Relation $\trianglerighteq$ contains all pairs of alternatives that the decision-maker is able to compare and has the following meaning:

- $x \trianglerighteq y$ and NOT $y \trianglerighteq x$: x is definitely strictly better than y; we write this as $x \vartriangleright y$.
- $x \trianglerighteq y$ and $y \trianglerighteq x$: the decision-maker is definitely indifferent between x and y; we write this as $x \equiv y$.
- NOT $x \trianglerighteq y$ and NOT $y \trianglerighteq x$: the decision-maker does not know how to compare x and y; we write this as $x \not\bowtie y$.

The set of all reflexive binary relations is denoted by $\mathcal{R}$.

An aggregation procedure is a mapping $\succeq: \mathcal{P} \times \mathcal{W} \times \mathcal{R} \to \mathcal{R} : (p, w, \trianglerighteq) \to {\succeq}(p, w, \trianglerighteq)$.

3. WITHOUT INITIAL PREFERENCES

In this section, we present some results where the initial preferences do not play a role. In the next section, the initial preferences will be central while the weights play no role. This will allow us to compare our modified framework (with $\trianglerighteq$) to the classical one (without $\trianglerighteq$).

Let us now formulate the axioms, before presenting the results.

A 1. Independence of initial preferences (IIP). $\succeq(p, w, \trianglerighteq_1) = \succeq(p, w, \trianglerighteq_2)$.

This axiom tells us that the initial preference relation will not be used in the aggregation procedure. The next one imposes that the result of the aggregation be a complete relation.

A 2. Completeness. *For all $x \neq y$, $x \succeq(p, w, \trianglerighteq) y$ or $y \succeq(p, w, \trianglerighteq) x$.*

The next four axioms are about weights.

A 3. Convexity. *$x \succeq(p, w, \trianglerighteq) y$ and $x \succeq(p, w', \trianglerighteq) y$ implies $x \succeq(p, w + w', \trianglerighteq) y$.*

At first sight, one could wonder why this axiom is called convexity, but if we work with normalized weights (summing up to 1), it would be stated as

$$x \succeq(p, w, \trianglerighteq) y \text{ and } x \succeq(p, w', \trianglerighteq) y \text{ implies } x \succeq(p, \tfrac{w+w'}{2}, \trianglerighteq) y.$$

Here, the convexity is evident. Since working with normalized weights or not is immaterial, in order to simplify the notation, we choose not to use normalized weights.

A 4. Monotonicity. *$x \succ(p, w, \trianglerighteq) y$ and $x \succeq(p, w', \trianglerighteq) y$ implies $x \succ(p, w + w', \trianglerighteq) y$.*

A 5. Archimedeanness. *$x \succ(p, w, \trianglerighteq) y$ implies that there is β such that, for any $\alpha > \beta$, $x \succ(p, \alpha w + w', \trianglerighteq) y$, where α and β are real numbers.*

By Archimedeanness, we know that, if we raise the weight of a criterion, we can make it into a kind of dictator.

3.1. Antisymmetric, additive and non transitive aggregation procedures

We say that an aggregation procedure is *additive and non transitive* if and only if, for each criterion i and each pair of alternatives (x, y), there is a mapping $s^i_{xy} : \mathcal{P} \times \mathcal{R} \to \mathbb{R} : (p, \trianglerighteq) \to s^i_{xy}(p, \trianglerighteq)$ such that

$$\bullet\ s^i_{xy}(p, \trianglerighteq) \geq 0 \text{ iff } y \succeq(p, u^i, \trianglerighteq) x \quad \text{and} \tag{1}$$

$$\bullet\ x \succeq(p, w, \trianglerighteq) y \text{ iff } \sum_{i=1}^{k} s^i_{yx}(p, \trianglerighteq) w_i \geq 0. \tag{2}$$

If, in addition, $s^i_{xy}(p, \trianglerighteq) = -s^i_{yx}(p, \trianglerighteq)$, then we say that the procedure is *antisymmetric.*

In (Jacquet-Lagrèze, 1982), Jacquet-Lagrèze describes a family of aggregation methods which is very similar to ours and he shows that many popular procedures are particular cases of his family of methods. It is very important to note that, contrary to what happens in the family considered by Jacquet-Lagrèze, s^i_{yx} depends on p and not just on p_i; s^i_{yx} corresponds to the i-th criterion, but can also be influenced by other criteria as well. Before characterizing the family of all antisymmetric, additive and non transitive aggregation procedures, let us have a look at some of its important procedures.

Promethee Let each assessment structure contain a real valued function v_i on X and a *preference function*, F_i, as defined in (Brans & Vincke, 1985), i.e. a non decreasing function from $\mathbb{R}$ to $[0,1]$ such that $F_i(0) = 0$. Let $s^i_{xy}(p, \trianglerighteq)$ be equal to $\Phi_i(y) - \Phi_i(x)$, where $\Phi_i(y)$ is the *single criterion net flow* of alternative y as defined in (Mareschal & Brans, 1988), i.e.

$$\Phi_i(y) = \sum_{x \neq y} F_i[v_i(y) - v_i(x)] - \sum_{x \neq y} F_i[v_i(x) - v_i(y)].$$

Then, the aggregation procedure that we obtain is exactly Promethee II.

Additive MAUT Let each assessment structure contain a mapping from X to some set and a utility function from that set to the reals. Let $s^i_{xy}(p, \trianglerighteq)$ be equal to $u_i(y) - u_i(x)$, where $u_i(y)$ is the single attribute utility of alternative y (for criterion i). This is nothing but an additive MAUT based aggregation procedure.

Weighted sum Let each assessment structure contain a mapping from X into the reals. To obtain a weighted sum, we just have to let $s^i_{xy}(p, \trianglerighteq)$ be equal to $v_i(y) - v_i(x)$, where $v_i(y)$ is the real number on which alternative y is mapped, for criterion i.

Simple weighted majority Let $s^i_{xy}(p, \trianglerighteq) = 1$ if $y\delta^i(p_i)x$ AND NOT $x\delta^i(p_i)y$. Let $s^i_{xy}(p, \trianglerighteq) = -1$ if $x\delta^i(p_i)y$ AND NOT $y\delta^i(p_i)x$. Let $s^i_{xy}(p, \trianglerighteq) = 0$ otherwise. We call this procedure *simple weighted majority* because $x \succeq(p, w, \trianglerighteq) y$ iff the sum of the weights of the criteria such that x is better than y is larger than or equal to the sum of the weights of the criteria such that y is better than x.

AHP Let each assessment structure contain a matrix of pairwise comparisons of the alternatives, evaluated on a ratio scale (Saaty, 1980). Let $s^i_{xy}(p, \trianglerighteq)$ be the x coordinate of the eigen vector of the matrix of assessment structure i minus the y coordinate of the same eigen vector. This is AHP.

Note that all above mentioned procedures share an additional characteristic: $s^i_{xy}(p, \trianglerighteq)$ depends only on p_i.

If we drop antisymmetry, we can obtain a procedure that we call *simple weighted majority with threshold*, described hereafter.

Simple weighted majority with threshold Let $s^i_{xy}(p, \trianglerighteq) = \mu > 0$ if $y\delta^i(p_i)x$ AND NOT $x\delta^i(p_i)y$. Let $s^i_{xy}(p, \trianglerighteq) = \nu < -\mu$ if $x\delta^i(p_i)y$ AND NOT $y\delta^i(p_i)x$. Let $s^i_{xy}(p, \trianglerighteq) = 0$ otherwise. We call this procedure *simple weighted majority with threshold* because $x \succeq(p, w, \trianglerighteq) y$ iff the sum of the weights of the criteria such that x is better than y is larger than or equal to the threshold multiplied by the sum of the weights of the criteria such that y is better than x. The threshold is equal to $-\nu/\mu$.

This procedure is additive and non transitive. But it is not antisymmetric. It is worth noting that simple weighted majority is a special case of simple weighted majority with threshold, where $\mu = -\nu$, i.e. the threshold is equal to 1. Let us also remark that simple weighted majority with threshold is very close to the concordance principle of ELECTRE(Roy, 1968).

PROPOSITION 1. *An aggregation procedure $\succeq$ satisfies completeness (A2), convexity (A3), monotonicity (A4) and Archimedeanness (A5) if and only if it is an antisymmetric, additive and non transitive aggregation procedure. If, in addition, independence of initial preferences (IIP, A1) is satisfied, then $s^i_{xy}(p, \trianglerighteq)$ doesn't depend on $\trianglerighteq$.*

Note that this proposition and its proof have strong links with a proposition in (Myerson, 1995) characterizing scoring rules in social choice. Note also that this proposition can easily be reformulated in a pure voting framework, disregarding initial preferences.

Compatibility with $\trianglerighteq$. It seems reasonable to assume that some consistency should exist between p, w and $\trianglerighteq$. For example, if $x\delta^i(p_i)y$ and NOT $y\delta^i(p_i)x$ for all criteria, then it would be strange for the decision-maker to consider y as strictly better than x. If some degree of consistency exists, it might be reasonable to expect that we can find values for the parameters of an aggregation procedure such that the final preference relation $\succeq(p, w, \trianglerighteq)$ is compatible with $\trianglerighteq$ in the following sense. We say that a preference relation $\succeq(p, w, \trianglerighteq)$ is *compatible* with $\trianglerighteq$ iff

- $x \rhd y \Rightarrow x \succ(p, w, \trianglerighteq) y$ and
- $x \equiv y \Rightarrow x \sim(p, w, \trianglerighteq) y$.

We will show that the family of aggregation procedures characterized by Proposition 1 is so large that it contains many aggregation procedures which are probably unreasonable. We show that there is almost always an aggregation procedure compatible with $\trianglerighteq$: the minimum consistency between p and $\trianglerighteq$ is very weak.

Let us now introduce a new axiom.

A 6. Faithfulness. $\succeq(p, u^i, \trianglerighteq) = \delta^i(p_i)$.

Thanks to faithfulness, if only one criterion is considered, the aggregation procedure will be coherent with the information contained in the assessment structure for that criterion.

PROPOSITION 2. *Given any $(p, w, \trianglerighteq)$, there is an aggregation procedure $\succeq$ satisfying completeness (A2), convexity (A3), monotonicity (A4) and Archimedeanness (A5) and such that $\succeq(p, w, \trianglerighteq)$ is compatible with $\trianglerighteq$.*

In addition, let us impose faithfulness (A6). Then, there is an aggregation procedure $\succeq$ such that $\succeq(p, w, \trianglerighteq)$ is compatible with $\trianglerighteq$ if and only if, whenever $x \rhd y$, we have $x\delta^i(p_i)y$ and NOT $y\delta^i(p_i)x$, for some i. Furthermore, for all i, $\delta^i(p_i)$ is complete.

Let us rephrase the second part of Proposition 2. When a decision-maker faces a given problem, i.e. a triplet $(p, w, \trianglerighteq)$, the initial preference $\trianglerighteq$ can be as "strange" as we want with respect to p. As long as x is not dominated by y when the decision-maker definitely prefers x to y, there is an aggregation procedure $\succeq$ such that $\succeq(p, w, \trianglerighteq)$ is compatible with $\trianglerighteq$. This (probably too) wide variety of aggregation procedures is due to the fact that the $s^i_{xy}(p, \trianglerighteq)$ in Proposition 1 depends on x and y.

Proposition 2 concerns only the case where $\trianglerighteq$ is a binary preference relation, but it can easily be rephrased for other cases.

Independence of the axioms of Proposition 1. In order to prove the independence of our axioms, we present four examples of aggregation procedures. In each example, three axioms are verified while one is not. As these examples can help the reader better understand what an antisymmetric, additive and non transitive aggregation procedure is, we do not defer the proof of the independence to the *Proofs* section.

Completeness Let $\succeq(p, w, \trianglerighteq) = \{(x, x) : x \in X\}$, for all p, w and $\trianglerighteq$.

Monotonicity We already met an example: simple weighted majority with threshold (when the threshold is strictly positive).

Archimedeanness Let ω be the sum of the weights. For all z and z' different of x and y, $z{\sim}(p, w, \trianglerighteq)z'$. For the pair (x, y),

$w_1/\omega > .5 \Rightarrow x{\succ}(p, w, \trianglerighteq)y$,

$w_1/\omega = .5$ and $w_2/\omega > .5 \Rightarrow x{\succ}(p, w, \trianglerighteq)y$,

$w_1/\omega = .5$ and $w_2/\omega = .5 \Rightarrow x{\sim}(p, w, \trianglerighteq)y$,

$w_1/\omega = .5$ and $w_2/\omega < .5 \Rightarrow y{\succ}(p, w, \trianglerighteq)x$,

$w_1/\omega < .5 \Rightarrow y{\succ}(p, w, \trianglerighteq)x$.

Convexity If $\omega < 1$, then $x{\sim}(p, w, \trianglerighteq)y$, for all x and y. If $\omega \geq 1$, then $\succeq(p, w, \trianglerighteq)$ is a given total order.

In the next section, we consider a particular case of Proposition 1.

3.2. A uniqueness result: simple weighted majority

Let us define some new axioms. The permutation of a profile, $\pi(p)$, where π is a permutation on X, is the profile defined by $(\pi(p))_i = \pi(p_i)$, for all i.

A 7. Neutrality. $\succeq(\pi(p), w, \pi(\trianglerighteq)) = \pi(\succeq(p, w, \trianglerighteq))$.

The next two axioms concern the roles of the criteria. Weighted anonimity tells us that all pairs of criteria and weight play the same role. Let σ be a permutation on C. We denote by $\sigma(w)$ the weight vector such that $\sigma(w)_i = w_{\sigma(i)}$.

A 8. Weighted anonymity. *Let p and q be two profiles. If there is a permutation σ on C such that $\delta^i(p_i) = \delta^{\sigma(i)}(q_{\sigma(i)})$ for all i, then $\succeq(p, w, \trianglerighteq) = \succeq(q, \sigma(w), \trianglerighteq)$.*

Let $D(p, q) = \{i \in C : p_i \neq q_i\}$.

A 9. Independence of Irrelevant Criteria (IIC). *If $w_i = 0$ for all criteria in $D(p, q)$, then $\succeq(p, w, \trianglerighteq) = \succeq(q, w, \trianglerighteq)$.*

The restriction of a profile p to a subset $Y \subset X$ is denoted by $p|_Y$ and defined by $(p|_Y)_i = p_i|_Y$, for all i in C. When Y contains only two elements, say x and y, we write $p|_{xy}$.

A 10. Independence of Irrelevant Alternatives (IIA). *If* $p|_{xy} = q|_{xy}$ *, then* $\succeq(p, w, \trianglerighteq)|_{xy} = \succeq(q, w, \trianglerighteq)|_{xy}$.

Our next axiom states that the result of the aggregation can depend only on the ordinal information contained in the assessment structures.

A 11. Ordinality. *If* $\delta^i(p_i) = \delta^i(q_i)$*, for all criteria, then* $\succeq(p, w, \trianglerighteq) = \succeq(q, w, \trianglerighteq)$.

PROPOSITION 3. *For each criterion, let the single-criterion preference relation be complete. The only aggregation method that satisfies completeness (A2), convexity (A3), monotonicity (A4), Archimedeanness (A5), neutrality (A7), weighted anonymity (A8), IIC (A9), IIA (A10), faithfulness (A6) and ordinality (A11) is the simple weighted majority.*

Note that IIP (A1), though it is satisfied by the simple weighted majority, doesn't appear in this characterization. It is easy to see that the simple weighted majority with threshold violates only one of the axioms of Proposition 3: monotonicity. Note also that, because of completeness and faithfulness, $\delta^i(p_i)$ must be complete for all i.

4. SIMPLE WEIGHTED MAJORITY WITHOUT WEIGHTS

A very interesting result can be found in (Fishburn, 1973). It deals with the problem of binary choice in a committee. The primitives used by Fishburn are the same as those used in (May, 1952) to characterize simple majority; they can easily be reinterpreted in a decision aiding context and form a subset of our primitives. Therefore, it is straightforward to adapt Fishburn's result to our framework. Here are, slightly adapted, the axioms used by Fishburn.

A 12. Unanimity. *If* $x\delta^i(p_i)y$ *and NOT* $y\delta^i(p_i)x$*, for all criteria* i *in* C*, then* $x{\succ}(p, w, \trianglerighteq)y$.

This is also sometimes called Pareto. The next axiom is a kind of monotonicity axiom. Unlike monotonicity (A4), it deals with changes in p and not in w.

A 13. Non-negative responsiveness. *If, for all* i *in* C*,*
$[NOT\ y\delta^i(p_i)x \Rightarrow NOT\ y\delta^i(q_i)x]$ *and* $[x\delta^i(p_i)y \Rightarrow x\delta^i(q_i)y]$*,*
then
$[x{\succ}(p, w, \trianglerighteq)y \Rightarrow x{\succ}(q, w, \trianglerighteq)y]$ *and* $[x{\succeq}(p, w, \trianglerighteq)y \Rightarrow x{\succeq}(q, w, \trianglerighteq)y]$.

A 14. Strong duality. *Let us consider* m *profiles* $p^1, \dots p^j, \dots p^m$*. If, for all criteria, the number of profiles in* $\{p^1, \dots p^j, \dots p^m\}$ *such that* $x\delta^i(p_i^j)y$ *and NOT* $y\delta^i(p_i^j)x$ *is the same as the number of profiles such that* $y\delta^i(p_i^j)x$ *and NOT* $x\delta^i(p_i^j)y$*, then* $x{\succ}(p^j, w, \trianglerighteq)y$ *for some* j *and* $y{\succ}(p^{j'}, w, \trianglerighteq)x$ *for some* j'.

PROPOSITION 4. **[Fishburn]** *Let* $X = \{x, y\}$*. For each criterion, let the single-criterion preference relation be complete. If an aggregation method satisfies completeness (A2), ordinality (A11), unanimity (A12), non-negative responsiveness (A13) and strong duality (A14), then, for all* w *in* $\mathcal{W}$ *and all* $\trianglerighteq$ *in* $\mathcal{R}$*, there are non negative real numbers* $c_i^{w,\trianglerighteq}$ *such that*

$$x \succeq(p, w, \trianglerighteq) y \quad \textit{iff} \sum_{i: x\delta^i(p_i) y} c_i^{w,\trianglerighteq} \geq \sum_{i: y\delta^i(p_i) x} c_i^{w,\trianglerighteq}$$

and

$$c_i^{w,\trianglerighteq} > 0 \textit{ for some } i.$$

Our goal in this section is to show what happens when we include $\trianglerighteq$ in the primitives and when the parameters are derived from $\trianglerighteq$ (as usually happens in reality and should always happen). We will therefore no longer assume Independence of Initial Preferences (A1) but will require the weights to play no role. For this reason, let us introduce a new axiom.

A 15. Independence of Weights. $\succeq(p, w, \trianglerighteq) = \succeq(p, w\prime, \trianglerighteq)$.

From the result of Fishburn (Proposition 4), we find the following.

Proposition 5. *For each criterion, let the single-criterion preference relation be complete. If an aggregation method satisfies completeness (A2), independence of irrelevant alternatives (A10), ordinality (A11), unanimity (A12), non-negative responsiveness (A13), strong duality (A14) and independence of weights (A15), then, for all $\trianglerighteq$ in $\mathcal{R}$, there are non negative real numbers $c_i^{\trianglerighteq}(x, y)$ such that*

$$x \succeq(p, w, \trianglerighteq) y \quad \textit{iff} \sum_{i: x\delta^i(p_i) y} c_i^{\trianglerighteq}(x, y) \geq \sum_{i: y\delta^i(p_i) x} c_i^{\trianglerighteq}(x, y),$$

$$c_i^{\trianglerighteq}(x, y) = c_i^{\trianglerighteq}(y, x)$$

and

$$\forall x, y \in X, \forall \trianglerighteq \in \mathcal{R}, \textit{ there is a criterion such that } c_i^{\trianglerighteq}(x, y) > 0.$$

Note that, even if IIA is satisfied, $c_i^{\trianglerighteq}(x, y)$ can depend on $z \neq x, y$ through $\trianglerighteq$.

The following result, like Proposition 2, shows that the family of aggregation procedures characterized by Proposition 5 contains many aggregation procedures that are probably not reasonable.

Proposition 6. *For each criterion, let the single-criterion preference relation be complete. Given any $(p, w, \trianglerighteq)$, there is an aggregation procedure $\succeq$ satisfying completeness (A2), independence of irrelevant alternatives (A10), ordinality (A11), unanimity (A12), non-negative responsiveness (A13), strong duality (A14) and independence of weights (A15) and such that $\succeq(p, w, \trianglerighteq)$ is compatible with $\trianglerighteq$ if and only if, whenever $x \triangleright y$, we have $x\delta^i(p_i)y$ and NOT $y\delta^i(p_i)x$, for some i.*

This (probably too) wide variety of aggregation procedures is due to the fact that the parameters $c_i^{\trianglerighteq}(x, y)$ of Proposition 5 depend on x and y. One possible way of avoiding this is to impose neutrality. But the full strength of neutrality can only be used if we also impose the following condition.

A 16. Extended Independence of Irrelevant Alternatives (EIIA). *If $p|_{xy} = q|_{xy}$ and $\trianglerighteq|_{xy} = \trianglerighteq'|_{xy}$, then $\succeq(p, w, \trianglerighteq)|_{xy} = \succeq(q, w, \trianglerighteq)|_{xy}$.*

Proposition 7. *For each criterion, let the single-criterion preference relation be complete. If an aggregation method satisfies completeness (A2), neutrality (A7), ordinality (A11), unanimity (A12), non-negative responsiveness (A13), strong duality (A14), independence of weights (A15) and extended independence of irrelevant alternatives (A16), then, for all* $\trianglerighteq$ *in* $\mathcal{R}$*, there are non negative real numbers* $c_i^{\trianglerighteq}$ *such that*

$$x \succeq (p, w, \trianglerighteq) y \quad \textit{iff} \sum_{i : x\delta^i(p_i)y} c_i^{\trianglerighteq} \geq \sum_{i : y\delta^i(p_i)x} c_i^{\trianglerighteq}, \tag{3}$$

and

$$\forall \trianglerighteq \in \mathcal{R}, \textit{ there is a criterion such that } c_i^{\trianglerighteq} > 0.$$

5. DISCUSSION

5.1. With and without $\trianglerighteq$

Equation (2), in the case of simple weighted majority, can be rewritten as

$$x \succeq (p, w, \trianglerighteq) y \quad \text{iff} \sum_{i : x\delta^i(p_i)y} w_i \geq \sum_{i : y\delta^i(p_i)x} w_i. \tag{4}$$

Comparing (3) and (4), it is evident that the family of aggregation procedures characterized by Proposition 7 is very much like simple weighted majority (Section 3.2) but there is a tremendous difference: in Proposition 7, the "weights" $c_i^{\trianglerighteq}$ are part of the aggregation procedure while, in Proposition 3, they are part of the primitives. We call the aggregation procedures characterized by Proposition 7 *simple unspecified weighted majority* in order to make clear that the weights are not specified a priori.

The characterization in Proposition 3 is very classical in the sense that the initial preferences play no role. It resembles other results in social choice theory very much. In Proposition 7, on the contrary, the initial preferences play a major role while the weights play no role at all because of independence of weights (A15). This is much more relevant to decision aiding because the parameters (weights, importance coefficients, indifference thresholds, . . .) depend on the preferences of the decision-maker and upon the aggregation procedure used in the decision aiding process. This is very natural for weights, for example, do not have the same meaning in different procedures.

The effect of adding a new primitive in order to obtain what we consider to be a more relevant characterization is, at least in the case of simple weighted majority, dramatic. Proposition 3 uses 10 axioms; Proposition 7 uses 6. Only three are common to both propositions. Therefore, if we try to understand an aggregation procedure by considering the axioms of a characterization to be the basic principles underlying that procedure, we get very different pictures in the two frameworks. We claim that one of them, the one using $\trianglerighteq$, is more relevant than the other.

We insisted on the fact that parameters such as weights, utility functions, . . . should be part of the aggregation procedures and not of the primitives. Yet we included weight vectors in our primitives. This might look like a contradiction. In fact, we included the weight vectors in the primitives just to be able to characterize the simple weighted majority in two different ways and to illustrate the benefits of including $\trianglerighteq$ in the primitives. Our motivation was didactical. We are convinced that an adequate framework for describing aggregation procedures in decision aiding should contain alternatives, criteria, profiles, intitial preferences but no weights.

5.2. Choice

Throughout this paper, we considered the problem of comparing (not necessarily completely) the alternatives in X. Our results can be transposed to the problem of choosing whether we consider the choice function defined as follows: x belongs to the choice set iff $x{\succeq}(p, w, \trianglerighteq)y$ for all y in X. We also have to modify some axioms in a quite straightforward way. Those involving two different weight vectors, profiles or initial preference relations can be rephrased in terms of the union or intersection of the choice set corresponding to each of the two structures. For example, Archimedeanness would become something like this: if $\mathcal{C}(p, w, \trianglerighteq) \cap \mathcal{C}(p, w', \trianglerighteq) \neq 0$, then $\mathcal{C}(p, w+w', \trianglerighteq) = \mathcal{C}(p, w, \trianglerighteq) \cap \mathcal{C}(p, w', \trianglerighteq)$, where $\mathcal{C}$ denotes the choice function.

5.3. Further research

5.3.1. Uniqueness

Proposition 3 characterizes a unique aggregation procedure, while Proposition 7 characterizes a family of aggregation procedures. Indeed, Proposition 7 doesn't tell us how the parameters $c_i^{\trianglerighteq}$ depend on $\trianglerighteq$. This dependence can take many different forms, each one corresponding to a particular aggregation procedure. It would be most interesting to have some results in the spirit of Proposition 7, i.e. where the parameters depend upon $\trianglerighteq$, but characterizing a unique aggregation procedure, i.e. such that the way the parameters depend on $\trianglerighteq$ is specified.

5.3.2. Generalization

Ultimately, a still more general framework is needed: a framework in which even the set of criteria is not a primitive. Some procedures, taking into account the interaction between criteria (Grabisch, 1996), can probably be used with sets of criteria in which some criteria are correlated and/or dependent (see (Roy & Bouyssou, 1993)). Other procedures cannot. Therefore, the construction of the set of criteria must be coupled to the choice of an aggregation procedure and the set of criteria cannot be considered as a primitive.

It is most likely that we could even find some reasons to reject the set of alternatives as a primitive. But we believe that we are far from being able to derive any interesting result in such a framework.

Another possible modification to our framework is the following: we could define $\succeq$ as a mapping from some subset of $\mathcal{P}\times\mathcal{W}\times\mathcal{R}$ into $\mathcal{R}$. Indeed, there is no reason for a decision-maker to impose that conditions such as IIC be satisfied for all profiles. Only some profiles are possible in the particular problem that the decision-maker faces. Most profiles have nothing to do with his problem. Therefore, axioms need not be satisfied for these irrelevant profiles. The same reasoning applies to the initial preference relations. Obviously, such a framework would be extremely difficult to handle.

5.4. Other theories

Social Choice theory is not the only theory that has been used to try to better understand aggregation procedures in decision aiding. Measurement theory (Fishburn, 1979) and rough sets (Słowinski, 1992) are other examples. In this paper, we focused on social choice theory and showed how this theory can be improved. In another paper (in preparation), we will show why we consider the framework proposed here as an improvement of measurement theory.

6. PROOFS

The proofs of propositions 1 and 3 are presented in the next two sections. The other proofs are simple and left as an exercise for the reader.

6.1. Antisymmetric, additive and non transitive aggregation procedures

Before proving Proposition 1, we are going to introduce a new condition and prove a serie of five lemmas.

A 17. Homogeneity. *For any positive real number* α, $\succeq(p, w, \trianglerighteq) = \succeq(p, \alpha w, \trianglerighteq)$.

Note that convexity, together with a continuity condition, implies homogeneity.

LEMMA 1. *Let* $\succeq$ *be an aggregation method satisfying the following conditions: completeness (A2), convexity (A3) and Archimedeanness (A5). Then it satisfies homogeneity (A17).*

Proof. Suppose that homogeneity is not satisfied. Then, by completeness, there are $w, p, \trianglerighteq, x, y, \gamma$ such that $x{\succ}(p, w, \trianglerighteq)y$ and $y{\succeq}(p, \gamma w, \trianglerighteq)x$, where γ is a real number. By successive applications of convexity, $y{\succeq}(p, \gamma wr, \trianglerighteq)x$, for all positive and integer r. By Archimedeanness, $x{\succ}(p, \alpha w + \gamma w, \trianglerighteq)y$, for all real α larger than some β. For some large r, there is $\alpha > \beta$ and such that $\gamma wr = \alpha w + \gamma w$. Therefore, we obtain a contradiction. □

LEMMA 2. *Let* $\succeq$ *be an aggregation method satisfying the following conditions: completeness (A2), convexity (A3) and homogeneity (A17). For all* $x \neq y$*, there are real valued mappings* $s^i_{xy}, i = 1 \dots k : (p, \trianglerighteq) \rightarrow s^i_{xy}(p, \trianglerighteq)$ *such that*

- $x{\succ}(p, w, \trianglerighteq)y$ *and* $y{\succ}(p, w', \trianglerighteq)x \Rightarrow \sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq)[w'_i - w_i] \geq 0$,
- $x{\succeq}(p, u^i, \trianglerighteq)y$*, for all* i*, implies*
  - $s^i_{yx}(p, \trianglerighteq) = 1$ *if* $x{\succ}(p, u^i, \trianglerighteq)y$,
  - $s^i_{yx}(p, \trianglerighteq) = 0$ *if* $x{\sim}(p, u^i, \trianglerighteq)y$,
- $s^i_{yx}(p, \trianglerighteq) = -s^i_{xy}(p, \trianglerighteq)$,
- *if* $x{\succ}(p, u^j, \trianglerighteq)y$ *or* $y{\succ}(p, u^j, \trianglerighteq)x$*, for some* j*, then there is at least one criterion* i *such that* $s^i_{yx}(p, \trianglerighteq) \neq 0$.

Proof. By completeness, we can distinguish the two following cases.

1. $y \succ (p, u^j, \trianglerighteq) x$ for some j. Let

$$A_{xy}(p, \trianglerighteq) = \{w - w' : x \succ (p, w, \trianglerighteq) y \text{ and } y \succ (p, w', \trianglerighteq) x\}.$$

Let $B_{xy}(p, \trianglerighteq)$ be the convex hull of $A_{xy}(p, \trianglerighteq)$. Suppose that the k-dimensional vector 0 belongs to $B_{xy}(p, \trianglerighteq)$. Then there are weight vectors $w^1, \ldots, w^M$ and $v^1, \ldots, v^M$, with $x \succ (p, w^m, \trianglerighteq) y$ and $y \succ (p, v^m, \trianglerighteq) x, m = 1 \ldots M$, such that the system

$$\sum_{m=1}^{M} \lambda_m (w_i^m - v_i^m) = 0, \forall i,$$

has non negative solutions with at least one λ_m strictly positive. Therefore,

$$\sum_{m=1}^{M} \lambda_m w^m = \sum_{m=1}^{M} \lambda_m v^m.$$

By convexity and homogeneity,

$$y \succ (p, \textstyle\sum_{m=1}^{M} \lambda_m v^m, \trianglerighteq) x$$

and

$$x \succ (p, \textstyle\sum_{m=1}^{M} \lambda_m w^m, \trianglerighteq) y.$$

This is a contradiction. Hence, 0 does not belong to $B_{xy}(p, \trianglerighteq)$.

By the Supporting Hyperplane Theorem, we can choose

$$(s_{xy}^1(p, \trianglerighteq), \ldots, s_{xy}^k(p, \trianglerighteq)) \text{ in } (\mathbb{R}^+)^C$$

such that

- $s_{xy}^i(p, \trianglerighteq) \neq 0$, for some i,
- $x \succ (p, w, \trianglerighteq) y$ and $y \succ (p, w', \trianglerighteq) x \Rightarrow \sum_{i=1}^{k} s_{xy}^i(p, \trianglerighteq)[w_i' - w_i] \geq 0$,

To satisfy these conditions when the roles of x and y are reversed, we can simply let $s_{yx}^i(p, \trianglerighteq) = -s_{xy}^i(p, \trianglerighteq)$.

2. If $x \succeq (p, u^i, \trianglerighteq) y$, for all i, then we are free to choose

$$s_{yx}^i(p, \trianglerighteq) = \begin{cases} 1 & \text{if } x \succ (p, u^i, \trianglerighteq) y \\ 0 & \text{if } x \sim (p, u^i, \trianglerighteq) y. \end{cases}$$

□

Lemma 3. *Let $\succeq$ be an aggregation method satisfying the following conditions: completeness (A2), convexity (A3) and Archimedeanness (A5). Let $s_{xy}^i, i = 1 \ldots k$, satisfy the conditions of Lemma 2.*

$$y \succ (p, w, \trianglerighteq) x \text{ implies } \sum_{i=1}^{k} s_{xy}^i(p, \trianglerighteq) w_i \geq 0.$$

Proof. By Lemma 1, homogeneity is satisfied as well. There are two possible cases.

1. There is j such that $x \succ (p, u^j, \trianglerighteq) y$. Then, by Archimedeanness,

$$y \succ (p, \alpha w + u^j, \trianglerighteq) x, \text{ for some real number } \alpha.$$

Hence,

$$\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq)[\alpha w_i + u^j_i - u^j_i] \geq 0.$$

2. There is no j such that $x \succ (p, u^j, \trianglerighteq) y$. Then, by Lemma 2, $s^i_{xy}(p, \trianglerighteq) \geq 0$, for all i. Therefore, $\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq) w_i \geq 0$.

□

LEMMA 4. *Let $\succeq$ be an aggregation method satisfying the following conditions: completeness (A2), convexity (A3), monotonicity (A4) and Archimedeanness (A5). Let $s^i_{xy}, i = 1 \dots k$, satisfy the conditions of Lemma 2.*

$$y \succeq (p, w, \trianglerighteq) x \text{ implies } \sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq) w_i \geq 0.$$

Proof. There are two possible cases.

1. There is a criterion j such that $y \succ (p, u^j, \trianglerighteq) x$. By monotonicity and homogeneity (Lemma 1) for any $\alpha > 0$, o$y \succ (p, \alpha w + u^j, \trianglerighteq) x$. By Lemma 3,

$$\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq)[\alpha w_i + u^j_i] \geq 0.$$

Hence,

$$\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq) w_i \geq 0.$$

2. There is no j such that $y \succ (p, u^j, \trianglerighteq) x$. In other words, $x \succeq (p, u^i, \trianglerighteq) y$ for all i in C. Therefore, for all i, $x \sim (p, u^i, \trianglerighteq) y$ or $w_i = 0$. If this was not true, then there would be a criterion j such that $x \succ (p, u^j, \trianglerighteq) y$ and $w_j > 0$. By Monotonicity, we would have $x \succ (p, w, \trianglerighteq) y$. This is a contradiction. By Lemma 2, we know that $s^i_{xy}(p, \trianglerighteq) = 0$ for all i such that $w_i \neq 0$. Thus, $\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq) w_i = 0$.

□

LEMMA 5. *Let $\succeq$ be an aggregation method satisfying the following conditions: completeness (A2), convexity (A3) and Archimedeanness (A5). Let $s^i_{xy}, i = 1 \dots k$, satisfy the conditions of Lemma 2. If $y \succ (p, u^{i^*}, \trianglerighteq) x$ for some i^*, then there is a weight vector w^y such that*

$$y \succ (p, w^y, \trianglerighteq) x \text{ and } \sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq) w^y_i > 0.$$

Proof. By Lemma 2, there is a criterion j such that $s^j_{xy}(p, \trianglerighteq) \neq 0$. By homogeneity and Archimedeanness, there is a positive real number α such that

$$y \succ (p, \alpha u^{i^*}, \trianglerighteq) x, \quad y \succ (p, \alpha u^{i^*} + u^j, \trianglerighteq) x, \quad y \succ (p, \alpha u^{i^*} + 2u^j, \trianglerighteq) x.$$

By Lemma 3,

$$\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq)[\alpha u^{i^*}_i + m u^j_i] \geq 0, \quad m \in \{0, 1, 2\}.$$

Let us rewrite this expression:

$$\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq)\alpha u^{i^*}_i + m s^j_{xy}(p) \geq 0, \quad m \in \{0, 1, 2\}.$$

But $s^j_{xy}(p, \trianglerighteq) \neq 0$. This is possible only if

$$\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq)\alpha u^{i^*}_i + s^j_{xy}(p, \trianglerighteq) > 0.$$

The proof is complete if we let $w^y = \alpha u^{i^*} + m u^j$. □

Proof of Proposition 1, part 1. We need to prove that, if s^i_{xy}, $i = 1 \ldots k$, satisfy the conditions of Lemma 2, then

$$\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq) w_i \geq 0 \text{ implies } y \succeq (p, w, \trianglerighteq) x.$$

There are two cases.

1. There is j such that $y \succ (p, u^j, \trianglerighteq) x$. Let w^y be as in Lemma 5. For any positive real number α,

   $$\sum_{i=1}^{k} s^i_{xy}(p, \trianglerighteq)[\alpha w_i + w^y_i] > 0.$$

   By Lemma 2,

   $$\sum_{i=1}^{k} s^i_{yx}(p, \trianglerighteq)[\alpha w_i + w^y_i] < 0.$$

   By Lemma 4, for all $\alpha > 0$, $y \succ (p, \alpha w + w^y, \trianglerighteq) x$. Consequently, it is not the case that $x \succeq (p, w, \trianglerighteq) y$. By completeness, $y \succeq (p, w, \trianglerighteq) x$.

2. There is no j such that $y \succ (p, u^j, \trianglerighteq) x$. Hence, $x \succeq (p, w, \trianglerighteq) y$ and, by Lemma 4,

   $$\sum_{i=1}^{k} s^i_{yx}(p, \trianglerighteq) w_i \geq 0.$$

   For all i such that $x \sim (p, u^i, \trianglerighteq) y$, we know by Lemma 2 that $s^i_{yx}(p, \trianglerighteq) = 0$. Let $Q \subseteq C$, be the set of all criteria j such that $x \succ (p, u^j, \trianglerighteq) y$. For all j in Q, $s^j_{yx}(p, \trianglerighteq) > 0$. If, for all j in Q, $w_j = 0$, then, by convexity, $x \sim (p, w, \trianglerighteq) y$. If, on the contrary, $w_j > 0$ for some j in Q, then, by monotonicity, $x{\succ}(p, w, \trianglerighteq)y$. But

$$\sum_{i=1}^{k} s^i_{yx}(p, \trianglerighteq)w_i > 0.$$

This is a contradiction. Hence, $w_j = 0$ for all j in Q and the proof is complete.

□

Proof of Proposition 1, part 2. If independence of initial preferences (IIP) is satisfied, it is obvious that $s^i_{yx}(p, \trianglerighteq)$ depends only on p and not on $\trianglerighteq$. □

6.2. Simple weighted majority

Two more lemmas will be necessary before proving Proposition 3.

LEMMA 6. *Let $\succeq$ be an aggregation method satisfying the following conditions: IIC (A9), IIA (A10) and ordinality (A11). If $w_i = 0$ for all i such that*

$$\succeq(p, u^i, \trianglerighteq)|_{xy} \neq \succeq(q, u^i, \trianglerighteq)|_{xy},$$

then $\succeq(p, w, \trianglerighteq)|_{xy} = \succeq(q, w, \trianglerighteq)|_{xy}$.

Proof. Let p' and q' be profiles such that

- $\delta^i(p'_i) = \delta^i(p_i)$ for all i in C,
- $\delta^i(q'_i) = \delta^i(q_i)$ for all i in C and
- $p'_i|_{xy} = q'_i|_{xy}$ for all i such that $\succeq(p, u^i, \trianglerighteq)|_{xy} = \succeq(q, u^i, \trianglerighteq)|_{xy}$.

These two profiles necessarily exist because of the fourth condition that we impose on the assessment structures. By ordinality, $\succeq(p, w, \trianglerighteq) = \succeq(p', w, \trianglerighteq)$ and $\succeq(q, w, \trianglerighteq) = \succeq(q', w, \trianglerighteq)$ for any w.

Let p'' and q'' be profiles such that

- $p''_i = p'_i$ and $q''_i = q'_i$ for all i such that $p'_i|_{xy} = q'_i|_{xy}$ and
- $p''|_{xy} = q''|_{xy}$ for all i in C.

By IIA, $\succeq(p'', w, \trianglerighteq)|_{xy} = \succeq(q'', w, \trianglerighteq)|_{xy}$. Let W be the set of all weight vectors such that $w_i = 0$ if $p'_i|_{xy} \neq q'_i|_{xy}$. By IIC,

$$\succeq(p', w, \trianglerighteq) = \succeq(p'', w, \trianglerighteq) \quad \text{and} \quad \succeq(q', w, \trianglerighteq) = \succeq(q'', w, \trianglerighteq),$$

for all $w \in W$. Therefore,

$$\begin{aligned} \succeq(p, w, \trianglerighteq)|_{xy} &= \succeq(p', w, \trianglerighteq)|_{xy} = \succeq(p'', w, \trianglerighteq)|_{xy} = \succeq(q'', w, \trianglerighteq)|_{xy} \\ &= \succeq(q', w, \trianglerighteq)|_{xy} = \succeq(q, w, \trianglerighteq)|_{xy} \end{aligned}$$

for all $w \in W$. □

Lemma 7. *Let $\succeq$ be an aggregation method satisfying the following conditions: weighted anonimity (A8), IIC (A9), IIA (A10) and ordinality (A11). If there is a permutation σ on C such that, for each criterion i,*

$$\succeq(p, u^i, \trianglerighteq)|_{xy} = \succeq(q, u^{\sigma(i)}, \trianglerighteq)|_{xy} \qquad or \qquad w_i = w_{\sigma(i)} = 0,$$

then $\succeq(p, w, \trianglerighteq)|_{xy} = \succeq(q, \sigma(w), \trianglerighteq)|_{xy}$.

Proof. Let p' be a profile such that

- $\succeq(p, u^i, \trianglerighteq)|_{xy} = \succeq(p', u^{\sigma(i)}, \trianglerighteq)|_{xy}$ for all i such that $w_i > 0$ and
- $\delta^i(p'_i) = \delta^{\sigma(i)}(p'_{\sigma(i)})$ for all i.

Such a profile p' necessarily exists, because of the fourth condition that we impose on the assessment structures. By Lemma 6, $\succeq(p, w, \trianglerighteq)|_{xy} = \succeq(p', w, \trianglerighteq)|_{xy}$ and by weighted anonimity, $\succeq(p', w, \trianglerighteq) = \succeq(q, \sigma(w), \trianglerighteq)$. □

Proof of Proposition 3. To complete the proof, all we need to do is to show that there are s^i_{yx} satisfying the conditions of Lemma 2 and such that $s^i_{yx}(p, \trianglerighteq) =$
$$\begin{cases} 1 & \text{if } x\delta^i(p_i)y & \text{and NOT } y\delta^i(p_i)x, \\ 0 & \text{if } x\delta^i(p_i)y & \text{and } y\delta^i(p_i)x, \\ -1 & \text{if NOT } x\delta^i(p_i)y & \text{and } y\delta^i(p_i)x. \end{cases}$$
(a) If $x\delta^i(p_i)y$ for all i in C, then, by faithfulness, $x{\succeq}(p, u^i, \trianglerighteq)y$ for all i. By Lemma 2, we find the following.

- $s^i_{yx}(p, \trianglerighteq) = 1$ if $x{\succ}(p, u^i, \trianglerighteq)y$. By faithfulness, $s^i_{yx}(p, \trianglerighteq) = 1$ if $x\delta^i(p_i)y$ and NOT $y\delta^i(p_i)x$.
- $s^i_{yx}(p, \trianglerighteq) = 0$ if $x{\sim}(p, u^i, \trianglerighteq)y$. By faithfulness, $s^i_{yx}(p, \trianglerighteq) = 0$ if $x\delta^i(p_i)y$ and $y\delta^i(p_i)x$.

(b) If we are not in case (a), then there are j and j^* such that NOT $y\delta^j(p_j)x$ and NOT $x\delta^{j^*}(p_{j^*})y$. By faithfulness, $x{\succ}(p, u^j, \trianglerighteq)y$ and $y{\succ}(p, u^{j^*}, \trianglerighteq)x$. Suppose that $s^j_{yx}(p, \trianglerighteq) \neq -s^{j^*}_{yx}(p, \trianglerighteq)$. Then, by Proposition 1, NOT $x{\sim}(p, u^j + u^{j^*}, \trianglerighteq)y$. Let π be a permutation on X such that $\pi(x) = y$ and $\pi(y) = x$. By neutrality,

$$\succeq(\pi(p), w, \pi(\trianglerighteq))|_{xy} \text{ is the converse of } \succeq(p, w, \trianglerighteq)|_{xy}. \tag{5}$$

In other words, $x{\succ}(\pi(p), w, \pi(\trianglerighteq))y \Leftrightarrow y{\succ}(p, w, \trianglerighteq)x$ and $y{\succ}(\pi(p), w, \pi(\trianglerighteq))x \Leftrightarrow x{\succ}(p, w, \trianglerighteq)y$.

Let q be a profile such that $\delta^j(p_j)|_{xy} = \delta^{j^*}(q_{j^*})|_{xy}$ and $\delta^{j^*}(p_{j^*})|_{xy} = \delta^j(q_j)|_{xy}$. By faithfulness, $\succeq(p, u^j, \trianglerighteq)|_{xy} = \succeq(q, u^{j^*}, \trianglerighteq)|_{xy}$ and $\succeq(p, u^{j^*}, \trianglerighteq)|_{xy} = \succeq(q, u^j, \trianglerighteq)|_{xy}$. By Lemma 7,

$$\succeq(p, u^j + u^{j^*}, \trianglerighteq)|_{xy} = \succeq(q, u^j + u^{j^*}, \trianglerighteq)|_{xy}. \tag{6}$$

Let us remark that $\succeq(q, u^j, \trianglerighteq)|_{xy} = \succeq(\pi(p), u^j, \pi(\trianglerighteq))|_{xy}$ and $\succeq(q, u^{j^*}, \trianglerighteq)|_{xy} = \succeq(\pi(p), u^{j^*}, \pi(\trianglerighteq))|_{xy}$. So, by Lemma 6,

$$\succeq(q, u^j + u^{j^*}, \trianglerighteq)|_{xy} = \succeq(\pi(p), u^j + u^{j^*}, \pi(\trianglerighteq))|_{xy}. \tag{7}$$

If we combine equations (6) and (7), we obtain a contradiction with respect to Equation (5), due to the fact that NOT $x{\sim}(p, u^j + u^{j^*}, \trianglerighteq)y$.

Because of this contradiction, we know that $s^j_{yx}(p, \trianglerighteq) = -s^{j^*}_{yx}(p, \trianglerighteq)$. For any criterion g such that $x \succ (p, u^g, \trianglerighteq) y$, we can use j^* to find that $s^j_{yx}(p, \trianglerighteq) = s^g_{yx}(p, \trianglerighteq) = -s^{j^*}_{yx}(p, \trianglerighteq)$. And for any criterion g^* such that $y \succ (p, u^{g^*}, \trianglerighteq) x$, we can use j to find that $s^j_{yx}(p, \trianglerighteq) = -s^{g^*}_{yx}(p, \trianglerighteq) = -s^{j^*}_{yx}(p, \trianglerighteq)$. Hence, $s^i_{yx}(p, \trianglerighteq)$ is either 0, a constant or the opposite of that constant. It is clear that the value of this constant is not important and we can choose it equal to 1. □

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