Multiattribute preference models with reference points[☆]

Denis Bouyssou[a,b], Thierry Marchant[c]

[a] *CNRS, LAMSADE, UMR 7243, F-75775 Paris Cedex 16, France*
[b] *Université Paris Dauphine, F-75775 Paris Cedex 16, France*
[c] *Ghent University, H. Dunantlaan, 1, B-9000 Gent, Belgium*

Abstract

In the context of multiple attribute decision making, preference models making use of reference points in an ordinal way have recently been introduced in the literature. This text proposes an axiomatic analysis of such models, with a particular emphasis on the case in which there is only one reference point. Our analysis uses a general conjoint measurement model resting on the study of traces induced on attributes by the preference relation and using conditions guaranteeing that these traces are complete. Models using reference points are shown to be a particular case of this general model. The number of reference points is linked to the number of equivalence classes distinguished by the traces. When there is only one reference point, the induced traces are quite rough, distinguishing at most two distinct equivalence classes. We study the relation between the model using a single reference point and other preference models proposed in the literature, most notably models based on concordance and models based on a discrete Sugeno integral.

Keywords: Multiple Criteria Decision Making, Reference point, Conjoint Measurement

1. Introduction

In a series of papers, Rolland (2003, 2006a,b, 2008, 2013) (see also Perny and Rolland 2006, in the related context of decision making under uncertainty) has suggested to use reference points[1] in an ordinal way to build preference models for multiattributed alternatives. This idea can be traced back to Fargier and Perny (1999) and Dubois et al.

[☆] Authors are listed alphabetically. They have equally contributed to this paper. We wish to thank Patrice Perny, Antoine Rolland, and anonymous referees for their very helpful comments on earlier drafts of this text. The usual caveat applies.

Email addresses: `bouyssou@lamsade.dauphine.fr` (Denis Bouyssou), `thierry.marchant@UGent.be` (Thierry Marchant)

[1] The notion of "reference point" is unfortunately used in the literature with many different meanings. The interpretation of the reference points in the models studied in this paper is discussed below. These reference points have little to do with the reference point used in prospect theory to distinguish gains from losses (Kahneman and Tversky 1979, Tversky and Kahneman 1992) or from the reference points used as a crucial element in the framing of decisions (Tversky and Kahneman 1986).

(2003, p. 247). In these models, the preference between alternatives x and y rests on a comparison in terms of "importance" of the sets of attributes for which x and y are above the reference points. Rolland has analyzed the interest of such models and has proposed axioms that could characterize them. Most of his axiomatic analysis supposes that the reference points are known beforehand[2]. Including reference points in the primitives of the model is a strong hypothesis and raises observational questions. Moreover, he invokes conditions that seem to be quite specific to models using reference points. It is therefore not easy to use them in order to compare these models with other ones that have been proposed and characterized in the literature.

The aim of this text is to propose an axiomatic analysis of preference models with reference points using the traditional primitives of conjoint measurement, i.e., a preference relation on the set of alternatives. Our analysis uses a general conjoint measurement model resting on the study of traces induced on attributes by the preference relation and using conditions guaranteeing that these traces are complete (Bouyssou and Pirlot 2004b). We show that preference models with reference points are a particular case of this general model. This will allow us to characterize preference models with reference points using conditions that will facilitate their comparison with other preference models proposed in the literature.

We put a special emphasis on preference models that use a single reference point. On each attribute, these models induce traces that are quite rough, distinguishing at most two distinct equivalence classes. Our characterization of these models allows us to compare them with other types of preference models introduced in the literature. In particular, we will show that they are a particular case of models based on a discrete Sugeno integral and study their relations with models based on the notion of concordance.

Our general strategy will be similar to the one used in Bouyssou and Pirlot (2005b, 2007) to analyze models based on the notion of concordance (see also Bouyssou et al. 1997, Greco et al. 2001, and Bouyssou and Pirlot 2002b). They have shown that such models could be seen as particular cases of the general conjoint measurement models developed in Bouyssou and Pirlot (2002a, 2004a) that generate complete traces on differences between levels in which these traces are "rough", i.e., only distinguishing a limited number of equivalence classes. We show here that models using reference points are a particular case of models inducing complete traces on levels developed in Bouyssou and Pirlot (2004b) in which these traces are "rough" (for a general overview of preference models based on different kinds of traces, we refer to Bouyssou and Pirlot 2005a).

The paper is organized as follows. Section 2 introduces our notation and setting. Section 3 formalizes and illustrates preference models using a single reference point. Section 4 recalls the main ingredients of the general conjoint measurement models introduced in Bouyssou and Pirlot (2004b). Section 5 characterizes preference models using a single reference point. Section 6 studies the links between preference models using a single reference point and other preference models introduced in the literature. Section 7 is devoted to the study of preference models using a single reference point that are weak

[2]For exceptions, see Rolland (2003, Th. 3) and Rolland (2008, Section 2.1.2).

orders. It also outlines an elicitation technique of the parameters of the model. Section 8 extends our results to preference models using several reference points. A final section discusses our findings. For space reasons and with apologies to the reader, most proofs are relegated to the supplementary material to this paper.

2. Background

2.1. Binary relations

A binary relation $\mathcal{K}$ on a set A is a subset of $A \times A$. We often write $a \mathcal{K} b$ instead of $(a, b) \in \mathcal{K}$. We define the symmetric and asymmetric parts of $\mathcal{K}$ as is usual.

An *equivalence* is a reflexive ($a \mathcal{K} a$), symmetric ($a \mathcal{K} b \Rightarrow b \mathcal{K} a$) and transitive ($[a \mathcal{K} b$ and $b \mathcal{K} c] \Rightarrow a \mathcal{K} c$) binary relation on A. An equivalence relation partitions A into equivalence classes. The set of equivalence classes induced by the equivalence $\mathcal{K}$ is denoted by $A/\mathcal{K}$.

A *weak order* is a complete ($a \mathcal{K} b$ or $b \mathcal{K} a$) and transitive binary relation. When $\mathcal{K}$ is a weak order on A, it is clear that the symmetric part of $\mathcal{K}$ is an equivalence. We often abuse terminology and speak of equivalence classes of the weak order $\mathcal{K}$ instead of the equivalence classes of its symmetric part. In this case, we also speak of the first, second, ..., last equivalence class of $\mathcal{K}$.

A *semiorder* is a reflexive ($a \mathcal{K} a$), Ferrers ($[a \mathcal{K} b$ and $c \mathcal{K} d]$ imply $[a \mathcal{K} d$ or $c \mathcal{K} b]$) and semitransitive ($[a \mathcal{K} b$ and $b \mathcal{K} c]$ imply $[a \mathcal{K} d$ or $d \mathcal{K} c]$) binary relation. If $\mathcal{K}$ is a semiorder, it is well known (see, e.g., Aleskerov et al. 2007, pp. 208 & 224) that the relation $\mathcal{K}^\circ$ defined letting, for all $a, b, c \in A$,

$$a \mathcal{K}^\circ b \Leftrightarrow \big[[b \mathcal{K} c \Rightarrow a \mathcal{K} c] \text{ and } [c \mathcal{K} a \Rightarrow c \mathcal{K} b]\big],$$

is a weak order.

2.2. Notation

In this paper, $\succsim$ will always denote a binary relation on a set $X = \prod_{i=1}^n X_i$ with $n \geq 2$. Elements of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes and $\succsim$ as an "at least as good as" relation between these alternatives. We denote by $\succ$ (resp. $\sim$) the asymmetric (resp. symmetric) part of $\succsim$. A similar convention holds when $\succsim$ is starred, superscripted and/or subscripted.

For any nonempty subset J of the set of attributes N, we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \in N \setminus J} X_i$). When $x, y \in X$, with customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. We sometimes omit braces around sets. For instance, when $J = \{i\}$ we write X_{-i} and (x_i, y_{-i}).

We say that attribute $i \in N$ is *influential* (for $\succsim$) if there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(z_i, a_{-i}) \not\succsim (w_i, b_{-i})$ and *degenerate* otherwise. A degenerate attribute has no influence whatsoever on the comparison of the elements of X and may be suppressed from N. As in Bouyssou and Pirlot (2005b), in order to avoid unnecessary minor complications, we suppose henceforth that *all attributes in N are influential.*

Let $J \subset N$ be a proper nonempty subset of attributes. We say that $\succsim$ is *independent* (see, e.g., Wakker 1989, p. 30) for J if, for all $x_J, y_J \in X_J$,

$$(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J} \Rightarrow$$
$$(x_J, w_{-J}) \succsim (y_J, w_{-J}), \text{ for all } w_{-J} \in X_{-J}.$$

If $\succsim$ is independent for all proper nonempty subsets of N, we say that $\succsim$ is *independent.* It is clear that $\succsim$ is independent iff $\succsim$ is independent for $N \setminus \{i\}$, for all $i \in N$.

A *capacity* on N is a real valued function μ on 2^N such that, for all $A, B \in 2^N$, $A \supseteq B \Rightarrow \mu(A) \geq \mu(B)$. The capacity μ on N is *normalized* if, furthermore, $\mu(\varnothing) = 0$ and $\mu(N) = 1$. All capacities used in this text will be normalized.

The *Möbius inverse* of a capacity is the the real valued function m on 2^N such that, for all $S \subseteq N$, $m(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \mu(T)$ (see, e.g., Chateauneuf and Jaffray 1989). A capacity is said to be k-additive (Grabisch 1997) if its Möbius inverse is null for all subsets containing $k+1$ elements or more. Capacities that are 2-additive are known to be of manageable complexity, whereas already allowing much flexibility w.r.t. additive capacities, (i.e., 1-additive capacities, see Grabisch 1997, Marichal and Roubens 2000).

3. Preference models with a single reference point

3.1. Motivation

The model that we study was introduced by Rolland (2003, 2006a,b, 2008, 2013). It has close connections with ELECTRE TRI (Roy and Bouyssou 1993, Chap. 6). Remember that ELECTRE TRI is a technique used to assign alternatives to ordered categories. Suppose that there are only two categories: $\mathscr{A}$ and $\mathscr{U}$, $\mathscr{A}$ being the best category. The limit between these two categories is indicated by a profile p that is at the same time the lower limit of $\mathscr{A}$ and the upper limit of $\mathscr{U}$. In the pessimistic version of ELECTRE TRI, an alternative $x \in X$ belongs to category $\mathscr{A}$ iff this alternative is declared at least as good as p. The central originality of ELECTRE TRI lies in the definition of this "at least as good as" relation that is based on the notions of concordance and non-discordance. Ignoring here the non-discordance condition, an alternative $x \in X$ is "at least as good as" the profile p if a "sufficient majority" of attributes support this assertion. When preference and indifference thresholds are equal, this is done as follows. A semiorder T_i is defined on each attribute. The set of attributes supporting the proposition that $x \in X$ is at least as good as p is simply $\mathcal{T}(x) = \{i \in N : x_i \; T_i \; p_i\}$. A positive weight w_i is assigned to each attribute. These weights are supposed to be normalized so that $\sum_{i=1}^{n} w_i = 1$. The test for deciding whether the subset of attributes $\mathcal{T}(x)$ is "sufficiently important" is done comparing $\sum_{i \in \mathcal{T}(x)} w_i$ to a majority threshold $\lambda \in [0.5, 1]$. We have:

$$x \in \mathscr{A} \Leftrightarrow \sum_{i \in \mathcal{T}(x)} w_i \geq \lambda.$$

Ordered partitions $\langle \mathscr{A}, \mathscr{U} \rangle$ of this type have been studied and characterized in Bouyssou and Marchant (2007a). For the sequel, it will be useful to note that the concordance condition for testing if x is "at least as good as" p only distinguishes two kind of attributes: the ones for which $x_i \; T_i \; p_i$ and the ones for which this is not true. It does not make further distinctions among the attributes and, in particular, does not make use of the preference difference between x_i and p_i. Hence, the assignment of an alternative mainly rests on "ordinal considerations".

3.2. Definition

The model defined below is close to ELECTRE TRI. The main difference with ELECTRE TRI is that the aim of this model is to compare alternatives rather than assigning them to ordered categories. In this model, there is a semiorder S_i on each attribute. In order to compare the alternatives x and y, we first compare each of them to a "reference point" π only using "ordinal considerations". Hence, we compute the subsets of attributes $\mathcal{S}(x) = \{i \in N : x_i \; S_i \; \pi_i\}$ and $\mathcal{S}(y) = \{i \in N : y_i \; S_i \; \pi_i\}$. The comparison of x and y is based on the comparison of the subsets $\mathcal{S}(x)$ and $\mathcal{S}(y)$. This comparison uses an "importance relation" that is only required to be monotonic w.r.t. inclusion. The following definition, inspired by Rolland (2003, 2006a,b, 2008, 2013), formalizes this idea.

Definition 1. *A binary relation $\succsim$ is a Relation with a Single Reference Point (or, more briefly, is a RSRP) if:*

- *for all $i \in N$, there is a semiorder S_i on X_i (with symmetric part I_i and asymmetric part P_i),*
- *there is an element $\pi \in X$,*
- *there is a binary relation $\trianglerighteq$ on 2^N (with symmetric part $\triangleq$ and asymmetric part $\triangleright$), that is* monotonic *w.r.t. inclusion, i.e., for all $A, B, C, D \subseteq N$,*

$$A \trianglerighteq B \Rightarrow C \trianglerighteq D,$$

whenever, $C \supseteq A$, $B \supseteq D$, and there are $x, y, z, w \in X$ such that $\mathcal{S}(x) = A$, $\mathcal{S}(y) = B$, $\mathcal{S}(z) = C$, and $\mathcal{S}(w) = D$,

such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow \mathcal{S}(x) \trianglerighteq \mathcal{S}(y), \tag{RSRP}$$

where $\mathcal{S}(x) = \{i \in N : x_i \; S_i \; \pi_i\}$.

The above model uses three parameters: the reference point π, the semiorders S_i, $i = 1, 2, \ldots, n$, and the importance relation $\trianglerighteq$. Although this presentation allows to easily grasp the intuition of the model, it is possible to reformulate it using less parameters. Indeed, define, for all $i \in N$, $\mathcal{A}_i = \{x_i \in X_i : x_i \; S_i \; \pi_i\}$. It is clear that, for all $x \in X$, we have $\mathcal{S}(x) = \{i \in N : x_i \in \mathcal{A}_i\}$. Hence, we can alternatively write the above model only using the following parameters: the sets $\mathcal{A}_i$, $i = 1, 2, \ldots, n$, and the importance relation

$\trianglerighteq$. In this reformulation, alternative x is at least as good as alternative y if the subset of attributes for which x has an evaluation that is "acceptable" (i.e., attributes i such that $x_i \in \mathcal{A}_i$) is "more important" (according to the relation $\trianglerighteq$) than the subset of attributes for which y has an evaluation that is acceptable. In Section 8, we generalize this model to include multiple reference points.

3.3. Elementary properties

The following lemma shows that a RSRP has a *unique* representation in terms of the sets $\mathcal{A}_i$ and the importance relation $\trianglerighteq$. It makes use of the fact that all attributes have been supposed influential[3].

Lemma 1. *A RSRP has a unique representation in terms of the sets $\mathcal{A}_i$, $i \in N$, and the relation $\trianglerighteq$. In this representation we have, for all $i \in N$, $\varnothing \subsetneq \mathcal{A}_i \subsetneq X_i$.*

Proof. See Section A1 in the supplementary material. □

Lemma 2. *Let $\succsim$ be RSRP with representation $\mathcal{A}_i$, $i \in N$, and $\trianglerighteq$.*

1. *$\succsim$ is reflexive iff $\trianglerighteq$ is reflexive,*
2. *$\succsim$ is complete iff $\trianglerighteq$ is complete,*
3. *$\succsim$ is transitive iff $\trianglerighteq$ is transitive,*
4. *$\succsim$ is independent iff, for all $i \in N$ and all $A, B \subseteq N$ such that $i \notin A$ and $i \notin B$, $A \trianglerighteq B$ iff $A \cup \{i\} \trianglerighteq B \cup \{i\}$.*

Proof. See Section A2 in the supplementary material. □

Suppose that $\succsim$ is a RSRP that is a weak order. It follows from the above lemma that $\trianglerighteq$ must be a weak order. Moreover, since $\trianglerighteq$ is monotonic w.r.t. inclusion and N is finite, there is a normalized capacity μ on N such that $A \trianglerighteq B \Leftrightarrow \mu(A) \geq \mu(B)$. Hence, we have $x \succsim y \Leftrightarrow \mu(\mathcal{S}(x)) \geq \mu(\mathcal{S}(y))$. This shows that a RSRP being a weak order has at most 2^n distinct equivalence classes.

3.4. Example

We illustrate the above model using an example that draws on a similar one presented in Marichal and Roubens (2000) who study a different problem related to the possibility to represent weak ordered preferences with the help of a Choquet integral w.r.t. to a 2-additive capacity.

Four alternatives are evaluated on three criteria as follows:

[3] Observe that the lemma implies that, for all $A \in 2^N$, there is $x \in X$ such that $\mathcal{S}(x) = A$. Hence, we could have omitted the qualification "there are $x, y, z, w \in X$ such that $\mathcal{S}(x) = A$, $\mathcal{S}(y) = B$, $\mathcal{S}(z) = C$, and $\mathcal{S}(w) = D$" from Definition 1.

	1	2	3
x	α_1	α_2	α_3
y	β_1	β_2	α_3
z	β_1	β_2	β_3
w	α_1	α_2	β_3

The preference information given by the decision-maker is in the form of a linear order:

$$x \succ y \succ z \succ w.$$

This preference information is easily seen to be compatible with our model. Indeed, take the reference point $\pi = (\alpha_1, \beta_2, \alpha_3) \in X$. Moreover, consider the following semiorders (remembering that P_i is the asymmetric part of S_i): $\alpha_1 \; P_1 \; \beta_1$, $\beta_2 \; P_2 \; \alpha_2$, and $\alpha_3 \; P_3 \; \beta_3$. This leads to: $\mathcal{A}_1 = \{\alpha_1\}$, $\mathcal{A}_2 = \{\beta_2\}$, and $\mathcal{A}_3 = \{\alpha_3\}$. Moreover, let the importance relation $\trianglerighteq$ be the following weak order:

$$\Big[\{1,2,3\} \triangleq \{1,3\}\Big] \rhd \Big[\{2,3\} \triangleq \{3\}\Big] \rhd \Big[\{1,2\} \triangleq \{2\}\Big] \rhd \Big[\{1\} \triangleq \varnothing\Big],$$

(remembering that $\triangleq$ and $\rhd$ denote the symmetric and asymmetric part of $\trianglerighteq$).

We obtain $\mathcal{S}(x) = \{1,3\}$, $\mathcal{S}(y) = \{2,3\}$, $\mathcal{S}(z) = \{2\}$, and $\mathcal{S}(w) = \{1\}$, which clearly allows to recover the linear order given by the decision maker. For instance, we have $x \succ y$ since $\mathcal{S}(x) = \{1,3\}$, $\mathcal{S}(y) = \{2,3\}$, and $\{1,3\} \rhd \{2,3\}$.

Observe that the above importance relation can be represented by a 2-additive capacity (Grabisch 1997) such that: $\mu(\{1,2,3\}) = \mu(\{1,3\}) = 1$, $\mu(\{2,3\}) = \mu(\{3\}) = 2/3$, $\mu(\{1,2\}) = \mu(\{2\}) = 1/3$, $\mu(\{1\}) = \mu(\varnothing) = 0$, with the corresponding Möbius inverse: $m(\{2\}) = 1/3$, $m(\{3\}) = 2/3$, $m(\{1,3\}) = 1/3$, and $m(\{2,3\}) = -1/3$, all other terms being null.

It is easy to check that the information given by the decision maker violates independence. Indeed, we have $x \succ y$ while x and y share a common evaluation on attribute 3. Changing this common evaluation from α_3 to β_3 should not affect preference in case independence holds. This fails here since we obtain $z \succ w$, whereas going from x to w and from y to z amounts to going from α_3 to β_3 on attribute 3. This failure of independence is reflected by the fact that the representing capacity exhibited above is not additive since we have $\mu(\{1,3\}) > \mu(\{2,3\})$ but $\mu(\{2\}) > \mu(\{1\})$, in line with the conclusion of Lemma 2.4.

4. Models using traces on levels

Our central tool for the analysis of RSRP will be the models introduced in Bouyssou and Pirlot (2004b) that induce complete traces on levels on each attribute (see also Greco et al. 2004). We recall here the essential elements of these models.

Definition 2. *Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. We define the binary relations $\succsim_i^+$, $\succsim_i^-$ and $\succsim_i^\pm$ on X_i letting, for all $x_i, y_i \in X_i$,*

$$\begin{aligned}
x_i \succsim_i^+ y_i &\Leftrightarrow \forall a, b \in X, [(y_i, a_{-i}) \succsim b \Rightarrow (x_i, a_{-i}) \succsim b],\\
x_i \succsim_i^- y_i &\Leftrightarrow \forall a, b \in X, [a \succsim (x_i, b_{-i}) \Rightarrow a \succsim (y_i, b_{-i})],\\
x_i \succsim_i^\pm y_i &\Leftrightarrow [x_i \succsim_i^+ y_i \text{ and } x_i \succsim_i^- y_i].
\end{aligned}$$

The relations $\succsim_i^+$, $\succsim_i^-$ and $\succsim_i^\pm$ are *traces* on the levels of attribute $i \in N$ generated by the relation $\succsim$. It is easy to check that these relations are always reflexive and transitive.

As shown below, the traces on levels combine nicely with the relation $\succsim$.

Lemma 3 (Bouyssou and Pirlot 2004b, Lemma 2)**.** *For all $i \in N$ and $x, y, z, w \in X$:*

$$[x \succsim y, z_i \succsim_i^+ x_i] \Rightarrow (z_i, x_{-i}) \succsim y, \tag{1}$$

$$[x \succsim y, y_i \succsim_i^- w_i] \Rightarrow x \succsim (w_i, y_{-i}), \tag{2}$$

$$[z_i \succsim_i^\pm x_i, y_i \succsim_i^\pm w_i] \Rightarrow \begin{cases} x \succsim y \Rightarrow (z_i, x_{-i}) \succsim (w_i, y_{-i}), \\ \text{and} \\ x \succ y \Rightarrow (z_i, x_{-i}) \succ (w_i, y_{-i}), \end{cases} \tag{3}$$

$$[x_i \sim_i^\pm z_i, y_i \sim_i^\pm w_i \text{ for all } i \in N] \Rightarrow \begin{cases} x \succsim y \Leftrightarrow z \succsim w, \\ \text{and} \\ x \succ y \Leftrightarrow z \succ w. \end{cases} \tag{4}$$

The following conditions[4] will imply the completeness of marginal traces on levels.

Definition 3. *We say that $\succsim$ satisfies:*

$$AC1_i \text{ if } \left.\begin{array}{c} (x_i, a_{-i}) \succsim c \\ \text{and} \\ (y_i, b_{-i}) \succsim d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \succsim c \\ \text{or} \\ (x_i, b_{-i}) \succsim d, \end{array}\right.$$

$$AC2_i \text{ if } \left.\begin{array}{c} c \succsim (y_i, a_{-i}) \\ \text{and} \\ d \succsim (x_i, b_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} c \succsim (x_i, a_{-i}) \\ \text{or} \\ d \succsim (y_i, b_{-i}), \end{array}\right.$$

$$AC3_i \text{ if } \left.\begin{array}{c} (x_i, a_{-i}) \succsim c \\ \text{and} \\ d \succsim (x_i, b_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \succsim c \\ \text{or} \\ d \succsim (y_i, b_{-i}), \end{array}\right.$$

for all $a, b, c, d \in X$ and all $x_i, y_i \in X_i$. We say that $\succsim$ satisfies $AC1$ (resp. $AC2$, $AC3$) if it satisfies $AC1_i$ (resp. $AC2_i$, $AC3_i$) for all $i \in N$.

[4] These conditions are named following Bouyssou and Pirlot (2004b). The rationale for these names is that these conditions are intr<u>A</u> attribute <u>C</u>ancellation conditions. This explains the names $AC1$, $AC2$, and $AC3$. The conditions $AC1^*$, $AC2^*$, $AC3^*$, $AC4^*$, $AC1^{**}$, $AC2^{**}$, $AC3^{**}$, and $AC4^{**}$ introduced later in the paper follow this naming scheme. It may have the virtue of avoiding any unwanted interpretation that would possibly come with more intuitive names.

Lemma 4 (Bouyssou and Pirlot 2004b, Lemma 3)**.** *We have:*

1. $AC1_i \Leftrightarrow \succsim_i^+$ *is complete* $\Leftrightarrow [Not[y_i \succsim_i^+ x_i] \Rightarrow x_i \succsim_i^+ y_i]$.
2. $AC2_i \Leftrightarrow \succsim_i^-$ *is complete* $\Leftrightarrow [Not[y_i \succsim_i^- x_i] \Rightarrow x_i \succsim_i^- y_i]$.
3. $AC3_i \Leftrightarrow [Not[y_i \succsim_i^+ x_i] \Rightarrow x_i \succsim_i^- y_i] \Leftrightarrow [Not[x_i \succsim_i^- y_i] \Rightarrow y_i \succsim_i^+ x_i]$.
4. $[AC1_i,\ AC2_i$ *and* $AC3_i] \Leftrightarrow \succsim_i^\pm$ *is complete.*
5. *In the class of all semiorders on X (and, therefore, in any class of relations containing this class), $AC1$, $AC2$ and $AC3$ are independent conditions.*

The following shows the consequences of having complete traces on each attribute in terms of numerical representation.

Proposition 1 (Bouyssou and Pirlot 2004b, Theorem 2)**.** *Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. Suppose that, for all $i \in N$, the set $X_i/\sim_i^\pm$ is at most countably infinite. Then there are real-valued functions u_i on X_i and a real-valued function F on $[\prod_{i=1}^n u_i(X_i)]^2$ such that, for all $x, y \in X$:*

$$x \succsim y \Leftrightarrow F(u_1(x_1), u_2(x_2), \ldots, u_n(x_n), u_1(y_1), u_2(y_2), \ldots, u_n(y_n)) \geq 0, \tag{M}$$

where F is increasing *in its first n arguments and* decreasing *in its last n arguments iff $\succsim$ satisfies $AC1$, $AC2$ and $AC3$.*

We refer to Bouyssou and Pirlot (2004b) for the analysis of model (M) in the general case, i.e., when the set $X_i/\sim_i^\pm$ can be uncountable (this requires conditions guaranteeing that the relations $\succsim_i^\pm$ have a numerical representation). This will not be useful here. Indeed, a characteristic feature of the models studied here is that they generate complete traces $\succsim_i^\pm$ that are "rough", only distinguishing a very limited number of distinct equivalence classes.

Let us conclude with a brief analysis of the very particular situation in which $\succsim$ is a weak order.

Definition 4. *Let $\succsim$ be a weak order on X. Attribute $i \in N$ is said to be weakly separable if $(x_i, a_{-i}) \succ (y_i, a_{-i})$ for some $x_i, y_i \in X_i$ and some $a_{-i} \in X_{-i}$ implies $(x_i, b_{-i}) \succsim (y_i, b_{-i})$, for all $b_{-i} \in X_{-i}$. The relation $\succsim$ is said to be weakly separable if all attributes $i \in N$ are weakly separable.*

The following lemma shows that for weak orders, the three conditions $AC1_i$, $AC2_i$, and $AC3_i$ are equivalent. Furthermore, they are equivalent to requiring that attribute $i \in N$ is weakly separable.

Lemma 5 (Bouyssou and Pirlot 2004b, Lemma 5)**.** *Let $\succsim$ be a weak order on X. Conditions $AC1_i$, $AC2_i$ and $AC3_i$ are equivalent. They hold iff attribute $i \in N$ is weakly separable.*

When $\succsim$ is a weakly separable weak order, i.e., a weak order satisfying $AC1$, $AC2$ and $AC3$, it is possible to further specify the numerical representation given by model (M). The following appears in Bouyssou and Pirlot (2004b, Proposition 8) and Greco et al. (2004, Theorem 1):

Proposition 2. *Let* $\succsim$ *be a weak order on a set* $X = \prod_{i=1}^{n} X_i$ *such that the set* $X/\sim$ *is at most countably infinite. Then there are real-valued functions* u_i *on* X_i *and a real-valued function* U *on* $\prod_{i=1}^{n} u_i(X_i)$ *such that, for all* $x, y \in X$*:*

$$x \succsim y \Leftrightarrow U(u_1(x_1), u_2(x_2), \ldots, u_n(x_n)) \geq U(u_1(y_1), u_2(y_2), \ldots, u_n(y_n)), \tag{U}$$

with U *is* nondecreasing *in each of its arguments iff* $\succsim$ *is weakly separable.*

5. Results

5.1. A characterization of RSRP

Our analysis of RSRP is based on the following two lemmas.

Lemma 6. *If* $\succsim$ *is a RSRP, then, for all* $i \in N$*, the relation* $\succsim_i^{\pm}$ *is a weak order having two distinct equivalence classes.*

Proof. See Section A3 in the supplementary material. □

Remark 1. Lemmas 4 and 6 imply that a RSRP satisfies $AC1$, $AC2$ and $AC3$. Furthermore all sets $X_i/\sim_i^{\pm}$ are finite, having two distinct elements. Hence, a RSRP also has a representation in (M). It is not difficult to figure out the particular form of model (M) that leads to a RSRP. Indeed, it suffices to consider, on each $i \in N$, a function u_i such that, for all $x_i \in X_i$,

$$u_i(x_i) = \begin{cases} 1 & \text{if } x_i \in \mathcal{A}_i, \\ 0 & \text{if } x_i \notin \mathcal{A}_i. \end{cases}$$

With the above definition, we clearly have $u_i(x_i) \geq u_i(y_i) \Leftrightarrow x_i \succsim_i^{\pm} y_i$.

We define the function F letting:

$$F([u_i(x_i)]; [u_i(y_i)]) = \begin{cases} + \exp(\sum_{i=1}^{n} (u_i(x_i) - u_i(y_i))) & \text{if } x \succsim y, \\ - \exp(\sum_{i=1}^{n} (u_i(y_i) - u_i(x_i))) & \text{otherwise.} \end{cases}$$

Using the definition of the functions u_i and the fact that $\trianglerighteq$ is monotonic w.r.t. inclusion, it is easy to show, using (4), that F is well defined and increasing (resp. decreasing) in its first (resp. last) n arguments.

Lemma 7. *If* $\succsim$ *is a relation on* X *such that, for all* $i \in N$*, the relation* $\succsim_i^{\pm}$ *is a weak order having two distinct equivalence classes, then* $\succsim$ *is a RSRP.*

Proof. See Section A4 in the supplementary material. □

In view of the above two lemmas, a characterization of RSRP will be at hand if we impose conditions guaranteeing that all relations $\succsim_i^{\pm}$ are weak orders having two distinct equivalence classes.

Definition 5 (Conditions $AC1^*$, $AC2^*$, $AC3^*$, $AC4^*$). *We say that $\succsim$ satisfies:*

$$AC1_i^* \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \succsim c\\ and\\ (y_i, b_{-i}) \succsim d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c\\ or\\ (z_i, b_{-i}) \succsim d,\end{array}\right.$$

$$AC2_i^* \text{ if } \left.\begin{array}{c}c \succsim (y_i, a_{-i})\\ and\\ d \succsim (z_i, b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}c \succsim (x_i, a_{-i})\\ or\\ d \succsim (y_i, b_{-i}),\end{array}\right.$$

$$AC3_i^* \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \succsim c\\ and\\ d \succsim (z_i, b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c\\ or\\ d \succsim (y_i, b_{-i}),\end{array}\right.$$

$$AC4_i^* \text{ if } \left.\begin{array}{c}c \succsim (y_i, a_{-i})\\ and\\ (y_i, b_{-i}) \succsim d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}c \succsim (x_i, a_{-i})\\ or\\ (z_i, b_{-i}) \succsim d,\end{array}\right.$$

for all $a, b, c, d \in X$ and all $x_i, y_i, z_i \in X_i$. We say that $\succsim$ satisfies $AC1^*$ (resp. $AC2^*$, $AC3^*$, $AC4^*$) if it satisfies $AC1_i^*$ (resp. $AC2_i^*$, $AC3_i^*$, $AC4_i^*$) for all $i \in N$.

The interpretation of the above condition will become clear considering their consequences on the relations $\succsim_i^+$, $\succsim_i^-$, and $\succsim_i^\pm$ (see Lemma 8 below). Take, for instance, condition $AC1_i^*$. Suppose that $(x_i, a_{-i}) \succsim c$ and that $(y_i, a_{-i}) \succsim c$ does not hold. This is an indication that it is not true that $y_i \succsim_i^+ x_i$. Condition $AC1_i^*$ then implies that y_i is below all other elements of X_i w.r.t. the relation $\succsim_i^+$, i.e., that $(y_i, b_{-i}) \succsim d$ implies $(z_i, b_{-i}) \succsim d$. Condition $AC2_i^*$ says a similar thing considering now the relation $\succsim_i^-$: if it is not true that $y_i \succsim_i^- x_i$ then y_i is below all other elements of X_i w.r.t. the relation $\succsim_i^-$. Conditions $AC3_i^*$ and $AC4_i^*$ connects what happens with the relations $\succsim_i^+$ and $\succsim_i^-$. Consider, for instance, condition $AC3_i^*$. Suppose that $(x_i, a_{-i}) \succsim c$, while $(y_i, a_{-i}) \succsim c$ does not hold. This is an indication that it is not true that $y_i \succsim_i^+ x_i$. Condition $AC3_i^*$ then implies that y_i is below all other elements of X_i w.r.t. the relation $\succsim_i^-$, i.e., that $d \succsim (z_i, b_{-i})$ implies $d \succsim (y_i, b_{-i})$. This intuition is formalized below.

Lemma 8. *For all $x_i, y_i, z_i \in X_i$,*

1. $AC1_i^* \Leftrightarrow [Not[y_i \succsim_i^+ x_i] \Rightarrow z_i \succsim_i^+ y_i]$,
2. $AC2_i^* \Leftrightarrow [Not[y_i \succsim_i^- x_i] \Rightarrow z_i \succsim_i^- y_i]$,
3. $AC3_i^* \Leftrightarrow [Not[y_i \succsim_i^+ x_i] \Rightarrow z_i \succsim_i^- y_i]$,
4. $AC4_i^* \Leftrightarrow [Not[y_i \succsim_i^- x_i] \Rightarrow z_i \succsim_i^+ y_i]$.

Proof. See Section A5 in the supplementary material. □

The following lemma shows that the above four conditions hold for a RSRP.

Lemma 9. *If $\succsim$ is a RSRP then it satisfies $AC1^*$, $AC2^*$, $AC3^*$, and $AC4^*$.*

Proof. See Section A6 in the supplementary material. □

Combining Lemmas 4 and 8 proves the following.

Lemma 10.

1. $AC1_i^* \Rightarrow AC1_i$,
2. $AC2_i^* \Rightarrow AC2_i$,
3. $AC3_i^* \Rightarrow AC3_i$,
4. $AC4_i^* \Rightarrow AC3_i$.

A crucial consequence of the combination of $AC1^*$, $AC2^*$, $AC3^*$ and $AC4^*$ is presented below.

Lemma 11. *The relation $\succsim$ satisfies $AC1_i^*$, $AC2_i^*$, $AC3_i^*$, and $AC4_i^*$ iff the binary relation $\succsim_i^{\pm}$ is a weak order having two distinct equivalence classes.*

Proof. See Section A7 in the supplementary material. □

Our first characterization of RSRP is as follows.

Theorem 1. *Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. The relation $\succsim$ is a Relation with a Single Reference Point iff it satisfies $AC1^*$, $AC2^*$, $AC3^*$, and $AC4^*$.*

Proof. Necessity results from Lemma 9. Sufficiency results from Lemmas 7 and 11. □

The conditions used in the above theorem are independent in a class of well behaved relations on X, as shown in the following lemma.

Lemma 12. *In the class of all semiorders on X (and, therefore, in any class of relations containing this class), conditions $AC1^*$, $AC2^*$, $AC3^*$, and $AC4^*$ are independent.*

Proof. See Section A8 in the supplementary material. □

5.2. An alternative characterization of RSRP

We have seen above a RSRP is characterized by the conjunction of conditions $AC1_i^*$, $AC2_i^*$, $AC3_i^*$ and $AC4_i^*$. A drawback of this result is that it does not explicitly use conditions $AC1_i$, $AC2_i$ and $AC3_i$ that characterize model (M). We show here how to weaken conditions $AC1_i^*$, $AC2_i^*$, $AC3_i^*$, $AC4_i^*$ so as to make them independent from conditions $AC1_i$, $AC2_i$ and $AC3_i$. This will allow us to exactly state what must be added to the conditions characterizing model (M) to obtain RSRP.

Definition 6 (Conditions $AC1^{**}$, $AC2^{**}$, $AC3^{**}$, $AC4^{**}$)**.** *We say that $\succsim$ satisfies:*

$$AC1_i^{**} \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \succsim c \\ \text{and} \\ (y_i, b_{-i}) \succsim d \\ \text{and} \\ (x_i, b_{-i}) \succsim d\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim c \\ \text{or} \\ (z_i, b_{-i}) \succsim d,\end{array}\right.$$

$$AC2_i^{**} \text{ if } \left.\begin{array}{c} c \succsim (y_i, a_{-i}) \\ \text{and} \\ d \succsim (z_i, b_{-i}) \\ \text{and} \\ c \succsim (z_i, a_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} c \succsim (x_i, a_{-i}) \\ \text{or} \\ d \succsim (y_i, b_{-i}), \end{array}\right.$$

$$AC3_i^{**} \text{ if } \left.\begin{array}{c} (x_i, a_{-i}) \succsim c \\ \text{and} \\ d \succsim (z_i, b_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \succsim c \\ \text{or} \\ d \succsim (y_i, b_{-i}) \\ \text{or} \\ d \succsim (x_i, b_{-i}), \end{array}\right.$$

$$AC4_i^{**} \text{ if } \left.\begin{array}{c} c \succsim (y_i, a_{-i}) \\ \text{and} \\ (y_i, b_{-i}) \succsim d \\ \text{and} \\ (x_i, b_{-i}) \succsim d \end{array}\right\} \Rightarrow \left\{\begin{array}{c} c \succsim (x_i, a_{-i}) \\ \text{or} \\ (z_i, b_{-i}) \succsim d, \end{array}\right.$$

for all $a, b, c, d \in X$ and all $x_i, y_i, z_i \in X_i$. We say that $\succsim$ satisfies $AC1^{**}$ (resp. $AC2^{**}$, $AC3^{**}$, $AC4^{**}$) if it satisfies $AC1_i^{**}$ (resp. $AC2_i^{**}$, $AC3_i^{**}$, $AC4_i^{**}$) for all $i \in N$.

The interpretation of these four new conditions is similar to that of conditions $AC1_i^*$, $AC2_i^*$, $AC3_i^*$ and $AC4_i^*$. Indeed, $AC1_i^{**}$, $AC2_i^{**}$, and $AC4_i^{**}$ (resp. $AC3_i^{**}$) are obtained from $AC1_i^*$, $AC2_i^*$, and $AC4_i^*$ by the addition of a new premise (resp. from $AC3_i^*$ by the addition of a new possible conclusion). Take, for instance, condition $AC1_i^{**}$. Suppose that $(x_i, a_{-i}) \succsim c$ and that $(y_i, a_{-i}) \succsim c$ does not hold. This is an indication that it is not true that $y_i \succsim_i^+ x_i$. Condition $AC1_i^{**}$ then implies that, whenever $(x_i, b_{-i}) \succsim d$ holds, y_i is below all other elements of X_i w.r.t. the relation $\succsim_i^+$ i.e., that $(y_i, b_{-i}) \succsim d$ implies $(z_i, b_{-i}) \succsim d$.

The relation between these four new conditions and the ones used earlier is detailed below.

Lemma 13. *Let $\succsim$ be a binary relation on X. We have:*

1. *$\succsim$ satisfies $AC1_i^*$ iff it satisfies $AC1_i$ and $AC1_i^{**}$,*
2. *$\succsim$ satisfies $AC2_i^*$ iff it satisfies $AC2_i$ and $AC2_i^{**}$,*
3. *$\succsim$ satisfies $AC3_i^*$ iff it satisfies $AC3_i$ and $AC3_i^{**}$,*
4. *$\succsim$ satisfies $AC4_i^*$ iff it satisfies $AC3_i$ and $AC4_i^{**}$.*

Proof. See Section A9 in the supplementary material. □

Combining the above lemma with Theorem 1 proves the following:

Theorem 2. *Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. The relation $\succsim$ is a Relation with a Single Reference Point iff it satisfies $AC1$, $AC2$, $AC3$, $AC1^{**}$, $AC2^{**}$, $AC3^{**}$, and $AC4^{**}$.*

The conditions used in the above theorem are independent in a class of well behaved relations on X, as shown in the following lemma.

Lemma 14. *In the class of all semiorders on X (and, therefore, in any class of relations containing this class), conditions $AC1_i$, $AC2_i$, $AC3_i$, $AC1^{**}$, $AC2^{**}$, $AC3^{**}$, and $AC4^{**}$ are independent.*

Proof. See Section A10 in the supplementary material. □

6. Relation to other preference models

In the previous section, we have introduced conditions that characterize preference models using a single reference point. One possible virtue of this analysis is that it allows to relate, in a simple way, these preference models with other preference models that have been introduced in the literature.

6.1. Relation to concordance relations

The idea of concordance is vital in the ELECTRE methods (see, e.g., Figueira et al. 2005, 2010, Roy 1991, Roy and Bouyssou 1993). We make use here of literature on the axiomatic analysis of concordance relations (Bouyssou and Pirlot 2002b, 2005b, 2007, Bouyssou et al. 1997, Greco et al. 2001) to relate this type of comparison to RSRP.

In view of Lemma 2.4, it is easy to build examples showing that a RSRP does not have to be independent. It is well known that concordance relations are independent (Bouyssou and Pirlot 2005b, 2007). Using this observation, Rolland (2003, 2006a, 2008) has concluded that models using reference points were more "flexible" than concordance relations. This section is devoted to a precise analysis of the links between concordance relations and RSRP. These links are more complex than what was suggested by Rolland.

We first recall the definition of a concordance relation with attribute transitivity (Bouyssou and Pirlot 2005b, 2007) when $\succsim$ is reflexive (since a concordance relation is independent, it is easy to check that it is either reflexive or irreflexive).

Definition 7. *Let $\succsim$ be a* reflexive *binary relation on X. We say that $\succsim$ is a* concordance relation with attribute transitivity *(or, more briefly, that $\succsim$ is a CR-AT) if there are:*

- *a* semiorder *T_i on each X_i ($i = 1, 2, \dots, n$),*
- *a binary relation $\trianglerighteq_C$ between subsets of N having N for union that is* monotonic *w.r.t. inclusion, i.e., for all $A, B, C, D \subseteq N$ such that $A \cup B = N$ and $C \cup D = N$,*

$$[A \trianglerighteq_C B, C \supseteq A, B \supseteq D] \Rightarrow C \trianglerighteq_C D, \tag{5}$$

such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow T(x, y) \trianglerighteq_C T(y, x), \tag{6}$$

where $T(x, y) = \{i \in N : x_i \, T_i \, y_i\}$.

Hence, when $\succsim$ is a CR-AT, the preference between x and y only depends on the subsets of attributes favoring x or y in terms of the semiorder T_i. It does not depend on preference differences between the various levels on each attribute besides the distinction between levels indicated by T_i.

Definition 8. *Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. We define the binary relations $\succsim_i^*$ and $\succsim_i^{**}$ on X_i^2 letting, for all $x_i, y_i, z_i, w_i \in X_i$,*

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \Leftrightarrow \\ [\textit{for all } a_{-i}, b_{-i} \in X_{-i}, (z_i, a_{-i}) \succsim (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succsim (y_i, b_{-i})],$$

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow [(x_i, y_i) \succsim_i^* (z_i, w_i) \textit{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)].$$

It is clear that the relations $\succsim_i^*$ and $\succsim_i^{**}$ are always reflexive and transitive.

Bouyssou and Pirlot (2005b, 2007) have shown that CR-AT are reflexive relations that are characterized by the fact that, for all $i \in N$, the relation $\succsim_i^{**}$ is a weak order having at most three distinct equivalence classes and $\succsim_i^{\pm}$ is a weak order. They have given necessary and sufficient conditions on $\succsim$ ensuring that this happens.

We are now in position to analyze the relations between CR-AT and RSRP. We start with a result showing that there are CR-AT that are not RSRP. This result exploits the fact that in a CR-AT the weak orders $\succsim_i^{\pm}$ may have more than two distinct equivalence classes. This cannot be the case in a RSRP.

Lemma 15. *A CR-AT may fail to be a RSRP.*

Proof. See Section A11 in the supplementary material. □

Our next result shows that a reflexive RSRP that is independent is a CR-AT.

Lemma 16. *If a reflexive RSRP is independent then it is a CR-AT.*

Proof. See Section A12 in the supplementary material. □

Observe finally that an independent relation being a RSRP is a very particular CR-AT since, in this case, all relations $\succsim_i^{\pm}$ have only two distinct equivalence classes.

Summarizing, there are CR-AT that are not RSRP (the ones in which the relations $\succsim_i^{\pm}$ have more than two equivalence classes) and there are reflexive RSRP that are not CR-AT (the ones that are not independent).

6.2. Relation to noncompensatory sorting models

In Bouyssou and Marchant (2007a), we study a sorting model called the "noncompensatory sorting model". This model was conceived so as to have close links with ELECTRE TRI. Hence, it is not surprising that it also has links with RSRP. We briefly study them below.

The following definition is taken from Bouyssou and Marchant (2007a).

Definition 9. *We say that a partition* $\langle \mathscr{A}, \mathscr{U} \rangle$ *of* X *has a representation in the* noncompensatory sorting model *if:*

- *for all* $i \in N$ *there is a set* $\mathscr{A}_i \subseteq X_i$*,*
- *there is a subset* $\mathscr{F}$ *of* 2^N *such that, for all* $I, J \in 2^N$*,*

$$[I \in \mathscr{F} \text{ and } I \subseteq J] \Rightarrow J \in \mathscr{F}, \tag{7}$$

such that, for all $x \in X$*,*

$$x \in \mathscr{A} \Leftrightarrow \{i \in N : x_i \in \mathscr{A}_i\} \in \mathscr{F}. \tag{8}$$

In this case, we say, that $\langle \mathscr{F}, \mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n \rangle$ *or, for short,* $\langle \mathscr{F}, \langle \mathscr{A}_i \rangle_{i \in N} \rangle$ *is a representation of* $\langle \mathscr{A}, \mathscr{U} \rangle$ *in the noncompensatory sorting model. When there is no risk of confusion on the underlying sets* $\mathcal{A}_i$*, we write* $A(x)$ *instead of* $\{i \in N : x_i \in \mathscr{A}_i\}$*. In this section, we write* $\mathscr{U}_i = X_i \setminus \mathscr{A}_i$*.*

Suppose that we have a twofold partition $\langle \mathscr{A}, \mathscr{U} \rangle$ of X that has a representation $\langle \mathscr{F}, \langle \mathscr{A}_i \rangle_{i \in N} \rangle$ in the noncompensatory sorting model. Let us show that it is possible to build a RSRP that models this twofold partition in the following sense. The relation $\succsim$ is a weak order having two equivalence classes; the first equivalence class contains all elements in $\mathscr{A}$, while the second contains all elements in $\mathscr{U}$. Notice that in this interpretation all elements in $\mathscr{A}$ (resp. $\mathscr{U}$) are judged indifferent, which is *not* required in the noncompensatory sorting model.

Define, for all $i \in N$, $\mathcal{A}_i = \mathscr{A}_i$. The relation $\trianglerighteq$ is defined as follows. For all $A, B \in 2^N$ such that $A, B \in \mathscr{F}$ or $A, B \notin \mathscr{F}$, we have $A \triangleq B$. For all $A, B \in 2^N$ such that $A \in \mathscr{F}$ and $B \notin \mathscr{F}$, we have $A \triangleright B$. It is simple to check that this defines a weak order $\trianglerighteq$ on 2^N. Since $\mathscr{F}$ satisfies (7), it is clear that this weak order is monotonic w.r.t. inclusion. By construction, we have $x \sim y \Leftrightarrow \mathcal{S}(x) \triangleq \mathcal{S}(y)$ and $x \succ y \Leftrightarrow \mathcal{S}(x) \triangleright \mathcal{S}(y)$.

Conversely, consider a weak order $\succsim$ that is a RSRP. Take any $a \in X$ and define $\mathscr{A} = \{x \in X : x \succsim a\}$ and $\mathscr{U} = \{x \in X : a \succ x\}$. Whenever a does not belong to the last equivalence class of $\succsim$, this defines a partition $\langle \mathscr{A}, \mathscr{U} \rangle$ of X.

Let us show that this induced twofold partition has a representation in the noncompensatory sorting model. Define, for all $i \in N$, $\mathscr{A}_i = \mathcal{A}_i$. Let $\mathcal{S}(a) = A$ and define $\mathscr{F} = \{B \in 2^N : B \trianglerighteq A\}$. Because $\trianglerighteq$ is monotonic w.r.t. inclusion, it is simple to check that $\mathscr{F}$ satisfies (7). Suppose that $x \in \mathscr{A}$. By construction, this means that $x \succsim a$, so that $\mathcal{S}(x) \trianglerighteq \mathcal{S}(a)$ which implies $\mathcal{S}(x) = \{i \in N : x_i \in \mathscr{A}_i\} \in \mathscr{F}$. Conversely if $\{i \in N : x_i \in \mathscr{A}_i\} \in \mathscr{F}$, we know that $\mathcal{S}(x) \trianglerighteq \mathcal{S}(a)$ so that $a \in \mathscr{A}$.

A weak order $\succsim$ on X that is a RSRP can be interpreted as a nested family of twofold partitions that all have a representation in the noncompensatory sorting model. Indeed, we know that the weak order $\succsim$ can only have a finite number of equivalence classes: $C^1, C^2, \ldots, C^k$. We have shown above that there is a twofold partition $\langle \mathscr{A}, \mathscr{U} \rangle$ such that $\mathscr{A} = C^1$ and $\mathscr{U} = \bigcup_{j=2}^{k} C^k$ that has a representation in the noncompensatory sorting model. Similarly, there is a twofold partition $\langle \mathscr{A}', \mathscr{U}' \rangle$ such that $\mathscr{A}' = C^1 \cup C^2$ and

$\mathscr{U} = \bigcup_{j=3}^{k} C^k$ that also has a representation in the noncompensatory sorting model. Using the above construction, for all $i \in N$, we have $\mathscr{A}_i = \mathscr{A}'_i = \mathcal{A}_i$. The only change happens with the sets $\mathscr{F}$ and $\mathscr{F}'$ that are such that $\mathscr{F} \subset \mathscr{F}'$. Hence a weak order that is a RSRP can be obtained using an noncompensatory sorting model by keeping the sets $\mathscr{A}_i$ fixed and progressively enlarging the set $\mathscr{F}$, i.e., be less and less strict on the size of the coalition of attributes that is sufficiently important to conclude that an alternative is "acceptable".

Ordered partitions admitting such a representation have been studied in Bouyssou and Marchant (2007b, Proposition 31). They are a particular case of the general noncompensatory sorting model for r-fold partitions studied in Bouyssou and Marchant (2007b, Theorem 22).

7. The case of weak orders

7.1. Results

Although all the conditions needed to characterize RSRP are independent in the class of all semiorders on X, the situation is vastly different when we turn to weak orders. Indeed, we know from Lemma 5 that conditions $AC1_i$, $AC2_i$, and $AC3_i$ are equivalent for weak orders. The same turns out to be true for conditions $AC1^*_i$, $AC2^*_i$, $AC3^*_i$ and $AC4^*_i$.

Lemma 17. *Let $\succsim$ be a weak order on a set X. Then conditions $AC1^*_i$, $AC2^*_i$, $AC3^*_i$ and $AC4^*_i$ are equivalent.*

Proof. See Section A13 in the supplementary material. □

We have shown above that $AC1^*_i$ (resp. $AC2^*_i$, $AC3^*_i$, $AC4^*_i$) was equivalent to the conjunction of $AC1_i$ and $AC1^{**}_i$ (resp. $AC2_i$ and $AC2^{**}_i$, $AC3_i$ and $AC3^{**}_i$, $AC3_i$ and $AC4^{**}_i$). The independence of conditions $AC1_i$ and $AC1^{**}_i$ is discussed in the following lemma.

Lemma 18. *In the class of all weak orders on X, conditions $AC1_i$ and $AC1^{**}_i$ are independent.*

Proof. See Section A14 in the supplementary material. □

Summarizing the above observations, we have:

Proposition 3. *Let $\succsim$ be a weak order on X. It is a Relation with a Single Reference Point iff it satisfies $AC1^*$ iff it satisfies $AC1$ and $AC1^{**}$.*

On the basis of Lemmas 5, 13 and 17 and building obvious variations of Lemma 18, it is possible to formulate many alternative equivalent results for the case of weak orders. We leave the details to the interested reader.

7.2. Relation to models based on a discrete Sugeno integral

We have shown in Section 3 that a RSRP that is a weak order has a numerical representation such that:

$$x \succsim y \Leftrightarrow \mu(\mathcal{S}(x)) \geq \mu(\mathcal{S}(y))$$

where μ is a normalized capacity on N.

This representation is reminiscent of representation of weak orders based on a discrete Sugeno integral. This model was proposed in Sugeno (1974). It was axiomatically studied in the context of MCDM by Greco et al. (2004) and Bouyssou et al. (2009). We show below that a RSRP that is a weak order always has a representation using a discrete Sugeno integral.

The following definitions are taken from Bouyssou et al. (2009).

Definition 10. *Let $\beta = (\beta_1, \beta_2, \dots, \beta_p) \in [0,1]^p$. The discrete Sugeno integral of the vector $(\beta_1, \beta_2, \dots, \beta_p) \in [0,1]^p$ w.r.t. the normalized capacity ν on $P = \{1, 2, \dots, p\}$ is defined by:*

$$\mathrm{Sug}_\nu[\beta] = \bigvee_{T \subseteq P} \left[\nu(T) \wedge \left(\bigwedge_{i \in T} \beta_i \right) \right].$$

Definition 11. *A weak order $\succsim$ on X has a representation in the discrete Sugeno integral model if there are a normalized capacity μ on N and functions $u_i : X_i \to [0,1]$ such that, for all $x, y \in X$,*

$$x \succsim y \Leftrightarrow \mathrm{Sug}_\mu[(u_1(x_1), u_2(x_2), \dots, u_n(x_n))] \geq \mathrm{Sug}_\mu[(u_1(y_1), u_2(y_2), \dots, u_n(y_n))].$$

Definition 12. *The relation $\succsim$ on X is said to be strongly 2-graded on attribute $i \in N$ (condition 2^*-graded$_i$) if, for all $a, b, c, d \in X$ and all $x_i, y_i, z_i \in X_i$,*

$$\left.\begin{array}{c} (x_i, a_{-i}) \succsim c \\ \textit{and} \\ (y_i, b_{-i}) \succsim d \\ \textit{and} \\ d \succsim c \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \succsim c \\ \textit{or} \\ (z_i, b_{-i}) \succsim d, \end{array}\right.$$

The relation $\succsim$ on X is said to be 2-graded on attribute $i \in N$ (condition 2-graded$_i$) if, for all $a, b, c, d \in X$ and all $x_i, y_i, z_i \in X_i$,

$$\left.\begin{array}{c} (x_i, a_{-i}) \succsim c \\ \textit{and} \\ (y_i, b_{-i}) \succsim d \\ \textit{and} \\ (x_i, b_{-i}) \succsim d \\ \textit{and} \\ d \succsim c \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \succsim c \\ \textit{or} \\ (z_i, b_{-i}) \succsim d, \end{array}\right.$$

A binary relation is said to be strongly 2-graded *(condition 2^*-graded) if it is strongly 2-graded on all attributes $i \in N$. Similarly, a binary relation is said to be* 2-graded *(condition 2-graded) if it is 2-graded on all attributes $i \in N$.*

Bouyssou et al. (2009, Lemma 1) have shown that condition 2^*-graded$_i$ holds iff conditions 2-graded$_i$ and $AC1_i$ hold.

It is clear that condition 2^*-graded$_i$ is a weakening of $AC1_i^*$ since it amounts to adding a premise to this condition. Similarly, condition 2-graded$_i$ is a weakening of $AC1_i^{**}$. Greco et al. (2004) and Bouyssou et al. (2009) have shown the following (this result can be extended to cover sets of arbitrary cardinality adding a condition imposing that the weak order has a numerical representation. This will not be useful here.).

Proposition 4. *Let $\succsim$ be a weak order on X such that $X/\sim$ is at most countably infinite. Then $\succsim$ has a representation in the discrete Sugeno integral model iff it satisfies condition 2^*-graded iff it satisfies conditions $AC1$ and 2-graded.*

The following lemma is a direct consequence of the fact that condition 2^*-graded$_i$ is a weakening of $AC1_i^*$.

Lemma 19. *Let $\succsim$ be a weak order. If it is a RSRP, then it has a representation in the discrete Sugeno integral model.*

Proof. See Section A15 in the supplementary material. □

The above lemma has shown that, for weak orders, a RSRP always has a representation in the discrete Sugeno integral model.

Suppose that the weak order $\succsim$ can be represented as a RSRP using sets $\mathcal{A}_i$ and an importance relation $\trianglerighteq$ (such a representation is unique, as shown by Lemma 1). We know from Lemma 2 that $\trianglerighteq$ is a weak order on N, so that it can be represented by a normalized capacity μ on N. It is then easy to devise a representation of this weak order in the discrete Sugeno integral model. (such a representation is clearly not unique).

Define, for all $i \in N$ and all $x_i \in X_i$,

$$u_i(x_i) = \begin{cases} 1 & \text{if } x_i \in \mathcal{A}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Using such functions u_i, it is easy to see that, for all $x \in X$, and all $J \subseteq N$, we have:

$$\mu(J) \wedge \left(\bigwedge_{i \in J} u_i(x_i) \right) = \begin{cases} \mu(J) & \text{if } x_i \in \mathcal{A}_i \text{ for all } i \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, when $\mathcal{S}(x) = J$ we have $\mathrm{Sug}_\mu[(u_1(x_1), u_2(x_2), \ldots, u_n(x_n))] = \mu(J)$, so that

$$\begin{aligned} x \succsim y &\Leftrightarrow \mathcal{S}(x) \trianglerighteq \mathcal{S}(y) \\ &\Leftrightarrow \mu(\mathcal{S}(x)) \geq \mu(\mathcal{S}(y)) \\ &\Leftrightarrow \mathrm{Sug}_\mu[(u_1(x_1), u_2(x_2), \ldots, u_n(x_n))] \geq \mathrm{Sug}_\mu[(u_1(y_1), u_2(y_2), \ldots, u_n(y_n))]. \end{aligned}$$

The representation of weak order that is a RSRP in the Sugeno integral model is clearly not unique.

We will see in Section 8 that the situation is different when considering relations using *several* reference points: such relations may not have a representation in the discrete Sugeno integral model.

7.3. Elicitation of a 2-additive capacity

Suppose that a decision maker has given some preference statements on the alternatives in X. Suppose that these preference statements are compatible with a weak order on X, i.e., are such that there is a weak order on X that contains them all, as was the case in the example given in Section 3.4.

Let us describe how one can use mathematical programming techniques to test whether the information provided by the decision maker is compatible with a RSRP, i.e., whether it has a representation in terms of sets $\mathcal{A}_i$, $i = 1, 2, \ldots, n$, and an importance relation $\trianglerighteq$. We will only investigate the case in which the importance relation $\trianglerighteq$ can be represented by a normalized 2-additive capacity (see Section 2.2) We only sketch our procedure since it rests on fairly standard techniques.

Let us define variables a_i, $i = 1, 2, \ldots, n$. These variables will represent the Möbius inverse of the capacity on singletons and, hence, be such that $0 \leq a_i \leq 1$, $i = 1, 2, \ldots, n$. Define analogously, variables $b_{ij}, b^+_{ij}, b^-_{ij}$, $i, j = 1, 2, \ldots, n$ with $i \neq j$, the variables b^+_{ij}, b^-_{ij} being non-negative. The b_{ij} variable will represent the Möbius inverse of the capacity for the pair $\{i, j\}$ and will be expressed as $b_{ij} = b^+_{ij} - b^-_{ij}$, exploiting the fact that any real variable can be written as the difference between two non-negative variables. It is well-known (Grabisch 1997) that we always have $b_{ij} \in [-1, 1]$. Hence, we may always impose that $0 \leq b^+_{ij} \leq 1$ and $0 \leq b^-_{ij} \leq 1$, the variable b_{ij} being unrestricted.

The conditions on the variables a_i and b_{ij} ensuring that they can be interpreted as the Möbius inverse of a 2-additive capacity are well known (Marichal and Roubens 2000, p. 645) and are linear:

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} b_{ij} = 1,$$

$$a_i + \sum_{j \in T} b_{ij} \geq 0, \forall i \in N, \forall T \subseteq N \setminus \{i\},$$

together with the fact that the Möbius inverse associated to the empty set is 0. This last requirement will always be ensured in our model.

To each element $x_i \in X_i$, we associate a binary variable that will model the fact that the element belongs or not to $\mathcal{A}_i$. We grossly abuse notation below and we write the binary variable as x_i. Notice here that, in fact, it is necessary to introduce a variable x_i corresponding to $x_i \in X_i$ only when the evaluation $x_i \in X_i$ is used in one of the alternatives that have been compared by the decision maker.

To each pair of elements $x_i \in X_i$ and $x_j \in X_j$, we associate a binary variable y_{ij} that will model the fact that the element $x_i \in X_i$ belongs to $\mathcal{A}_i$ and the element $x_j \in X_j$ belongs to $\mathcal{A}_j$. Of course, we must impose conditions ensuring that y_{ij} takes the value 1 iff both x_i and x_j take the value 1. This is easily done using linear constraints, e.g., requiring that

$$y_{ij} \geq x_i + x_j - 1, \quad y_{ij} \leq 0.5 x_i + 0.5 x_j.$$

Consider now an alternative $(x_1, x_2, \ldots, x_n) \in X$ that has been compared by the decision maker to other ones. The value of the 2-additive capacity corresponding to $\mathcal{S}((x_1, x_2, \ldots, x_n))$ can be written as:

$$\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} b_{ij} y_{ij}, \tag{9}$$

an expression that should be linearized. This is easily done using standard tricks.

We first introduce for each element $x_i \in X_i$ a variable $\hat{x}_i$ that will be equal to a_i if $x_i = 1$ and to 0 otherwise. This is a standard trick that can be done imposing, e.g., the following linear constraints:

$$\hat{x}_i \leq x_i, \quad \hat{x}_i \geq 0, \quad \hat{x}_i \leq a_i, \quad \hat{x}_i \geq x_i - 1 + a_i.$$

Hence, we may replace in Expression (9) all terms $a_i x_i$ by $\hat{x}_i$. Notice that the above trick only works if it is known that $a_i \geq 0$. This is not the case for the variables b_{ij}, so that we have recourse here to the two non-negative variables b_{ij}^+ and b_{ij}^-.

We then proceed as above. We introduce a variable $\hat{y}_{ij}^+$ that will be equal to b_{ij}^+ iff $y_{ij} = 1$ and to 0 otherwise. Similarly, we introduce a variable $\hat{y}_{ij}^-$ that will be equal to b_{ij}^- iff $y_{ij} = 1$ and to 0 otherwise. This can be done linearly, e.g., imposing that:

$$\hat{y}_{ij}^+ \leq y_{ij}, \quad \hat{y}_{ij}^+ \geq 0, \quad \hat{y}_{ij}^+ \leq b_{ij}^+, \quad \hat{y}_{ij}^+ \geq y_{ij} - 1 + b_{ij}^+,$$

and, similarly,

$$\hat{y}_{ij}^- \leq y_{ij}, \quad \hat{y}_{ij}^- \geq 0, \quad \hat{y}_{ij}^- \leq b_{ij}^-, \quad \hat{y}_{ij}^- \geq y_{ij} - 1 + b_{ij}^-.$$

With these new variables at hand, we may replace in Equation (9) all terms $b_{ij} y_{ij}$ by $\hat{y}_{ij}^+ - \hat{y}_{ij}^-$.

Expression (9) can now be expressed as the linear expression:

$$\sum_{i=1}^{n} \hat{x}_i + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} (\hat{y}_{ij}^+ - \hat{y}_{ij}^-), \tag{10}$$

which gives all we need.

The rest of the procedure is quite standard. For each indifference statement issued by the decision maker, we associate an equality constraint between two terms of the type given by Expression (10). For each strict preference statement this equality becomes a strict inequality and we replace it by a non-strict one with the help of an auxiliary non-negative variable ε.

If the resulting set of linear constraints is compatible, we have found a 2-additive capacity compatible with this information, as given by the Möbius terms a_i and b_{ij}.

Using a_i and b_{ij}, it is indeed easy to recover the 2-additive capacity (see, e.g., Marichal and Roubens 2000, p. 645) letting:

$$\begin{aligned}
\mu(\{i\}) &= a_i, \forall i \in N, \\
\mu(\{i,j\}) &= a_i + a_j + b_{ij}, \forall i,j \in N \text{ with } i \neq j, \\
\mu(S) &= \sum_{i \in S} a_i + \sum_{\substack{i \in S, j \in S \\ i \neq j}} b_{ij}, \forall S \subseteq N \text{ with } |S| \geq 2.
\end{aligned}$$

The sets $\mathcal{A}_i$ consist in the evaluations for which we have $x_i = 1$.

This test of compatibility of the linear constraints can be done using an MILP solver. The objective function can, e.g., consist in maximizing the auxiliary variable ε. If the LP has a feasible solution, the compatibility of the constraints is ensured[5].

The capacity exhibited in the analysis of the example analyzed in section 3.4 has been obtained using the above technique. In this example, the following preferences:

$$[x \sim y] \succ z \succ w,$$

lead to the following solution: $\mathcal{A}_1 = \{a_1\}$, $\mathcal{A}_2 = \{b_2\}$, and $\mathcal{A}_3 = \{a_3\}$, together with the 2-additive capacity having the Möbius inverse such that: $m(\{2\}) = 1/2$, $m(\{3\}) = 1$, and $m(\{2,3\}) = -1/2$, all other terms being null.

Unsurprisingly, in the same example, the preference information:

$$[x \sim y] \succ [z \sim w],$$

is compatible with a representation that uses an additive capacity, e.g., with $\mathcal{A}_1 = \{a_1\}$, $\mathcal{A}_2 = \{b_2\}$, and $\mathcal{A}_3 = \{a_3\}$, with the 2-additive capacity with a Möbius inverse such that: $m(\{3\}) = 1$, all other terms being null.

Clearly the above procedure can easily be adapted to cope with other type of information given by the decision maker, e.g., in the form of an a priori weak order on each attribute or giving constraints on the importance relation.

It should be clear that the solution found by the above elicitation procedure will be unique only in exceptional cases. Hence, using this procedure for decision aiding purposes should take this non-uniqueness into account, for instance in the spirit of the robust ordinal regression advocated in Greco et al. (2008). We leave the details to the interested reader since this is not our main point here.

Let us finally notice that Mousseau et al. (2012) and Zheng et al. (2012) have also investigated elicitation techniques for models with reference points. On the one hand, they study a case that is more general than the one we cover here since they deal with

[5]Notice that an optimal solution may be found in which we have at the same time $b^+_{ij} > 0$ and $b^-_{ij} > 0$. This can be easily avoided, since the only role of these two variables is to define the variable $b_{ij} = b^+_{ij} - b^-_{ij}$. Hence, if this happens, we may fix the value of the objective function to its optimal value and have recourse to a secondary optimization that will minimize the sum of all b^+_{ij} and b^-_{ij}. Doing so, we will obtain an optimal solution in which at least one of b^+_{ij} and b^-_{ij} is zero.

multiple reference points (each reference point leads to a preference structure; these structures are then aggregated in a lexicographic way). On the other hand, they confine themselves, contrary to what we do here, to the case in which the importance relation w.r.t. each reference point that is additive. We have worked independently of them on this topic.

8. Multiple reference points

8.1. Definition

The model defined below, building on Rolland (2003, 2006a,b, 2008, 2013), extends Definition 1 to deal with the case of multiple reference points. In this model, there is a semiorder R_i on each attribute. In order to compare the alternatives x and y, we first compare each of them to a number of "reference points" $\pi^1, \pi^2, \ldots, \pi^\ell$ only using "ordinal considerations". For each profile π^k, we compute the subsets of attributes $\mathcal{R}^k(x) = \{i \in N : x_i \; R_i \; \pi^k_i\}$ and $\mathcal{R}^k(y) = \{i \in N : y_i \; R_i \; \pi^k_i\}$. The comparison of x and y is based on the two ℓ-tuples $\mathcal{R}^L(x) = (\mathcal{R}^1(x), \mathcal{R}^2(x), \ldots, \mathcal{R}^\ell(x))$ and $\mathcal{R}^L(y) = (\mathcal{R}^1(y), \mathcal{R}^2(y), \ldots, \mathcal{R}^\ell(y))$. This comparison uses an "importance relation" that will only be required to be monotonic w.r.t. inclusion.

Definition 13. *A binary relation $\succsim$ is a Relation with Multiple Reference Points (or more briefly is a RMRP) if:*

- *for all $i \in N$, there is a semiorder R_i on X_i,*
- *there are $\ell \in \mathbb{N}^+$ elements of X, $\pi^1, \pi^2, \ldots, \pi^\ell$, interpreted as ℓ "reference points",*
- *there is a binary relation $\trianglerighteq_L$ on $(2^N)^\ell$ that is* monotonic *w.r.t. inclusion, i.e., for all $A^1, B^1, C^1, D^1, \ldots, A^\ell, B^\ell, C^\ell, D^\ell \subseteq N$,*

$$(A^1, \ldots, A^\ell) \trianglerighteq_L (B^1, \ldots, B^\ell) \Rightarrow (C^1, \ldots, C^\ell) \quad \trianglerighteq_L (D^1, \ldots, D^\ell),$$

whenever, for all $k \in L = \{1, 2, \ldots, \ell\}$, $C^k \supseteq A^k$, $B^k \supseteq D^k$, and there are $x, y, z, w \in X$ such that $\mathcal{R}^L(x) = (A^1, \ldots, A^\ell)$, $\mathcal{R}^L(y) = (B^1, \ldots, B^\ell)$, $\mathcal{R}^L(z) = (C^1, \ldots, C^\ell)$, and $\mathcal{R}^L(w) = (D^1, \ldots, D^\ell)$,

such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow (\mathcal{R}^1(x), \mathcal{R}^2(x), \ldots, \mathcal{R}^\ell(x)) \trianglerighteq_L (\mathcal{R}^1(y), \mathcal{R}^2(y), \ldots, \mathcal{R}^\ell(y)), \tag{RMRP}$$

where $\mathcal{R}^k(x) = \{i \in N : x_i \; R_i \; \pi^k_i\}$ and $\mathcal{R}^L(x) = (\mathcal{R}^1(x), \mathcal{R}^2(x), \ldots, \mathcal{R}^\ell(x))$.

We write $\mathcal{R}^L(x) \sqsupseteq \mathcal{R}^L(y)$ to mean that, for all $k \in L$, $\mathcal{R}^k(x) \supseteq \mathcal{R}^k(y)$. Observe that, contrary to what was the case with Definition 1 the fact that each attribute is influential does not imply that, for all for all $A^1, A^2, \ldots, A^\ell \subseteq N$, we have $\mathcal{R}^L(x) = (A^1, \ldots, A^\ell)$, for some $x \in X$.

Rolland (2008, 2013) defines and studies many particular cases of this general model. We do not consider them here.

8.2. Results

We start by showing that a RMRP is a particular case of model (M).

Lemma 20. *If a binary relation on X is a RMRP with ℓ reference points, it satisfies $AC1$, $AC2$ and $AC3$, so that all relations $\succsim_i^{\pm}$ are weak orders. Furthermore, for all $i \in N$, the weak order $\succsim_i^{\pm}$ has at most $\ell + 1$ distinct equivalence classes.*

Proof. See Section A16 in the supplementary material. $\square$

The above lemma shows that a RMRP satisfies $AC1$, $AC2$ and $AC3$ and is such that, for all $i \in N$, $X_i/\sim_i^{\pm}$ is finite. Hence, it has a representation in model (M). We show below that the converse is true, as soon as all sets $X_i/\sim_i^{\pm}$ are finite.

Lemma 21. *Let $\succsim$ be a binary relation on X. If $\succsim$ satisfies $AC1$, $AC2$, and $AC3$ and, for all $i \in N$, the sets $X_i/\sim_i^{\pm}$ are finite then $\succsim$ is a RMRP.*

Proof. See Section A17 in the supplementary material. $\square$

Lemmas 20 and 21 show that, whenever $X_i/\sim_i^{\pm}$ is finite, for all $i \in N$, the Model with Multiple Reference Points is equivalent to model (M). Moreover:

- if $\succsim$ has a representation in the Model with Multiple Reference Points, it also has a representation in the Model with Multiple Reference Points in which all relations R_i are weak orders,
- if $\succsim$ has a representation in the Model with Multiple Reference Points, it also has a representation in which reference points dominates each other according to the weak orders R_i, i.e., for all $k \in \{2, 3, \dots, \ell\}$ and all $i \in N$, $\pi_i^k \ R_i \ \pi_i^{k-1}$. In such a representation, for all $x \in X$, with $\mathcal{R}^L(x) = (A^1, \dots, A^\ell)$ we have $A^1 \supseteq A^2 \supseteq \cdots \supseteq A^\ell$ (this was already observed in Rolland 2008, p. 51).

Indeed, the reference points $\pi^1, \pi^2, \dots, \pi^\ell$ and the relations R_i that are built in the proof of Lemma 21 have these two properties.

Our findings are summarized below.

Theorem 3. *A binary relation $\succsim$ on X is a RMRP iff it satisfies $AC1$, $AC2$ and $AC3$ and, for all $i \in N$, the set $X_i/\sim_i^{\pm}$ is finite.*

If a relation $\succsim$ is a RMRP, it always has a representation in which all relations R_i are weak orders and in which the ℓ reference points are such that for all $k \in \{2, 3, \dots, \ell\}$ and all $i \in N$, $\pi_i^k \ R_i \ \pi_i^{k-1}$.

The uniqueness of the representation of RMRP is obviously quite weak. Since its study is cumbersome and does not appear to be particularly informative, we do not detail this point.

By definition, a RMRP only uses a finite number of reference points. This is reflected in the above result by the fact that in a RMRP, all relations $\succsim_i^{\pm}$ have a finite number of equivalence classes. We have seen above (see the proof of Lemma 15) that there are

CR-AT in which the relations $\succsim_i^{\pm}$ have infinitely many equivalence classes. Hence, there are CR-AT that are not reflexive RMRP. The converse is also true since a reflexive RMRP may fail to be independent whereas we know that a CR-AT always is. The claim of Rolland (2003, 2006a, 2008) that models using reference points are more "flexible" than concordance relations is nevertheless clearly true if attention is restricted to the case of finite sets X. Indeed, a CR-AT always satisfies conditions $AC1$, $AC2$ and $AC3$. Moreover, on finite sets, the condition stating that, for all $i \in N$, the set $X_i/\sim_i^{\pm}$ is finite trivially holds. Hence, in this particular but important case, all CR-AT are reflexive RMRP.

Using Lemma 5, it is easy to formulate a result characterizing model (RMRP) when $\succsim$ is a weak order. In this case, we know that the three conditions $AC1$, $AC2$ and $AC3$ are equivalent and may be replaced by weak separability. For the record, we state the following:

Proposition 5. *Let $\succsim$ be a weak order on X. It is a Relation with Multiple Reference Points iff it satisfies $AC1$ and, for all $i \in N$, the set $X_i/\sim_i^{\pm}$ is finite.*

Since the relation $\trianglerighteq_L$ is defined on the finite set $(2^N)^\ell$, it is simple to show that a RMRP that is a weak order can only have a finite number of equivalence classes.

Consider now a relation $\succsim$ defined on a *finite* set X. It will be a RMRP as soon as it satisfies $AC1$. In view of Proposition 4, it will have a representation in the discrete Sugeno integral model if, furthermore, it is 2-graded. Clearly, a relation satisfying $AC1$ does not have to be 2-graded. This shows that, contrary to the situation with RSRP, a RMRP may not have a representation in the discrete Sugeno integral model. This was already noted in Rolland (2008, Ex. 61, p. 138).

9. Discussion

This paper has shown how to use the general model developed in Bouyssou and Pirlot (2004b) to characterize preference models using a single reference point introduced by Rolland (2003, 2006a,b, 2008, 2013). This analysis was extended to the case of multiple reference points in Section 8. Basically, models using reference points are particular cases of the model inducing complete traces on the levels of each attribute. The number of reference points is linked to the number of distinct equivalence classes generated by these traces. Models using a single reference point generate traces that are quite rough. We have proposed a complete characterization of these models using a traditional conjoint measurement setting in which the only primitive is a preference relation $\succsim$ on X. This analysis has allowed us to compare models using reference points with several other preference models for multiattributed alternatives. In particular, we have shown that preference models using a single reference point that are weak orders are a particular case of preference models based on the discrete Sugeno integral. Moreover, we have analyzed the relations between preference models using a single reference point and concordance relations. None of these two models is a subclass of the other. Finally, since the conditions that we have exhibited are entirely phrased in terms of a preference relation $\succsim$ on X, they could be subjected to empirical tests.

We conclude with the indication of directions for future research on the subject. The first is to analyze models using reference points that would include an idea of "non-discordance" as in the ELECTRE TRI model. Using the results in Bouyssou and Marchant (2007a,b), this should not be overly difficult. The second is to pursue the analysis of preference models using several reference points. Indeed, Rolland (2003, 2006a,b, 2008, 2013) has proposed several particular cases of the general model that we study in Section 8. Finally, it is clearly of particular interest to further investigate elicitation techniques that would lead to specify the parameters of the models that we have studied.

References

F. Aleskerov, D. Bouyssou, and B. Monjardet. *Utility maximization, choice and preference.* Springer-Verlag, Berlin, 2nd edition, 2007.

D. Bouyssou and T. Marchant. An axiomatic approach to noncompensatory sorting methods in MCDM, I: The case of two categories. *European Journal of Operational Research*, 178(1): 217–245, 2007a.

D. Bouyssou and T. Marchant. An axiomatic approach to noncompensatory sorting methods in MCDM, II: More than two categories. *European Journal of Operational Research*, 178(1): 246–276, 2007b.

D. Bouyssou and M. Pirlot. Nontransitive decomposable conjoint measurement. *Journal of Mathematical Psychology*, 46(6):677–703, 2002a.

D. Bouyssou and M. Pirlot. A characterization of strict concordance relations. In D. Bouyssou, É. Jacquet-Lagrèze, P. Perny, R. Słowiński, D. Vanderpooten, and Ph. Vincke, editors, *Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy*, pages 121–145. Kluwer, Dordrecht, 2002b.

D. Bouyssou and M. Pirlot. 'Additive difference' models without additivity and subtractivity. *Journal of Mathematical Psychology*, 48(4):263–291, 2004a.

D. Bouyssou and M. Pirlot. Preferences for multiattributed alternatives: Traces, dominance, and numerical representations. *Journal of Mathematical Psychology*, 48(3):167–185, 2004b.

D. Bouyssou and M. Pirlot. Following the traces: An introduction to conjoint measurement without transitivity and additivity. *European Journal of Operational Research*, 163(2):287–337, 2005a.

D. Bouyssou and M. Pirlot. A characterization of concordance relations. *European Journal of Operational Research*, 167(2):427–443, 2005b.

D. Bouyssou and M. Pirlot. Further results on concordance relations. *European Journal of Operational Research*, 181(1):505–514, 2007.

D. Bouyssou, M. Pirlot, and Ph. Vincke. A general model of preference aggregation. In M. H. Karwan, J. Spronk, and J. Wallenius, editors, *Essays in Decision Making*, pages 120–134. Springer-Verlag, Berlin, 1997.

D. Bouyssou, T. Marchant, and M. Pirlot. A conjoint measurement approach to the discrete Sugeno integral. In S. Brams, W. V. Gehrlein, and F. S. Roberts, editors, *The Mathematics of Preference, Choice and Order. Essays in Honor of Peter C. Fishburn*, pages 85–109. Springer, 2009.

A. Chateauneuf and J.-Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

D. Dubois, H. Fargier, and P. Perny. Qualitative decision theory with preference relations and comparative uncertainty: An axiomatic approach. *Artificial Intelligence*, 148(1–2):219–260, 2003. Corrigendum: Artificial Intelligence, Volume 171(5–6), 2007, 361–362.

H. Fargier and P. Perny. Qualitative decision models under uncertainty without the commensurability assumption. In K. B. Laskey and H. Prade, editors, *Proceedings of Uncertainty in Artificial Intelligence*, pages 188–195. Morgan Kaufmann Publishers, 1999.

J. R. Figueira, V. Mousseau, and B. Roy. ELECTRE methods. In J. R. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 133–162. Springer Verlag, Boston, Dordrecht, London, 2005.

J. R. Figueira, S. Greco, B. Roy, and R. Słowiński. ELECTRE Methods: Main features and recent developments. In C. Zopounidis and P. M. Pardalos, editors, *Handbook of Multicriteria Analysis*, number 103 in Applied Optimization, pages 51–90. Springer-Verlag, Berlin, 2010. ISBN 978-3-540-92827-0.

M. Grabisch. k-order addive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

S. Greco, B. Matarazzo, and R. Słowiński. Axiomatic basis of noncompensatory preferences. Communication to *FUR X*, 30 May–2 June, Torino, Italy, 2001.

S. Greco, B. Matarazzo, and R. Słowiński. Axiomatic characterization of a general utility function and its particular cases in terms of conjoint measurement and rough-set decision rules. *European Journal of Operational Research*, 158(2):271–292, 2004.

S. Greco, V. Mousseau, and R. Słowiński. Ordinal regression revisited: Multiple criteria ranking using a set of additive value functions. *European Journal of Operational Research*, 191(2): 416–436, 2008.

D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–291, 1979.

J.-L. Marichal and M. Roubens. Determination of weights of interacting criteria from a reference set. *European Journal of Operational Research*, 124(3):641–650, 2000.

V. Mousseau, A. Rolland, and J. Zheng. Learning a reference based multicriteria ranking model from pairwise comparisons. Communication to MCDA 75, Taragona, Spain, 12–13 April 2012, 2012.

P. Perny and A. Rolland. Reference-dependent qualtitative models for decision making under uncertainty. In G. Brewka, S. Coradeschi, A. Perini, and P. Traverso, editors, *Proceedings ECAI-2006, 20th European Conference on Artificial Intelligence*, pages 422–426. IOS Press, 2006.

A. Rolland. Agrégation de préférences et point de référence. In *Actes de l'école d'automne de Recherche Opérationnelle*, pages 154–160. Université de Tours, 2003.

A. Rolland. Points de référence en décision multicritère. In *Actes de la 7ème Conférence ROADEF*, pages 153–167. Presses Universitaires de Valenciennes, 2006a.

A. Rolland. On bi-capacity-based concordance rules in multicriteria decision making. In B. Bouchon-Meunier, G. Coletti, and R. R. Yager, editors, *Modern Information Processing: From theory to applications*, pages 231–245. Elsevier, 2006b.

A. Rolland. *Procédures d'agrégation ordinale de préférences avec points de référence pour l'aide à la décision*. Thèse de doctorat, Université Pierre et Marie Curie, Paris, 2008. Available from `http://eric.univ-lyon2.fr/~arolland/index.php?choix=3`.

A. Rolland. Reference-based preferences aggregation procedures in multi-criteria decision making. *European Journal of Operational Research*, 225(3):479–486, 2013.

B. Roy. The outranking approach and the foundations of ELECTRE methods. *Theory and Decision*, 31(1):49–73, 1991.

B. Roy and D. Bouyssou. *Aide multicritère à la décision : méthodes et cas*. Economica, Paris, 1993.

M. Sugeno. *Theory of fuzzy integrals and its applications.* PhD thesis, Tokyo Institute of Technology, 1974.

A. Tversky and D. Kahneman. Rational choice and the framing of decisions. *Journal of Business*, 59(4):S251–S278, 1986.

A. Tversky and D. Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5(4):297–323, 1992.

P. P. Wakker. *Additive representations of preferences: A new foundation of decision analysis.* Kluwer, Dordrecht, 1989.

J. Zheng, A. Rolland, and V. Mousseau. Preference elicitation for a ranking method based on multiple reference points. Research Report CER 12–05, École Centrale Paris, Laboratoire de Génie Industriel, August 2012.

Supplementary material to the paper
Multiattribute preference models with reference points
Denis Bouyssou & Thierry Marchant

1 February 2013

A1. Proof of Lemma 1

Suppose that $\succsim$ is a RSRP having two distinct representations, i.e., that we have

$$x \succsim y \Leftrightarrow \mathcal{S}(x) \trianglerighteq \mathcal{S}(y) \Leftrightarrow \mathcal{S}'(x) \trianglerighteq' \mathcal{S}'(y),$$

where $\mathcal{S}(x) = \{i \in N : x_i \in \mathcal{A}_i\}$ and $\mathcal{S}'(x) = \{i \in N : x_i \in \mathcal{A}'_i\}$.

Suppose that for some $e_i \in X_i$, we have $e_i \in \mathcal{A}_i$ and $e_i \notin \mathcal{A}'_i$. Because attribute $i \in N$ is influential, there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(z_i, a_{-i}) \not\succsim (w_i, b_{-i})$.

Because $e_i \in \mathcal{A}_i$ and $(x_i, a_{-i}) \succsim (y_i, b_{-i})$, we must have $(e_i, a_{-i}) \succsim (y_i, b_{-i})$. Similarly, because $e_i \notin \mathcal{A}'_i$ and $(z_i, a_{-i}) \not\succsim (w_i, b_{-i})$, we must have $(e_i, a_{-i}) \not\succsim (w_i, b_{-i})$.

Suppose that $y_i \in \mathcal{A}_i$. Using $(e_i, a_{-i}) \succsim (y_i, b_{-i})$ and the monotonicity of $\trianglerighteq$, we obtain $(e_i, a_{-i}) \succsim (s_i, b_{-i})$, for all $s_i \in X_i$. This is contradictory since we know that $(e_i, a_{-i}) \not\succsim (w_i, b_{-i})$.

Suppose that $y_i \in \mathcal{A}'_i$. Using $(e_i, a_{-i}) \succsim (y_i, b_{-i})$ and the monotonicity of $\trianglerighteq'$, we obtain $(e_i, a_{-i}) \succsim (s_i, b_{-i})$, for all $s_i \in X_i$. This is contradictory since we know that $(e_i, a_{-i}) \not\succsim (w_i, b_{-i})$.

Hence, $y_i \notin \mathcal{A}_i$ and $y_i \notin \mathcal{A}'_i$. Since we have $(e_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(e_i, a_{-i}) \not\succsim (w_i, b_{-i})$, we must have $w_i \in \mathcal{A}_i$ and $w_i \in \mathcal{A}'_i$.

Because $(e_i, a_{-i}) \succsim (y_i, b_{-i})$, $y_i \notin \mathcal{A}'_i$ and $e_i \notin \mathcal{A}'_i$, we obtain $(e_i, a_{-i}) \succsim (e_i, b_{-i})$. Similarly, because $(e_i, a_{-i}) \not\succsim (w_i, b_{-i})$, $w_i \in \mathcal{A}_i$ and $e_i \in \mathcal{A}_i$, we obtain $(e_i, a_{-i}) \not\succsim (e_i, b_{-i})$, a contradiction.

Hence, we have shown that, for all $i \in N$, we have $\mathcal{A}_i = \mathcal{A}'_i$. It is easy to show that this implies $\trianglerighteq = \trianglerighteq'$.

Let us now prove that $\varnothing \subsetneq \mathcal{A}_i \subsetneq X_i$. Because attribute $i \in N$ is influential, there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(z_i, a_{-i}) \not\succsim (w_i, b_{-i})$. If $\mathcal{A}_i = \varnothing$ or $\mathcal{A}_i = X_i$, we have $\mathcal{S}((x_i, a_{-i})) = \mathcal{S}((z_i, a_{-i}))$ and $\mathcal{S}((y_i, b_{-i})) = \mathcal{S}((w_i, b_{-i}))$. Because $\succsim$ is a RSRP, this implies $(x_i, a_{-i}) \succsim (y_i, b_{-i}) \Leftrightarrow (z_i, a_{-i}) \succsim (w_i, b_{-i})$, a contradiction.

A2. Proof of Lemma 2

We know that $\succsim$ has a unique representation in which, for all $i \in N$, $\varnothing \subsetneq \mathcal{A}_i \subsetneq X_i$. Hence, for all $A, B \subseteq N$, there are $x, y \in X$ such that $\mathcal{S}(x) = A$ and $\mathcal{S}(y) = B$.

Part 1.Let $x \in X$. Since $\trianglerighteq$ is reflexive, we know that $\mathcal{S}(x) \trianglerighteq \mathcal{S}(x)$, so that $x \succsim x$. Hence, $\succsim$ is reflexive. Conversely, let $A \subseteq N$. We have $\mathcal{S}(x) = A$, for some $x \in X$. Since $x \succsim x$, we obtain $A \trianglerighteq A$. Hence, $\trianglerighteq$ is reflexive.

The proof of Parts 2 and 3 is similar.

Part 4. Suppose that $\succsim$ is not independent so that we have $(x_i, a_{-i}) \succsim (x_i, b_{-i})$ and $(y_i, a_{-i}) \not\succsim (y_i, b_{-i})$, for some $i \in N$, $x_i, y_i \in X_i$, and $a, b \in X$. Since $\succsim$ is a RSRP, it is impossible that $x_i, y_i \in \mathcal{A}_i$ or $x_i, y_i \notin \mathcal{A}_i$. Letting $A = \{j \in N \setminus \{i\} : a_j \in \mathcal{A}_j\}$ and $B = \{j \in N \setminus \{i\} : b_j \in \mathcal{A}_j\}$, we obtain either $A \trianglerighteq B$ and $A \cup \{i\} \not\trianglerighteq B \cup \{i\}$ or $A \not\trianglerighteq B$ and $A \cup \{i\} \trianglerighteq B \cup \{i\}$.

Conversely, suppose that we have $A \trianglerighteq B$ and $A \cup \{i\} \not\trianglerighteq B \cup \{i\}$. Let $a, b \in X$ be such that $A = \{j \in N \setminus \{i\} : a_j \in \mathcal{A}_j\}$ and $B = \{j \in N \setminus \{i\} : b_j \in \mathcal{A}_j\}$. Take $x_i \in \mathcal{A}_i$ and $y_i \notin \mathcal{A}_i$. We obtain $(x_i, a_{-i}) \not\succsim (x_i, b_{-i})$ and $(y_i, a_{-i}) \succsim (y_i, b_{-i})$. The case $A \not\trianglerighteq B$ and $A \cup \{i\} \trianglerighteq B \cup \{i\}$ is similar.

A3. Proof of Lemma 6

Let $\succsim$ be a RSRP. Lemma 1 has shown that it has a unique representation using the sets $\mathcal{A}_i$ and the relation $\trianglerighteq$. Using the definition of a RSRP, it is easy to see that if $x_i, y_i \in \mathcal{A}_i$ or if $x_i, y_i \notin \mathcal{A}_i$, we have $x_i \sim_i^{\pm} y_i$. Moreover, if $x_i \in \mathcal{A}_i$ and $y_i \notin \mathcal{A}_i$, the monotonicity of $\trianglerighteq$ w.r.t. inclusion implies $x_i \succsim_i^{\pm} y_i$. Hence, the relation $\succsim_i^{\pm}$ is a weak order having at most two distinct equivalence classes.

Because we have supposed that each $i \in N$ is influential, there are $x_i, y_i, z_i, w_i \in X_i$ and $a, b \in X$ such that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(z_i, a_{-i}) \not\succsim (w_i, b_{-i})$. If $z_i \succsim_i^{+} x_i$ and $y_i \succsim_i^{-} w_i$, $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ implies $(z_i, a_{-i}) \succsim (w_i, b_{-i})$, a contradiction. Hence, we must have either $x_i \succ_i^{+} z_i$ or $w_i \succ_i^{-} y_i$. Either case implies that the weak order $\succsim_i^{\pm}$ has at least two distinct equivalence classes.

A4. Proof of Lemma 7

For all $i \in N$, define $\mathcal{A}_i$ as the set of elements of X_i in the first equivalence class of $\succsim_i^{\pm}$. Define the relation $\trianglerighteq$ letting, for all $A, B \subseteq N$, $A \trianglerighteq B$ if $x \succsim y$, for some $x, y \in X$ such that $\mathcal{S}(x) = A$ and $\mathcal{S}(y) = B$. We have to show that, for all $x, y \in X$, $x \succsim y \Leftrightarrow \mathcal{S}(x) \trianglerighteq \mathcal{S}(y)$ and that $\trianglerighteq$ is monotonic w.r.t. inclusion. If $x \succsim y$, the definition of $\trianglerighteq$ implies that $\mathcal{S}(x) \trianglerighteq \mathcal{S}(y)$. Suppose now that $\mathcal{S}(x) \trianglerighteq \mathcal{S}(y)$. By construction, this implies that, for some $z, w \in X$ we have $z \succsim w$, $\mathcal{S}(z) = \mathcal{S}(x)$ and $\mathcal{S}(w) = \mathcal{S}(y)$. This implies that, for all $i \in N$, $x_i \sim_i^{\pm} z_i$ and $y_i \sim_i^{\pm} w_i$. Using (4), we obtain $x \succsim y$.

It remains to prove that $\trianglerighteq$ is monotonic w.r.t. inclusion.

Suppose that for some $x, y, z, w \in X$ we have $\mathcal{S}(z) \supseteq \mathcal{S}(x)$, $\mathcal{S}(y) \supseteq \mathcal{S}(w)$, and $\mathcal{S}(x) \trianglerighteq \mathcal{S}(y)$. By construction of the sets $\mathcal{A}_i$, $\mathcal{S}(z) \supseteq \mathcal{S}(x)$ and $\mathcal{S}(y) \supseteq \mathcal{S}(w)$ imply that, for all $i \in N$, we have $z_i \succsim_i^{\pm} x_i$ and $y_i \succsim_i^{\pm} w_i$. Since $\mathcal{S}(x) \trianglerighteq \mathcal{S}(y)$, we have $x \succsim y$. Using (3), we obtain $z \succsim w$, so that $\mathcal{S}(z) \trianglerighteq \mathcal{S}(w)$.

A5. Proof of Lemma 8

We prove Part 1, the proof of the other parts being similar. The negation of $AC1_i^*$ says that $(x_i, a_{-i}) \succsim c$, $(y_i, b_{-i}) \succsim d$, $(y_i, a_{-i}) \not\succsim c$, $(z_i, b_{-i}) \not\succsim d$, for some $a, b, c, d \in X$ and some $x_i, y_i, z_i \in X_i$. This is equivalent to saying that we have $Not[y_i \succsim_i^+ x_i]$ and $Not[z_i \succsim_i^+ y_i]$.

A6. Proof of Lemma 9

Suppose that $AC1_i^*$ is violated, so that we have $(x_i, a_{-i}) \succsim c$, $(y_i, b_{-i}) \succsim d$, $(y_i, a_{-i}) \not\succsim c$, and $(z_i, b_{-i}) \not\succsim d$, for some $a, b, c, d \in X$ and some $x_i, y_i, z_i \in X_i$. Since $(x_i, a_{-i}) \succsim c$ and $(y_i, a_{-i}) \not\succsim c$, we have $Not[y_i \succsim_i^+ x_i]$. In a RSRP, this can only happen if $y_i \notin \mathcal{A}_i$. Since $y_i \notin \mathcal{A}_i$, we have $\mathcal{S}((z_i, a_{-i})) \supseteq \mathcal{S}((y_i, a_{-i}))$. Because $(y_i, b_{-i}) \succsim d$, we know that $\mathcal{S}((y_i, b_{-i})) \trianglerighteq \mathcal{S}(d)$. Using the monotonicity of $\trianglerighteq$ w.r.t. inclusion, we obtain $\mathcal{S}((z_i, b_{-i})) \trianglerighteq \mathcal{S}(d)$, so that $(z_i, b_{-i}) \succsim d$, a contradiction. The proof of the other parts is similar.

A7. Proof of Lemma 11

$[AC1_i^*, AC2_i^*, AC3_i^*, AC4_i^*] \Rightarrow [\succsim_i^{\pm}$ is a weak order having at most two distinct equivalence classes]. Using Lemma 10, we know that $AC1_i$, $AC2_i$, and $AC3_i$ hold, so that, using Lemma 4, $\succsim_i^{\pm}$ is a weak order. Suppose that we have, for some $x_i, y_i, z_i \in X_i$, $x_i \succ_i^{\pm} y_i$ and $y_i \succ_i^{\pm} z_i$. By construction, $x_i \succ_i^{\pm} y_i$ implies either

$$[x_i \succ_i^+ y_i \text{ and } x_i \succsim_i^- y_i] \text{ or} \tag{A 11a}$$
$$[x_i \succsim_i^+ y_i \text{ and } x_i \succ_i^- y_i]. \tag{A 11b}$$

Similarly, $y_i \succ_i^{\pm} z_i$ implies either

$$[y_i \succ_i^+ z_i \text{ and } y_i \succsim_i^- z_i] \text{ or} \tag{A 11c}$$
$$[y_i \succsim_i^+ z_i \text{ and } y_i \succ_i^- z_i]. \tag{A 11d}$$

The combination of (A 11a) and (A 11c) violates Lemma 8.1 (i.e., Part 1 of Lemma 8) since it implies $Not[y_i \succsim_i^+ x_i]$ and $Not[z_i \succsim_i^+ y_i]$. The combination of (A 11b) and (A 11d) violates Lemma 8.2 since it implies $Not[y_i \succsim_i^- x_i]$ and $Not[z_i \succsim_i^- y_i]$. The combination of (A 11a) and (A 11d) violates Lemma 8.3 since it implies $Not[y_i \succsim_i^+ x_i]$ and $Not[z_i \succsim_i^- y_i]$. Finally, the combination of (A 11b) and (A 11c) violates Lemma 8.4 since it implies $Not[y_i \succsim_i^- x_i]$ and $Not[z_i \succsim_i^+ y_i]$.

$[\succsim_i^{\pm}$ is a weak order having at most two distinct equivalence classes$] \Rightarrow [AC1_i^*, AC2_i^*, AC3_i^*, AC4_i^*]$. Suppose that $(x_i, a_{-i}) \succsim c$, and $(y_i, b_{-i}) \succsim d$. If $(y_i, a_{-i}) \not\succsim c$, we know that $x_i \succ_i^+ y_i$, so that $x_i \succ_i^{\pm} y_i$. Since $\succsim_i^{\pm}$ is a weak order having only two distinct equivalence classes, this implies that, for all $z_i \in X_i$, $z_i \succsim_i^{\pm} y_i$, so that $z_i \succsim_i^+ y_i$. Hence, $(y_i, b_{-i}) \succsim d$ implies $(z_i, b_{-i}) \succsim d$. This shows that $AC1_i^*$ holds. The proof for the other three conditions is similar.

The proof is completed observing that, since all attributes are influential, all relations $\succsim_i^{\pm}$ have at least two equivalence classes.

In all examples in Sections A8, A10, and A14, we take $n = 3$ and $X_1 = \{x, y, z\}$, $X_2 = \{a, b\}$ and $X_3 = \{p, q\}$. To save space, we often write xap instead of (x, a, p).

A8. Proof of Lemma 12

We have to show that in the class of all semiorders on X, conditions $AC1^*$, $AC2^*$, $AC3^*$, and $AC4^*$ are independent.

We provide below the required four examples. In all these examples, we define $\succsim$ by its boolean matrix, so that it is easy to check that the relations considered below are semiorders. Indeed, if it is possible to arrange the rows and columns of the boolean matrix in the same order so that the boolean matrix is stepped, we know that the relation is a semiorder (Aleskerov et al. 2007, p. 80).

Example 1. Let $\succsim$ on X be defined by the following table:

	xaq	yaq	zaq	xap	yap	zap	ybq	xbp	xbq	ybp	zbp	zbq
xaq	1	1	1	1	1	1	1	1	1	1	1	1
yaq	1	1	1	1	1	1	1	1	1	1	1	1
zaq	1	1	1	1	1	1	1	1	1	1	1	1
xap	1	1	1	1	1	1	1	1	1	1	1	1
yap	1	1	1	1	1	1	1	1	1	1	1	1
zap	1	1	1	1	1	1	1	1	1	1	1	1
ybq	1	1	1	1	1	1	1	1	1	1	1	1
xbp	1	1	1	1	1	1	1	1	1	1	1	1
xbq	1	1	1	1	1	1	1	1	1	1	1	1
ybp	0	0	0	1	1	1	1	1	1	1	1	1
zbp	0	0	0	0	0	0	1	1	1	1	1	1
zbq	0	0	0	0	0	0	1	1	1	1	1	1

It is clear that this relation is a semiorder on X.

It is easy to check that we have $a \succ_2^+ b$, $a \succ_2^- b$, $q \succ_3^+ p$ and $q \succ_3^- p$. Using Lemma 8 shows that $AC1_i^*$, $AC2_i^*$, $AC3_i^*$, and $AC4_i^*$ hold for $i = 2, 3$.

On attribute 1, we have $x \sim_1^- y \sim_1^- z$ and $x \succ_1^+ y \succ_1^+ z$. Using Lemma 8 shows that $AC2_1^*$, $AC3_1^*$, and $AC4_1^*$ hold, while $AC1_1^*$ is clearly violated.

Transposing the boolean matrix in the above example, one easily obtain an example satisfying all our conditions except $AC2$ on one attribute. Indeed, on each attribute the consequence of this transposition is is to interchange the roles of $\succsim_i^+$ and $\succsim_i^-$ and to reverse them. This is detailed below.

Example 2. Let $\succsim$ on X be defined by the following table:

	zbq	zbp	ybp	xbq	xbp	ybq	zap	yap	xap	zaq	yaq	xaq
zbq	1	1	1	1	1	1	1	1	1	1	1	1
zbp	1	1	1	1	1	1	1	1	1	1	1	1
ybp	1	1	1	1	1	1	1	1	1	1	1	1
xbq	1	1	1	1	1	1	1	1	1	1	1	1
xbp	1	1	1	1	1	1	1	1	1	1	1	1
ybq	1	1	1	1	1	1	1	1	1	1	1	1
zap	0	0	1	1	1	1	1	1	1	1	1	1
yap	0	0	1	1	1	1	1	1	1	1	1	1
xap	0	0	1	1	1	1	1	1	1	1	1	1
zaq	0	0	0	1	1	1	1	1	1	1	1	1
yaq	0	0	0	1	1	1	1	1	1	1	1	1
xaq	0	0	0	1	1	1	1	1	1	1	1	1

It is clear that this relation is a semiorder on X.

It is easy to check that we have $b \succ_2^+ a$, $b \succ_2^- a$, $p \succ_3^+ q$ and $p \succ_3^- q$. Using Lemma 8 shows that $AC1_i^*$, $AC2_i^*$, $AC3_i^*$, and $AC4_i^*$ hold for $i = 2, 3$.

On attribute 1, we have $x \sim_1^+ y \sim_1^+ z$ and $z \succ_1^- y \succ_1^- x$. Using Lemma 8 shows that $AC1_1^*$, $AC3_1^*$, and $AC4_1^*$ hold, while $AC2_1^*$ is clearly violated.

Example 3. Let $\succsim$ on X be defined by the following table:

	xaq	yaq	xap	xbp	xbq	yap	ybq	zap	zaq	zbq	ybp	zbp
xaq	1	1	1	1	1	1	1	1	1	1	1	1
yaq	1	1	1	1	1	1	1	1	1	1	1	1
xap	1	1	1	1	1	1	1	1	1	1	1	1
xbp	1	1	1	1	1	1	1	1	1	1	1	1
xbq	1	1	1	1	1	1	1	1	1	1	1	1
yap	1	1	1	1	1	1	1	1	1	1	1	1
ybq	1	1	1	1	1	1	1	1	1	1	1	1
zap	1	1	1	1	1	1	1	1	1	1	1	1
zaq	1	1	1	1	1	1	1	1	1	1	1	1
zbq	1	1	1	1	1	1	1	1	1	1	1	1
ybp	0	0	1	1	1	1	1	1	1	1	1	1
zbp	0	0	1	1	1	1	1	1	1	1	1	1

It is clear that this relation is a semiorder on X.

It is easy to check that we have $a \succ_2^+ b$, $a \succ_2^- b$, $q \succ_3^+ p$ and $q \succ_3^- p$. Using Lemma 8 shows that $AC1_i^*$, $AC2_i^*$, $AC3_i^*$, and $AC4_i^*$ hold for $i = 2, 3$.

On attribute 1, we have $x \succ_1^+ [y \sim_1^+ z]$ and $[x \sim_1^- y] \succ_1^- z$. Using Lemma 8 shows that $AC1_1^*$, $AC2_1^*$, and $AC4_1^*$ hold, while $AC3_1^*$ is clearly violated.

Example 4. Let $\succsim$ on X be defined by the following table:

	xbp	xbq	xap	xaq	yap	yaq	ybp	ybq	zaq	zbq	zap	zbp
xbp	1	1	1	1	1	1	1	1	1	1	1	1
xbq	1	1	1	1	1	1	1	1	1	1	1	1
xap	1	1	1	1	1	1	1	1	1	1	1	1
xaq	1	1	1	1	1	1	1	1	1	1	1	1
yap	1	1	1	1	1	1	1	1	1	1	1	1
yaq	1	1	1	1	1	1	1	1	1	1	1	1
ybp	1	1	1	1	1	1	1	1	1	1	1	1
ybq	1	1	1	1	1	1	1	1	1	1	1	1
zaq	1	1	1	1	1	1	1	1	1	1	1	1
zbq	1	1	1	1	1	1	1	1	1	1	1	1
zap	0	0	1	1	1	1	1	1	1	1	1	1
zbp	0	0	1	1	1	1	1	1	1	1	1	1

It is clear that this relation is a semiorder on X.

It is easy to check that we have $b \sim_2^+ a$, $b \succ_2^- a$, $q \succ_3^+ p$ and $q \sim_3^- p$. Using Lemma 8 shows that $AC1_i^*$, $AC2_i^*$, $AC3_i^*$, and $AC4_i^*$ hold for $i = 2, 3$.

On attribute 1, we have $[x \sim_1^+ y] \succ_1^+ z$ and $x \succ_1^- [y \sim_1^- z]$. Using Lemma 8 shows that $AC1_1^*$, $AC2_1^*$, and $AC3_1^*$ hold, while $AC4_1^*$ is clearly violated.

Observe that going from Example 3 to Example 4does not amount to transposing the boolean matrix of the relation in Example 3. Transposing a relation violating only $AC3_1^*$ does not lead to a relation violating only $AC4_1^*$.

A9. Proof of Lemma 13

Part 1. It is clear that $AC1_i^*$ implies $AC1_i^{**}$ (since the latter condition amount to add a premise to the former condition) and $AC1_i$ (in view of Lemma 10). Let us show that the reverse implication holds. Suppose that $AC1_i^*$ is violated, so that we have: $(x_i, a_{-i}) \succsim c$, $(y_i, b_{-i}) \succsim d$, $(y_i, a_{-i}) \not\succsim c$, $(z_i, b_{-i}) \not\succsim d$, for some $a, b, c, d \in X$ and some $x_i, y_i, z_i \in X_i$. Since $AC1_i$ holds, we know that $\succsim_i^+$ is a weak order. In view of the fact that $(x_i, a_{-i}) \succsim c$ and $(y_i, a_{-i}) \not\succsim c$, we must have $x_i \succsim_i^+ y_i$. Since $(y_i, b_{-i}) \succsim d$, we obtain $(x_i, b_{-i}) \succsim d$. Hence, we have: $(x_i, a_{-i}) \succsim c$, $(y_i, b_{-i}) \succsim d$, $(x_i, b_{-i}) \succsim d$, $(y_i, a_{-i}) \not\succsim c$, $(z_i, b_{-i}) \not\succsim d$, violating $AC1_i^{**}$.

The proof of the other parts is similar.

A10. Proof of Lemma 14

We have to show that in the class of all semiorders on X, conditions $AC1_i$, $AC2_i$, $AC3_i$, $AC1^{**}$, $AC2^{**}$, $AC3^{**}$, and $AC4^{**}$ are independent. Seven examples are needed to do so.

Observe first that in Examples 1–4, conditions $AC1$, $AC2$, $AC3$ are satisfied. Hence, each of these examples violates exactly one of $AC1_i^{**}$, $AC2_i^{**}$, $AC3_i^{**}$, and $AC4_i^{**}$. It remains to find three more examples.

In order to check the following three examples, it will be helpful to observe that whenever X_i has only two elements, conditions $AC1_i^{**}$, $AC2_i^{**}$, $AC3_i^{**}$, and $AC4_i^{**}$ are always satisfied. Indeed, the premises of $AC1_i^{**}$ state that $(x_i, a_{-i}) \succsim c$, $(y_i, b_{-i}) \succsim d$, and $(x_i, b_{-i}) \succsim d$. The two possible conclusions of $AC1_i^{**}$ are $(y_i, a_{-i}) \succsim c$ or $(z_i, b_{-i}) \succsim d$. If $x_i = y_i$ then the first conclusion of $AC1_i^{**}$ trivially holds. If $x_i \neq y_i$ then the second possible conclusion of $AC1_i^{**}$ will trivially hold since z_i must be either x_i or y_i. A similar reasoning applies to $AC2_i^{**}$ and $AC4_i^{**}$. The premises of $AC3_i^{**}$ state that $(x_i, a_{-i}) \succsim c$, $d \succsim (z_i, b_{-i})$. The three possible conclusions are $(y_i, a_{-i}) \succsim c$, $d \succsim (y_i, b_{-i})$, or $d \succsim (x_i, b_{-i})$. If $x_i = z_i$, then the third conclusion of $AC3_i^{**}$ trivially holds. If $x_i \neq z_i$, then the y_i in the conclusion of $AC3_i^{**}$ must be either x_i or z_i. In the first (resp. second) case, the first (resp. second) conclusion trivially holds.

Example 5. Let $\succsim$ on X be defined by the following table:

	zbp	ybp	xbp	ybq	zbq	xbq	zap	yap	xap	yaq	zaq	xaq
zbp	1	1	1	1	1	1	1	1	1	1	1	1
ybp	1	1	1	1	1	1	1	1	1	1	1	1
xbp	1	1	1	1	1	1	1	1	1	1	1	1
ybq	1	1	1	1	1	1	1	1	1	1	1	1
zbq	0	0	0	1	1	1	1	1	1	1	1	1
xbq	0	0	0	1	1	1	1	1	1	1	1	1
zap	0	0	0	1	1	1	1	1	1	1	1	1
yap	0	0	0	0	0	0	1	1	1	1	1	1
xap	0	0	0	0	0	0	1	1	1	1	1	1
yaq	0	0	0	0	0	0	1	1	1	1	1	1
zaq	0	0	0	0	0	0	0	0	0	1	1	1
xaq	0	0	0	0	0	0	0	0	0	1	1	1

It is clear that this relation is a semiorder.

It is easy to check that the traces are as follows.

On attribute 2, we have $b \succ_2^+ a$ and $b \succ_2^- a$. Hence, $AC1_2$, $AC2_2$, and $AC3_2$ hold. Since X_2 has only two elements, $AC1_2^{**}$, $AC2_2^{**}$, $AC3_2^{**}$, and $AC4_2^{**}$ trivially hold.

On attribute 3, we have $p \succ_3^+ q$ and $p \succ_3^- q$. Hence, $AC1_3$, $AC2_3$, and $AC3_3$ hold. Since X_3 has only two elements, $AC1_3^{**}$, $AC2_3^{**}$, $AC3_3^{**}$, and $AC4_3^{**}$ trivially hold.

On attribute 1, $\succsim_1^-$ is a clique. Furthermore, we have $y \succ_1^+ x$ and $z \succ_1^+ x$, but neither $y \succsim_1^+ z$ nor $z \succsim_1^+ y$. It easy to check that this implies that $AC2_1$ and $AC3_1$ hold, while $AC1_1$ is violated.

It remains to check that $AC1_1^{**}$, $AC2_1^{**}$, $AC3_1^{**}$, and $AC4_1^{**}$ are satisfied.

Using Lemma 8, it is easy to check that $AC2_1^*$, $AC3_1^*$, and $AC4_1^*$ are satisfied. Indeed, since $\succsim_1^-$ is a clique, the premise of $AC2_1^*$ and $AC4_1^*$ is never satisfied, while the conclusion $AC3_1^*$ always holds.

Let us check that $AC1_1^{**}$ holds.

The second premise of $AC1_1^{**}$ is $(y_1, b_{-1}) \succsim d$. If $y_1 = x$ then since all elements of X_1 are above x according to $\succsim_1^+$, we will have that $(z_1, b_{-1}) \succsim d$, for all $z_1 \in X_1$.

Similarly, the first premise of $AC1_1^{**}$ is $(x_1, a_{-1}) \succsim c$. If $x_1 = x$ then since all elements of X_1 are above x according to $\succsim_1^+$, we will have that $(y_1, a_{-1}) \succsim c$, for all $y_1 \in X_1$.

This shows that the only possible violations of $AC1_1^{**}$ will occur if either $x_1 = y$ and $y_1 = z$ or $x_1 = z$ and $y_1 = y$.

Let us consider the first case with $x_1 = y$ and $y_1 = z$. The premises of $AC1_1^{**}$ state that $(y, a_{-1}) \succsim c$, $(z, b_{-1}) \succsim d$, and $(y, b_{-1}) \succsim d$. It is easy to check that whenever we have both $(z, b_{-1}) \succsim d$ and $(y, b_{-1}) \succsim d$, we also have that $(x, b_{-1}) \succsim d$. Hence, the second conclusion of $AC1_1^{**}$ is satisfied.

Let us now consider the second case in which $x_1 = z$ and $y_1 = y$. The premises of $AC1_1^{**}$ state that $(z, a_{-1}) \succsim c$, $(y, b_{-1}) \succsim d$, and $(z, b_{-1}) \succsim d$. As observed above, whenever we have both $(y, b_{-1}) \succsim d$ and $(z, b_{-1}) \succsim d$, we also have that $(x, b_{-1}) \succsim d$. Hence, the second conclusion of $AC1_1^{**}$ is satisfied.

Hence, $AC1_1^{**}$ holds.

Transposing the boolean matrix in the above example, one easily obtain an example satisfying all our conditions except $AC2$ on one attribute. This is detailed below.

Example 6. Let $\succsim$ on X be defined by the following table:

	xaq	zaq	yaq	xap	yap	zap	xbq	zbq	ybq	xbp	ybp	zbp
xaq	1	1	1	1	1	1	1	1	1	1	1	1
zaq	1	1	1	1	1	1	1	1	1	1	1	1
yaq	1	1	1	1	1	1	1	1	1	1	1	1
xap	0	0	1	1	1	1	1	1	1	1	1	1
yap	0	0	1	1	1	1	1	1	1	1	1	1
zap	0	0	1	1	1	1	1	1	1	1	1	1
xbq	0	0	0	0	0	1	1	1	1	1	1	1
zbq	0	0	0	0	0	1	1	1	1	1	1	1
ybq	0	0	0	0	0	1	1	1	1	1	1	1
xbp	0	0	0	0	0	0	0	0	1	1	1	1
ybp	0	0	0	0	0	0	0	0	1	1	1	1
zbp	0	0	0	0	0	0	0	0	1	1	1	1

It is clear that this relation is a semiorder.

It is easy to check that the traces are as follows.

On attribute 2, we have $a \succ_2^+ b$ and $a \succ_2^- b$. Hence, $AC1_2$, $AC2_2$, and $AC3_2$ hold. Since X_2 has only two elements, $AC1_2^{**}$, $AC2_2^{**}$, $AC3_2^{**}$, and $AC4_2^{**}$ trivially hold.

On attribute 3, we have $q \succ_3^+ p$ and $q \succ_3^- p$. Hence, $AC1_3$, $AC2_3$, and $AC3_3$ hold. Since X_3 has only two elements, $AC1_3^{**}$, $AC2_3^{**}$, $AC3_3^{**}$, and $AC4_3^{**}$ trivially hold.

On attribute 1, $\succsim_1^+$ is a clique. Furthermore, we have $x \succ_1^- y$ and $x \succ_1^- z$, but neither $y \succsim_1^- z$ nor $z \succsim_1^- y$. It easy to check that this implies that $AC1_1$, and $AC3_1$ hold, while $AC2_1$ is violated.

It remains to check that $AC1_1^{**}$, $AC2_1^{**}$, $AC3_1^{**}$, and $AC4_1^{**}$ are satisfied.

Using Lemma 8, it is easy to check that $AC1_1^*$, $AC3_1^*$, and $AC4_1^*$ are satisfied. Indeed, since $\succsim_1^+$ is a clique, the premise of $AC1_1^*$ and $AC3_1^*$ is never satisfied, while the conclusion $AC4_1^*$ always holds.

Let us check that $AC2_1^{**}$ holds.

The second premise of $AC2_1^{**}$ is $d \succsim (z_1, b_{-1})$. If $z_1 = x$ then since all elements of X_1 are below x according to $\succsim_1^-$, we will have that $d \succsim (y_1, b_{-i})$, for all $y_1 \in X_1$.

Similarly, the first premise of $AC2_1^{**}$ is $c \succsim (y_1, a_{-1})$. If $y_1 = x$ then, since all elements of X_1 are below x according to $\succsim_1^-$, we will have that $c \succsim (x_1, a_{-1})$, for all $x_1 \in X_1$.

This shows that the only possible violations of $AC2_1^{**}$ will occur if either $y_1 = y$ and $z_1 = z$ or $y_1 = z$ and $z_1 = y$.

Let us consider the first case with $y_1 = y$ and $z_1 = z$. The premises of $AC2_1^{**}$ state that $c \succsim (y, a_{-1})$, $d \succsim (z, b_{-1})$, and $c \succsim (z, a_{-1})$. It is easy to check that whenever we have both $c \succsim (y, a_{-1})$ and $c \succsim (z, a_{-1})$ we also have $c \succsim (x, a_{-1})$. Hence, the first conclusion of $AC2_1^{**}$ will hold.

Let us now consider the second case in which $y_1 = z$ and $z_1 = y$. The premises of $AC2_1^{**}$ state that $c \succsim (z, a_{-1})$, $d \succsim (y, b_{-1})$, and $c \succsim (y, a_{-1})$. As observed above, whenever we have both $c \succsim (z, a_{-1})$ and $c \succsim (y, a_{-1})$, we also have that $c \succsim (x, a_{-1})$. Hence, the first conclusion of $AC2_1^{**}$ is satisfied.

Hence, $AC2_1^{**}$ holds.

Example 7. Let $\succsim$ on X be defined by the following table:

	xaq	yaq	xap	xbp	xbq	yap	ybp	ybq	zap	zaq	zbp	zbq
xaq	1	1	1	1	1	1	1	1	1	1	1	1
yaq	1	1	1	1	1	1	1	1	1	1	1	1
xap	1	1	1	1	1	1	1	1	1	1	1	1
xbp	1	1	1	1	1	1	1	1	1	1	1	1
xbq	1	1	1	1	1	1	1	1	1	1	1	1
yap	1	1	1	1	1	1	1	1	1	1	1	1
ybp	1	1	1	1	1	1	1	1	1	1	1	1
ybq	1	1	1	1	1	1	1	1	1	1	1	1
zap	1	1	1	1	1	1	1	1	1	1	1	1
zaq	1	1	1	1	1	1	1	1	1	1	1	1
zbp	1	1	1	1	1	1	1	1	1	1	1	1
zbq	0	0	1	1	1	1	1	1	1	1	1	1

This relation is a semiorder.

It is easy to check that the traces are as follows.

On attribute 2, we have $a \succ_2^+ b$ and $a \succ_2^- b$. Hence, $AC1_2$, $AC2_2$, and $AC3_2$ hold. Since X_2 has only two elements, $AC1_2^{**}$, $AC2_2^{**}$, $AC3_2^{**}$, and $AC4_2^{**}$ trivially hold.

On attribute 1, we have $[x \sim_1^+ y] \succ_1^+ z$ and $[x \sim_1^- y] \succ_1^- z$. Using Lemma 8, it is easy to check that $AC1_1^*$, $AC2_1^*$, $AC3_1^*$, and $AC4_1^*$ hold. This shows that $AC1_1$, $AC2_1$, $AC3_1$, $AC1_1^{**}$, $AC2_1^{**}$, $AC3_1^{**}$, and $AC4_1^{**}$ hold.

On attribute 3, we have $p \succ_3^+ q$ and $q \succ_3^- p$. Hence, $AC1_3$ and $AC2_3$ hold but $AC3_3$ is violated. Since X_3 has only two elements, $AC1_3^{**}$, $AC2_3^{**}$, $AC3_3^{**}$, and $AC4_3^{**}$ trivially hold.

A11. Proof of Lemma 15

This is obvious since a CR-AT does not impose a limit on the number of equivalence classes of the relations $\succsim_i^{\pm}$. For instance, let $n = 3$, $X_1 = X_2 = X_3 = \mathbb{R}$, and $X = \prod_{i=1}^{n} X_i$. For all $i \in N$, let T_i be identical to $\geq$. Consider the relation $\succsim$ on X such that

$$x \succsim y \Leftrightarrow |T(x, y)| \geq |T(y, x)|$$

By construction, this relation is a CR-AT. It is not a RSRP. Indeed, it is easy to see that for all $i \in N$, we have $\succsim_i^{\pm} = \geq$. But we know that with a RSRP, all relations $\succsim_i^{\pm}$ have two distinct equivalence classes.

A12. Proof of Lemma 16

We know from Lemma 6 that in a RSRP all relations $\succsim_i^{\pm}$ are weak orders. Hence, in view of the comments following Definition 8, it is enough to show that all relations $\succsim_i^{**}$ are weak orders having at most three equivalence classes.

It is easy to show that, for all $i \in N$ and all $x_i, x'_i, y_i, y'_i, z_i, z'_i, w_i, w'_i \in X_i$, $(x_i, y_i) \succsim_i^{**} (z_i, w_i)$, $x'_i \succsim_i^{\pm} x_i$, $y_i \succsim_i^{\pm} y'_i$, $z_i \succsim_i^{\pm} z'_i$, $w'_i \succsim_i^{\pm} w_i$, imply $(x'_i, y'_i) \succsim_i^{**} (z'_i, w'_i)$ (Bouyssou and Pirlot 2005a, Lemma 3.8). In particular, using the reflexivity of $\succsim_i^{**}$, if $x_i \sim_i^{\pm} z_i$ and $y_i \sim_i^{\pm} w_i$ we have $(x_i, y_i) \sim_i^{**} (z_i, w_i)$.

Let $a_i, b_i \in X_i$. For the ordered pair (a_i, b_i), it is clear that we have one of the following four situations:

1. $a_i \in \mathcal{A}_i$ and $b_i \in \mathcal{A}_i$,
2. $a_i \in \mathcal{A}_i$ and $b_i \notin \mathcal{A}_i$,
3. $a_i \notin \mathcal{A}_i$ and $b_i \in \mathcal{A}_i$,
4. $a_i \notin \mathcal{A}_i$ and $b_i \notin \mathcal{A}_i$.

Consider two ordered pairs (x_i, y_i) and (z_i, w_i) of elements in X_i. If these two ordered pairs are in the same situation, we have $x_i \sim_i^{\pm} z_i$ and $y_i \sim_i^{\pm} w_i$, so that $(x_i, y_i) \sim_i^{**} (z_i, w_i)$.

All ordered pairs (x_i, y_i) in the second situation are above all ordered pairs (z_i, w_i) in the first situation in terms of $\succsim_i^{**}$. Indeed, in this case, we know that $[x_i \sim_i^{\pm} z_i \sim_i^{\pm} w_i] \succ_i^{\pm} y_i$. Since $\succsim_i^{**}$ is reflexive, we have $(z_i, w_i) \succsim_i^{**} (z_i, w_i)$. Using the fact that $x_i \succsim_i^{\pm} z_i$ and $w_i \succsim_i^{\pm} y_i$, we obtain $(x_i, y_i) \succsim_i^{**} (z_i, w_i)$. A similar reasoning shows that all ordered pairs (x_i, y_i) in the second situation are above all ordered pairs (z_i, w_i) in the in the fourth situation in terms of $\succsim_i^{**}$.

Similarly, it is easy to check that all ordered pairs (x_i, y_i) in the first situation are above all ordered pairs (z_i, w_i) in the in the third situation in terms of $\succsim_i^{**}$ and that all ordered pairs (x_i, y_i) in the fourth situation are above all ordered pairs (z_i, w_i) in the in the third situation in terms of $\succsim_i^{**}$.

Hence, the relation $\succsim_i^{**}$ will be complete iff the elements in the first and fourth situations are equivalent in terms of $\succsim_i^{**}$. Using Lemma 2.4, it is easy to see that this is equivalent to requiring that $\succsim$ is independent. Observe that, when this is the case, $\succsim_i^{**}$ has at most three distinct equivalence classes.

A13. Proof of Lemma 17

$[AC2_i^* \Rightarrow AC1_i^*]$ Suppose that $AC1_i^*$ is violated, so that $(x_i, a_{-i}) \succsim c$, $(y_i, b_{-i}) \succsim d$, $(y_i, a_{-i}) \not\succsim c$, $(z_i, b_{-i}) \not\succsim d$. Using the fact that $\succsim$ is a weak order, we obtain $(x_i, a_{-i}) \succ (y_i, a_{-i})$ and $(y_i, b_{-i}) \succ (z_i, b_{-i})$. Since $\succsim$ is reflexive, we have $(y_i, a_{-i}) \succsim (y_i, a_{-i})$ and $(z_i, b_{-i}) \succsim (z_i, b_{-i})$. Applying $AC2_i^*$, we obtain $(y_i, a_{-i}) \succsim (x_i, a_{-i})$ or $(z_i, b_{-i}) \succsim (y_i, b_{-i})$, a contradiction.

$[AC3_i^* \Rightarrow AC2_i^*]$ Suppose that $AC2_i^*$ is violated, so that $c \succsim (y_i, a_{-i})$, $d \succsim (z_i, b_{-i})$, $c \not\succsim (x_i, a_{-i})$, and $d \not\succsim (y_i, b_{-i})$. Using the fact that $\succsim$ is a weak order, we obtain $(x_i, a_{-i}) \succ (y_i, a_{-i})$ and $(y_i, b_{-i}) \succ (z_i, b_{-i})$. Since $\succsim$ is reflexive, we have $(x_i, a_{-i}) \succsim (x_i, a_{-i})$ and $(z_i, b_{-i}) \succsim (z_i, b_{-i})$. Applying $AC3_i^*$, we obtain $(y_i, a_{-i}) \succsim (x_i, a_{-i})$ or $(z_i, b_{-i}) \succsim (y_i, b_{-i})$, a contradiction.

$[AC4_i^* \Rightarrow AC3_i^*]$ Suppose that $AC3_i^*$ is violated, so that $(x_i, a_{-i}) \succsim c$, $d \succsim (z_i, b_{-i})$, $(y_i, a_{-i}) \not\succsim c$, $d \not\succsim (y_i, b_{-i})$. Using the fact that $\succsim$ is a weak order, we obtain $(y_i, b_{-i}) \succ (z_i, b_{-i})$ and $(x_i, a_{-i}) \succ (y_i, a_{-i})$. Since $\succsim$ is reflexive, we have $(y_i, a_{-i}) \succsim (y_i, a_{-i})$ and $(y_i, b_{-i}) \succsim (y_i, b_{-i})$. Applying $AC4_i^*$, we obtain $(y_i, a_{-i}) \succsim (x_i, a_{-i})$ or $(z_i, b_{-i}) \succsim (y_i, b_{-i})$, a contradiction.

$[AC1_i^* \Rightarrow AC4_i^*]$ Suppose that $AC4_i^*$ is violated, so that $c \succsim (y_i, a_{-i})$, $(y_i, b_{-i}) \succsim d$, $c \not\succsim (x_i, a_{-i})$, $(z_i, b_{-i}) \not\succsim d$. Using the fact that $\succsim$ is a weak order, we obtain $(x_i, a_{-i}) \succ (y_i, a_{-i})$ and $(y_i, b_{-i}) \succ (z_i, b_{-i})$. Since $\succsim$ is reflexive, we have $(x_i, a_{-i}) \succsim (x_i, a_{-i})$ and $(y_i, b_{-i}) \succsim (y_i, b_{-i})$. Applying $AC1_i^*$, we obtain $(y_i, a_{-i}) \succsim (x_i, a_{-i})$ or $(z_i, b_{-i}) \succsim (y_i, b_{-i})$, a contradiction.

A14. Proof of Lemma 18

We have to show that in the class of all weak orders on X, conditions $AC1$ and $AC1^{**}$ are independent. We need two examples. The first gives an example of a weak order satisfying $AC1$ and $AC1_i^{**}$ on all but one of the attributes. The second gives an example of a weak order satisfying $AC1^{**}$ and $AC1_i$ on all but one of the attributes.

Example 8. Let $\succsim$ on X be the weak order obtained using an additive value function model with the following value functions: $v_1(x) = 0$, $v_1(y) = 1$, $v_1(z) = 2$, $v_2(a) = 0$, $v_2(b) = 1$, $v_3(p) = 0$, and $v_3(q) = 1$. It is easy to check that, for this weak order, we have: $z \succ_1^+ y \succ_1^+ x$, $b \succ_2^+ a$, and $q \succ_3^+ p$. This shows that conditions $AC1$, $AC1_2^{**}$, and $AC1_3^{**}$ hold. By construction, $AC1_1^{**}$ is violated.

Example 9. Let $\succsim$ on X be defined by the following table:

	xbp	xbq	ybp	ybq	zap	zbp	zbq	xap	yap	yaq	xaq	zaq
xbp	1	1	1	1	1	1	1	1	1	1	1	1
xbq	1	1	1	1	1	1	1	1	1	1	1	1
ybp	1	1	1	1	1	1	1	1	1	1	1	1
ybq	1	1	1	1	1	1	1	1	1	1	1	1
zap	1	1	1	1	1	1	1	1	1	1	1	1
zbp	1	1	1	1	1	1	1	1	1	1	1	1
zbq	1	1	1	1	1	1	1	1	1	1	1	1
xap	0	0	0	0	0	0	0	1	1	1	1	1
yap	0	0	0	0	0	0	0	1	1	1	1	1
yaq	0	0	0	0	0	0	0	1	1	1	1	1
xaq	0	0	0	0	0	0	0	0	0	0	1	1
zaq	0	0	0	0	0	0	0	0	0	0	1	1

It is clear that this relation is the weak order on X such that:

$$[xbp, xbq, ybp, ybq, zap, zbp, zbq] \succ [xap, yap, yaq] \succ [xaq, zaq]$$

It is easy to check that we have $b \succ_2^+ a$ and $p \succ_3^+ q$. Using Lemma 8 shows that $AC1_i^*$ and, hence, $AC1_i$ and $AC1_i^{**}$, hold for $i = 2, 3$.

On attribute 1, we have $y \succ_1^+ x$, $z \succ_1^+ x$, but neither $y \succ_1^+ z$ nor $z \succ_1^+ y$, so that $AC1_1$ is violated.

Let us check that $AC1_1^{**}$ holds.

The second premise of $AC1_1^{**}$ is $(y_1, b_{-1}) \succsim d$. If $y_1 = x$ then, since all elements of X_1 are above x according to $\succsim_1^+$, we will have that $(z_1, b_{-1}) \succsim d$, for all $z_1 \in X_1$.

Similarly, the first premise of $AC1_1^{**}$ is $(x_1, a_{-1}) \succsim c$. If $x_1 = x$ then, since all elements of X_1 are above x according to $\succsim_1^+$, we will have that $(y_1, a_{-1}) \succsim c$, for all $y_1 \in X_1$.

This shows that the only possible violations of $AC1_1^{**}$ will occur if either $x_1 = y$ and $y_1 = z$ or $x_1 = z$ and $y_1 = y$.

Let us consider the first case with $x_1 = y$ and $y_1 = z$. The premises of $AC1_1^{**}$ state that $(y, a_{-1}) \succsim c$, $(z, b_{-1}) \succsim d$, and $(y, b_{-1}) \succsim d$. It is easy to check that whenever we have both $(z, b_{-1}) \succsim d$ and $(y, b_{-1}) \succsim d$, we also have that $(x, b_{-1}) \succsim d$. Hence, the second conclusion of $AC1_1^{**}$ is satisfied.

Let us now consider the second case in which $x_1 = z$ and $y_1 = y$. The premises of $AC1_1^{**}$ state that $(z, a_{-1}) \succsim c$, $(y, b_{-1}) \succsim d$, and $(z, b_{-1}) \succsim d$. As observed above, whenever we have both $(y, b_{-1}) \succsim d$ and $(z, b_{-1}) \succsim d$, we also have that $(x, b_{-1}) \succsim d$. Hence, the second conclusion of $AC1_1^{**}$ is satisfied.

Hence, $AC1_1^{**}$ holds.

A15. Proof of Lemma 19

We know that a RSRP that is a weak order only has a finite number of distinct equivalence classes. Hence, the proof follows from Proposition 4 and the fact that $AC1^*$ implies 2^*-graded.

A16. Proof of Lemma 20

Let us show that $AC1$ holds. Suppose that $(x_i, a_{-i}) \succsim c$ and $(y_i, b_{-i}) \succsim d$. Because R_i is a semiorder, we know (see the end of Section 2.1) that the relation R_i° is a weak order. By construction, we have either $x_i \; R_i^\circ \; y_i$ or $y_i \; R_i^\circ \; x_i$.

Suppose that $x_i \; R_i^\circ \; y_i$. By construction, for all $k \in L$, $y_i \; R_i \; \pi_i^k$ implies $x_i \; R_i \; \pi_i^k$. This implies $\mathcal{R}^L((x_i, b_{-i})) \sqsupseteq \mathcal{R}^L((y_i, b_{-i}))$. Since $(y_i, b_{-i}) \succsim d$, we know that $\mathcal{R}^L((y_i, b_{-i})) \trianglerighteq_L \mathcal{R}^L(d)$. Using the fact that $\trianglerighteq_L$ is monotonic w.r.t. inclusion, we obtain $\mathcal{R}^L((x_i, b_{-i})) \trianglerighteq_L \mathcal{R}^L(d)$, so that $(x_i, b_{-i}) \succsim d$. A similar proof shows that if $y_i \; R_i^\circ \; x_i$ then we have $(y_i, a_{-i}) \succsim c$. Hence, $AC1$ holds.

The proof for $AC2$ and $AC3$ is similar.

Let us show that the weak order $\succsim_i^\pm$ has at most $\ell + 1$ distinct equivalence classes. For all $k \in L$, define $\mathcal{A}_i^k = \{x_i \in X_i : x_i \; R_i \; \pi_i^k\}$. Because R_i is a semiorder, the relation R_i° is a weak order. Let $k, k' \in L$. We have either $\pi_i^k \; R_i^\circ \; \pi_i^{k'}$ or $\pi_i^{k'} \; R_i^\circ \; \pi_i^k$. Hence, we have either $\mathcal{A}_i^k \subseteq \mathcal{A}_i^{k'}$ or $\mathcal{A}_i^{k'} \subseteq \mathcal{A}_i^k$. This shows that, for all $i \in N$, the sets $\mathcal{A}_i^k$, $k \in L$ are nested. Moreover, it is clear that if $x_i, y_i \in X_i$ belong exactly to the same subsets $\mathcal{A}_i^k$, $k \in L$, we must have $x_i \sim_i^\pm y_i$. This completes the proof since the sets $\mathcal{A}_i^k$ are nested.

A17. Proof of Lemma 21

For all $i \in N$, we know that $\succsim_i^\pm$ is a weak order and that $X_i/{\sim_i^\pm}$ is finite. Let ℓ_i be the number of distinct equivalence classes of $X_i/{\sim_i^\pm}$. Let $\ell = \max_{i \in N} \ell_i$. The proof will be complete if we show that $\succsim$ is a RMRP.

By construction, there is at least one $i \in N$ such that $\succsim_i^\pm$ has exactly ℓ distinct equivalence classes. On all such attributes $i \in N$, we define E_i^k, for $k = 1, 2, \ldots, \ell$, as the set containing all elements in the kth equivalence class of $\succsim_i^\pm$. Consider now an attribute $j \in N$ such that $\succsim_j^\pm$ has ℓ_j distinct equivalence classes with $\ell_j < \ell$. For all such attributes $j \in N$, we define E_j^k, for $k = 1, 2, \ldots, \ell_j$, as the set containing all elements in the kth equivalence class of $\succsim_j^\pm$. For $k = \ell_j + 1, \ell_j + 2, \ldots, \ell$, we define $E_j^k = E_j^{\ell_j}$.

For all $i \in N$, let us take $R_i = \succsim_i^\pm$.

We use ℓ profiles $\pi^1, \pi^2, \ldots, \pi^\ell$ that are build as follows. For all $i \in N$ and $k \in L$, let π_i^k be any element belonging to $E_i^{\ell - k + 1}$.

We define the binary relation $\trianglerighteq_L$ on $(2^N)^\ell$ letting $(A^1, \ldots, A^\ell) \trianglerighteq_L (B^1, \ldots, B^\ell)$ whenever there are $x, y \in X$ such that $x \succsim y$, $\mathcal{R}^L(x) = (A^1, \ldots, A^\ell)$, and $\mathcal{R}^L(y) = (B^1, \ldots, B^\ell)$.

We claim that the family of reference points $\pi^1, \pi^2, \ldots, \pi^\ell$ together with the weak orders $R_i = \succsim_i^\pm$ and the relation $\trianglerighteq_L$ is a representation of $\succsim$ in model (RMRP).

Suppose first that $x \succsim y$. Then, by construction, we have $\mathcal{R}^L(x) \trianglerighteq_L \mathcal{R}^L(y)$.

Suppose now that $(A^1, \ldots, A^\ell) \trianglerighteq_L (B^1, \ldots, B^\ell)$ and consider $x, y \in X$ such that $\mathcal{R}^L(x) = (A^1, \ldots, A^\ell)$, and $\mathcal{R}^L(y) = (B^1, \ldots, B^\ell)$. Because $(A^1, \ldots, A^\ell) \trianglerighteq_L (B^1, \ldots, B^\ell)$, we know that there are $z, w \in X$ such that $\mathcal{R}^L(z) = (A^1, \ldots, A^\ell)$, $\mathcal{R}^L(w) = (B^1, \ldots, B^\ell)$, and $z \succsim w$. Given the construction of the profiles $\pi^1, \pi^2, \ldots, \pi^\ell$, $\mathcal{R}^L(x) = \mathcal{R}^L(z)$ implies that, for all $i \in N$, $x_i \sim_i^{\pm} z_i$. Similarly, $\mathcal{R}^L(y) = \mathcal{R}^L(w)$ implies that, for all $i \in N$, $y_i \sim_i^{\pm} w_i$. Using (4), we obtain $x \succsim y$.

It remains to show that $\trianglerighteq_L$ is monotonic w.r.t. inclusion.

Suppose that $(A^1, \ldots, A^\ell) \trianglerighteq_L (B^1, \ldots, B^\ell)$. This implies that $x \succsim y$, for some $x, y \in X$ such that $\mathcal{R}^L(x) = (A^1, \ldots, A^\ell)$, and $\mathcal{R}^L(y) = (B^1, \ldots, B^\ell)$. Take any $(C^1, \ldots, C^\ell)$ and $(D^1, \ldots, D^\ell)$ such that, $k \in L$, $C^k \supseteq A^k$, $B^k \supseteq D^k$, and there are $z, w \in X$ such that $\mathcal{R}^L(z) = (C^1, \ldots, C^\ell)$, and $\mathcal{R}^L(w) = (D^1, \ldots, D^\ell)$. By construction of the family of reference points $\pi^1, \pi^2, \ldots, \pi^\ell$, this implies that $z_i \succsim_i^{\pm} x_i$ and $y_i \succsim_i^{\pm} w_i$, for all $i \in N$. Using, (3), we obtain $z \succsim w$, so that $(C^1, \ldots, C^\ell) \trianglerighteq_L (D^1, \ldots, D^\ell)$, as required.