

Rank Dependent Weighted Average Utility Models for decision-making under Objective Ambiguity^{*†}

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September 7, 2026

JEL Codes: D80, D81

Keywords: Ambiguity, Prospects, Sets, Axioms, Ranks,
Weights, Utility

Abstract

This paper provides an axiomatic characterization of a family of rank-dependent weighted average utility criteria for comparing finite sets of objective probability distributions over consequences. These sets are interpreted as objectively ambiguous prospects: the relevant probability distributions are part of the description of the alternative,

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[†]The project leading to this article has received funding from the French government under the “France 2030” investment plan managed by the French National Research Agency (reference: ANR-17-EURE-0020) and from the Excellence Initiative of Aix-Marseille University - A*MIDEX. We are also grateful to Peter Wakker, associate editor Faruk Gul and two anonymous referees for their extremely useful comments on an earlier version. The usual disclaimer applies.

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but no likelihood is assigned to any of them. The resulting criteria evaluate a prospect by applying a utility function to each of its elements and taking a weighted average of the resulting utilities, where the weights depend on the rank of each element within the prospect. The main theorem characterizes the subclass of such criteria whose rank-dependent weights are generated by a single weighting function on the unit interval. A second characterization covers the larger family in which the weights may vary freely with the number of elements in the prospect. The paper also examines how comparative ambiguity aversion constrains utility functions and rank-dependent weights, and relates optimism and pessimism to the monotonicity of these weights.

1 Introduction

Consider the following variant of a decision problem discussed by Ahn (2008). A cancer patient has to choose between two treatments. The first is a conventional and widely used chemotherapeutic treatment associated with a five-year survival rate of 0.5. The second is a new targeted therapy that has only been tried on two small samples of patients of comparable size. In one sample, 80% of the patients were still alive after 5 years, but in the other sample, only 20% of the patients were alive after 5 years. This is an example of decision-making under *objective ambiguity*. There is *ambiguity* because the probabilities that appear in the description of the second treatment are not unique. The ambiguity, however, is *objective* because the probabilities, while multiple, are known to the decision-maker and are therefore part of the description of the decision problem. This contrasts with decision-making under subjective ambiguity studied in papers such as Gilboa and Schmeidler (1989), Epstein and Zhang (2001), Ghirardato and Marinacci (2002), Ghirardato, Maccheroni, and Marinacci (2004) or Klibanoff, Marinacci, and Mukerji (2005) in which the compared alternatives are described as Savagean

acts without any *a priori* probabilities.¹ Another well-known example of an objectively ambiguous decision is the sequence of two choices made in the Ellsberg (1961) experiment. A very simplified version of this experiment involves two urns *A* and *B*, each with two balls. Urn *A* has one black and one white ball, while *B* has an unknown distribution of its two balls between the two colors. The decision-maker is asked to choose the urn from which they would like a draw to be made on the understanding that they will get \$1,000 if the outcome is black and zero otherwise.

Decision-making under objective ambiguity amounts to ordering sets of objective probability distributions—or lotteries—over final consequences. It does not invoke the apparatus of Savagean acts and therefore differs from the above-mentioned literature. It also differs from the literature that examines how decision-makers compare Savagean acts mapping states of nature into sets of objective probability distributions (Ghirardato, 2001; Gajdos, Hayashi, Tallon, and Vergnaud, 2008; Gul and Pesendorfer, 2014; Vierø, 2009). The description of alternatives as sets of objects is also made in the literature on decision-making under radical uncertainty or ignorance surveyed, for example, in Barberà, Bossert, and Pattanaik (2004), in which the elements of the sets are interpreted as final consequences of the decisions.

This paper contributes to the literature on decision-making under ignorance or objective ambiguity—we shall concentrate on the latter—under the additional assumption that the compared alternatives are described as *finite* sets. Our approach therefore differs in that respect from Ahn (2008) and Olszewski (2007) in which the compared alternatives are depicted as *atomless* sets. In Ahn’s version of the above medical example, the patient is told nothing about the number of samples on which the new treatment has been tested. Instead the patient is simply told that the new treatment gives a probability of survival ranging between 0.2 and 0.8 and the choice the pa-

¹An excellent survey of the literature on subjective ambiguity is provided in Wakker (2010).

tient must make is between a non-ambiguous 0.5 survival probability and the whole continuum of survival probabilities ranging from 0.2 to 0.8. As argued in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2018), we believe that the description of alternatives as finite sets of probability distributions is quite natural. In the medical example, any evidence that a physician could have about the effectiveness of a treatment can only come from a finite collection of samples of patients on which it has been tried. The Ellsberg experiment, certainly, also involves finite sets.

In such a finite setting, Gravel, Marchant, and Sen (2012) axiomatically characterized the Uniform Expected Utility (UEU) family of criteria for comparing finite sets of objects. A UEU criterion compares sets on the basis of an unweighted (uniform) expected utility of their elements. Gravel, Marchant, and Sen (2018) generalized the UEU family of orderings of finite sets to the Conditional Expected Utility (CEU) family. Any CEU ordering assigns to every conceivable element of a set both a utility and a (strictly positive) likelihood, and compares sets on the basis of their expected utility, with expectations taken with respect to the relative likelihood of those elements conditional on their belonging to the set. A UEU ordering of sets is simply a CEU ordering whose likelihood function treats all conceivable elements as equally likely. CEU orderings can be viewed as the finite analogues of the orderings of atomless sets characterized by Ahn (2008) and, before him, by Bolker (1966) and Jeffrey (1983).

Both UEU and CEU criteria satisfy an axiom that may be at odds with actual decision-making behavior. This axiom, called *Averaging* by Gravel, Marchant, and Sen (2012, 2018) and Fishburn (1972) or disjoint set betweenness by Ahn (2008)², requires that the union of two disjoint sets be ranked between the two sets. In the only experimental study we are aware of in which the Averaging axiom has been tested (Vrijdags and Marchant, 2015),

²Weaker variants of this axiom are also satisfied by the orderings examined in Olszewski (2007) and, in another context, in Gul and Pesendorfer (2001).

it has been violated by a large majority of subjects.

In this paper, we characterize a family of criteria that generalizes the UEU family while dispensing with the Averaging axiom. Criteria of this kind have been discussed—but not characterized—by Vrijdags and Marchant (2015) under the name ‘Uniform Rank-Dependent Utility’ criteria. In this paper, we use the more explicit name of Rank-Dependent Weighted Average Utility (RDWAU) to designate them. The characterized family can be viewed as a generalized variant of the rank-dependent expected utility family originally proposed by Quiggin (1982) that is suitable to our finite set-theoretic setting.

An RDWAU criterion compares finite sets on the basis of their weighted average utility, for some utility function defined over all conceivable elements of the sets and some non-negative weights, summing to one, that depend both on the number of elements in the sets and on the rank of those elements if they were non-ambiguous. Our main result characterizes the somewhat narrower class of RDWAU criteria for which all rank-dependent weights result from a non-negative weighting function defined on all rational numbers in the unit interval. Orderings whose rank-dependent weights are generated by such a weighting function have been extensively discussed in the literature on rank-dependent expected utility models (e.g. Luce, 1988, 1991; Tversky and Wakker, 1995; Prelec, 1998) in contexts where decision-makers are comparing single lotteries rather than sets of outcomes.

The class of such criteria is quite large. It contains as particular cases all UEU criteria. It also contains rankings of sets based on their worst, or on their best, elements, as well as those that result from a weighted average of those two rankings (such as the α -maxmin rule discussed by Hartmann (2023) or Olszewski (2007), among others). We examine what the fact that one of two decision-makers is no less ambiguity averse than the other implies for their utility functions and rank-dependent weights. We also identify the restrictions on the rank-dependent weights implied by the optimism or the pessimism of the decision-maker.

The description of prospects as sets of possible probability distributions does not provide *a priori* information about the likelihoods of those probability distributions. This does not, of course, preclude the the decision-maker from subjectively assigning these likelihoods before ranking the prospects. After all, the rank-dependent weights used by RDWAU decision-makers can be interpreted as such likelihood assignments. There is a literature (see notably Segal, 1990) on two-stage lotteries, where the decision-maker ranks “lotteries over lotteries” and the first stage lottery is objective. In the current setting, one could view for example the first choice of the Ellsberg experiment—which we describe as the choice between $\{1/2\}$ and $\{0, 1/2, 1\}$ —as a choice between $\{(1, 1/2)\}$ and $\{(1/3, 0), (1/3, 1/2), (1/3, 1)\}$, where (x, y) denotes the outcome in which objective lottery y is assigned a probability x by the decision-maker. The approach of the paper could certainly apply to such a setting if the elements of the sets were interpreted in this way as pairs of probabilities. But we do not require such an interpretation. An Ellsberg decision-maker who considers the draw of a black ball from urn B in the above example may have no information whatsoever about the likelihood that, say, the two balls are white. She only knows that the three possibilities exist (i.e. both balls are black, both white or one of each) without any further capability of appraising their relative likelihood.

There are, however, plausible circumstances in which some of the possible probability distributions that describe a prospect could have more *a priori* evidence in their favor than others. In the medical example, one could imagine that the new treatment had been tested on three samples instead of two and had given a probability of survival of $4/5$ in two samples, and a probability of survival of $1/5$ in the third. In that case, the patient would be asked to choose between the standard treatment $\{1/2\}$, and the new treatment $\{1/5, 4/5, 4/5\}$. Presumably, the decision-maker would not make the same choice when facing these two alternatives as when facing $\{1/2\}$ and $\{1/5, 4/5\}$. Observe that the alternative $\{1/5, 4/5, 4/5\}$ is not a set, since

4/5 appears more than once. It is a multiset (see e.g. Blizard (1989)). In an Appendix of this paper, we show that the main characterization result of this paper obtained for comparisons of sets of objects (lotteries) also applies *mutatis mutandis* to multisets.

The organization of the remainder of the paper is as follows. The next section introduces the framework, notation and main definitions. Section 3 provides the main result, and Section 4 discusses the implications of aversion to ambiguity. Section 5 discusses some specific families of RDWAU orderings, in particular the optimistic and pessimistic families. Section 6 concludes.

2 The Framework

2.1 Notation and definitions

We let X be a universe of *outcomes* that we interpret either as possible final consequences of decisions (ignorance) or as probability distributions over a more fundamental set of final consequences (objective ambiguity). We nonetheless use the generic term of outcomes to designate these elements of X . Many results proved in this paper also rely on the assumption that X is a connected topological space, as would be the case for most sets of probability distributions over a set of final consequences that we can think of. An alternative or *prospect* is a finite non-empty subset D of X . We denote by $\mathcal{P}(X)$ the set of all such prospects. Prospects consisting of a single outcome (singletons) are naturally interpreted as *certain* or *non-ambiguous*.

For all integers m and n such that $m \leq n$, the set $\{m, m+1, \dots, n\}$ is denoted by $[m, n]$. When $m = 1$, we simply write $[n]$ instead of $[1, n]$. A set of the form $\{m, m+1, \dots\}$ is denoted by $[m, \cdot]$. The set $[1, \cdot]$ is also denoted by $\mathbb{N}$. The real and rational unit intervals are respectively denoted by $[0, 1]_{\mathbb{R}}$ and $[0, 1]_{\mathbb{Q}}$.

Prospects are compared by a binary relation $\succsim$ (“weakly preferred to”) on $\mathcal{P}(X)$. As usual, the asymmetric and symmetric factors of $\succsim$ are respectively

denoted by $\succ$ and $\sim$. For now, we do not impose any property on $\succsim$. For reasons that will soon become clear, whenever we write a prospect D in $\mathcal{P}(X)$ with n possible outcomes in the form $D = \{d_1, \dots, d_n\}$, we label the outcomes of D in such a way that $\{d_1\} \not\succ \dots \not\succ \{d_n\}$. There may of course be several such labellings and we choose once and for all any of the several labellings that could do. For any prospect $D \in \mathcal{P}(X)$ labelled in this way and any $x \in D$, we denote by $r_x^D \in [\#D]$ the rank of x in D defined by $r_x^D = i \iff x = d_i$ for $D = \{d_1, \dots, d_{\#D}\}$. For any $n \in \mathbb{N}$, we say that rank $i \in [n]$ is essential for prospects with n outcomes if there are two distinct outcomes x and $y \in X$ and a prospect $A \in \mathcal{P}(X)$ such that $\#A = n - 1$, $\{x, y\} \cap A = \emptyset$, $r_x^{A \cup \{x\}} = r_y^{A \cup \{y\}} = i$ and $A \cup \{x\} \succ A \cup \{y\}$. In plain English, rank i is essential for prospects of size n if modifying an outcome with rank i can sometimes influence the ranking of those prospects. For example, a maximin ordering of sets of n outcomes has only n as an essential rank.

This paper focuses on *Rank-Dependent Weighted Average Utility* (RDWAU) orderings³ of $\mathcal{P}(X)$ for which there is a continuous function $u : X \rightarrow \mathbb{R}$ and, for any $n \in \mathbb{N}$, n non-negative real numbers w_i^n satisfying $\sum_{i \in [n]} w_i^n = 1$ such that, for all $A = \{a_1, \dots, a_{\#A}\}$ and $B = \{b_1, \dots, b_{\#B}\}$,

$$A \succsim B \iff \sum_{i \in [\#A]} w_i^{\#A} u(a_i) \geq \sum_{i \in [\#B]} w_i^{\#B} u(b_i). \quad (1)$$

An RDWAU ordering of prospects can thus be thought of as resulting from the comparison of a weighted average utility, where the weights depend on the rank of the outcomes in the prospect. In the most general form of Expression (1), the rank-dependent weights w_i^n (for $i \in [n]$) are allowed to depend on the number n of possible outcomes in a completely arbitrary way. For example, a decision-maker whose preferences over $\mathcal{P}(X)$ are represented by (1) could rank prospects with two outcomes based on their worst outcomes (e.g. use

³An ordering is a reflexive, complete and transitive binary relation.

$w_1^2 = 0$ and $w_2^2 = 1$) but compare prospects with three outcomes based on their best outcomes ($w_1^3 = 1$ and $w_2^3 = w_3^3 = 0$).

While such a dependency of the rank-dependent weights on the number of possible outcomes may be plausible in certain circumstances, as discussed in Section 5.2, the main theorem of this paper focuses on the case where the rank-dependent weights w_i^n that appear in (1) are generated by a *weighting function*, that is, a non-decreasing mapping $\psi : [0, 1]_{\mathbb{Q}} \rightarrow [0, 1]_{\mathbb{Q}}$ satisfying $\psi(0) = 0$ and $\psi(1) = 1$ in such a way that $w_i^n = \psi\left(\frac{i}{n}\right) - \psi\left(\frac{i-1}{n}\right)$ or, equivalently, $\psi\left(\frac{i}{n}\right) = \sum_{h=1}^i w_h^n$. Most rank-dependent expected utility criteria examined in the literature (see e.g. Quiggin (1982), Quiggin (1993), Schmeidler (1989) and Yaari (1987))—albeit not in a finite set framework—also result from a weighting function.

2.2 Axioms

We now state and discuss the axioms that characterize the family of all RDWAU orderings generated by a weighting function. The first such axiom is, of course, the requirement that the binary relation $\succsim$ be reflexive, transitive and complete on $\mathcal{P}(X)$.

Axiom 1 *The binary relation $\succsim$ is an ordering.*

Before presenting the second axiom, we need a definition of *preference strength*, adapted from Wakker (1989) to our framework.

Definition 1 *Let $\succsim$ be a binary relation on $\mathcal{P}(X)$, with x, y, x' and y' outcomes in X . The preference strength for x over y is larger than that for x' over y' , formally written as $(x, y) \Delta^{\succsim} (x', y')$, if there are two sets A and B disjoint from $\{x, y, x', y'\}$, satisfying $\#A = \#B$ and $r_x^{A \cup \{x\}} = r_y^{B \cup \{y\}} = r_{x'}^{A \cup \{x'\}} = r_{y'}^{B \cup \{y'\}} = i$ for some essential rank $i \in [\#A + 1]$ such that*

$$A \cup \{x\} \succsim B \cup \{y\} \text{ and } A \cup \{x'\} \precsim B \cup \{y'\}. \quad (2)$$

If at least one of $\succsim$ and $\precsim$ in (2) is strict, we say that the preference strength for x over y is strictly larger than that for x' over y' , which we write as $(x, y) \Delta_s^{\succsim} (x', y')$.

If both $\succsim$ and $\precsim$ in (2) are replaced by $\sim$, we then say that the two preference strengths are equivalent, which we write as $(x, y) \Delta_e^{\succsim} (x', y')$.

In words, the preference strength for x over y is larger than that for x' over y' if x ‘does more’ with respect to y than x' does with respect to y' , when judged by their addition to some sets A, B that do not contain these outcomes. Simple algebra shows that, for a preference represented by (1) for some utility function u , we have $(x, y) \Delta_s^{\succsim} (x', y') \iff u(x) - u(y) \geq u(x') - u(y')$.

The second axiom is an adaptation to the present setting of Peter Wakker’s trade-off consistency condition (Wakker, 1989). It precludes any contradiction between preference strength statements corresponding to $\Delta_s^{\succsim}$ and $\Delta_s^{\precsim}$.

Axiom 2 *Preference Strength Consistency.* For no x, y, x' and y' in X should we observe both $(x, y) \Delta_s^{\succsim} (x', y')$ and $(x', y') \Delta_s^{\precsim} (x, y)$.

The third axiom is a continuity condition. Its statement requires X to be endowed with a topology.

Axiom 3 *Fixed Cardinality Continuity.* For any prospect D , the sets

$$\{(b_1, \dots, b_{\#D}) \in X^{\#D} : \{b_1, \dots, b_{\#D}\} \succsim D\}$$

and

$$\{(b_1, \dots, b_{\#D}) \in X^{\#D} : \{b_1, \dots, b_{\#D}\} \precsim D\}$$

are closed in the product topology.

We observe that this continuity axiom is limited to comparisons of prospects with the same number of outcomes. The last axiom is the only one that restricts the ordering of sets involving different numbers of outcomes.

Axiom 4 *Strong Internality.* For every $n, m \in \mathbb{N}$, and every collection of n disjoint prospects $B^1, \dots, B^n$ satisfying $\#B^j = m$ for $j \in [n]$, one has $\{b_1^1, \dots, b_1^n\} \succsim B^1 \cup \dots \cup B^n \succsim \{b_m^1, \dots, b_m^n\}$.

This axiom requires that the union of several disjoint sets of the same cardinality not be strictly better (resp. worse) than the set containing the best (resp. worst) elements of each.

3 Main result

The main theorem, proved in the Appendix like all formal results of this paper, is the following.

Theorem 1 *Assume X is a connected topological space and $\succsim$ is a binary relation on $\mathcal{P}(X)$. Then $\succsim$ is an ordering of $\mathcal{P}(X)$ satisfying Preference Strength Consistency, Fixed Cardinality Continuity and Strong Internality iff $\succsim$ has an RDWAU representation as in (1) in which the weights w_i^n ($i \in [n]$) are defined by $w_i^n = \psi\left(\frac{i}{n}\right) - \psi\left(\frac{i-1}{n}\right)$ for some weighting function ψ .*

The main part of the proof of Theorem 1 consists in characterizing the larger family of RDWAU orderings that are numerically represented by (1) for arbitrary weights w_i^n ($i \in [n]$), not necessarily produced by a weighting function. To do so, we replace Strong Internality by the weaker Internality axiom of Gneezy, List, and Wu (2006).

Axiom 5 *Internality.* For every prospect $B = \{b_1, \dots, b_n\} \in \mathcal{P}(X)$, one has $\{b_1\} \succsim B \succsim \{b_n\}$.

This axiom is a weakening of the Gärdenfors (1976) principle discussed in the literature on ignorance (Barberà and Pattanaik, 1984; Fishburn, 1984; Kannai and Peleg, 1984). Notice that Gneezy, List, and Wu (2006) provide

some experimental evidence against Internality in the context of decision making under risk.

Once this more general characterization is obtained, we show that the strengthening of Internality to Strong Internality enables us to single out those weights that are generated by a weighting function. The characterization of the larger family has an interest in its own right since it allows for a great flexibility in the choice of the weights. We therefore state it as follows.

Proposition 1 *Assume X is a connected topological space and $\succsim$ is a binary relation on $\mathcal{P}(X)$. Then $\succsim$ is an ordering of $\mathcal{P}(X)$ satisfying Preference Strength Consistency, Fixed Cardinality Continuity and Internality iff $\succsim$ has an RDWAU representation as in (1). The weights w_i^n ($i \in [n]$) are unique for any n .*

The function u is unique up to a positive affine transformation if there is a prospect size for which at least two ranks are essential. Otherwise u is unique up to an increasing transformation.

4 Comparative attitude to ambiguity

Ambiguity aversion is often considered to be a prevalent attitude that lies, for instance, behind the choices made by many subjects in Ellsberg experiments. It is therefore of interest to investigate how a difference in two RDWAU decision-maker's attitudes to ambiguity is reflected in the numerical representation (1).

Definition 2 *Given two orderings $\succsim^h$ and $\succsim^l$ on $\mathcal{P}(X)$, we say that $\succsim^l$ is at least as ambiguity averse as $\succsim^h$ if for any outcome $x \in X$ and any prospect $A \in \mathcal{P}(X)$, $\{x\} \succsim^h A \implies \{x\} \succsim^l A$ and $\{x\} \succ^h A \implies \{x\} \succ^l A$.*

Observe that restricting this definition to prospects A that are themselves non-ambiguous leads to the implication that if l is at least as ambiguity averse

as h , then l and h must order non-ambiguous (certain) outcomes in the same way.

It is common in decision theory also to discuss the notion of ambiguity aversion *tout court* rather than the comparative notion of Definition 2. The usual way to obtain such an absolute notion of ambiguity aversion is to select a reference ordering of prospects as a benchmark of neutrality to ambiguity, and to qualify as ambiguity averse any ordering that is at least as ambiguity averse—in the sense of Definition 2—as this benchmark.

For example, in decision-making under risk, the risk neutrality benchmark is the ordering of probability distributions on the basis of their mean. In our context, there does not seem to be an ordering of prospects that can serve as a natural neutrality to ambiguity benchmark. We therefore limit our analysis to comparative ambiguity aversion.

Numerical representations of RDWAU criteria involve two classes of parameters: the rank-dependent weights w_i^n and the utility function u . The requirement for a decision-maker l to exhibit at least as much ambiguity aversion as another decision-maker h is likely to imply that both the weights and the utility functions of l and h be connected in a certain way. This creates difficulties, noticed by several authors—including Chateauneuf and Cohen (1994), Hong, Karni, and Safra (1987) and Wakker (1994)—in the context of rank-dependent expected utility models. The difficulty lies in the possible interweaving of the connections between the two decision-makers' utility functions and their rank-dependent weights.

It is however easy to identify connections—one between the utility functions and the other between the weights—of two RDWAU decision-makers that are sufficient for one of them to be more ambiguity averse than the other. The connection between the utility functions of two decision-makers with different degrees of ambiguity aversion is standard in decision theory. The more ambiguity averse decision-maker must have a more concave utility function. The connection between the rank-dependent weights is simple

(reverse) first-order stochastic dominance. If the weights are generated by a weighting function, this reverse first-order stochastic dominance is equivalent to the requirement that the weighting function of the more ambiguity averse agent be *more convex* than that of the other agent. Specifically, the following proposition—whose proof is left to the reader—holds.

Proposition 2 *Let $\succsim^h$ and $\succsim^l$ be RDWAU orderings on $\mathcal{P}(X)$ represented numerically by (1) for the continuous utility function u^h (resp. u^l) and the weights generated by the weighting function ψ^h (resp. ψ^l).*

Assume that $u^l(x) = \Phi(u^h(x))$ for every $x \in X$ for some concave and increasing function Φ having the image of u^h as domain and that, for every $x \in [0, 1]$, $\psi^h(x) \geq \psi^l(x)$. Then $\succsim^l$ is at least as ambiguity averse as $\succsim^h$.

Consider now two RDWAU decision-makers h and l such that l 's utility function is *not* a concave transformation of that of h . Does this prevent l from being at least as ambiguity averse as h ? The answer to this question is clearly negative. Just to give an example, assume $X = \mathbb{R}_+$, $\psi^h(0) = 0$, $\psi^h(x) = 1$ for all $x \in]0, 1]$, $\psi^l(x) = 0$ for $x \in]0, 1]$, $\psi^l(1) = 1$, $u^h(x) = x$ and $u^l(x) = x^2$. Hence h assigns positive weight only to the best outcome of any prospect while l only assigns positive weight to the worst outcome of any prospect. It is clear that l is at least as ambiguity averse as h despite having a less concave utility function than h . The difficulty is that any relation of comparative concavity that can prevail between the utility functions of two RDWAU decision-makers can be undone by a relation of first-order stochastic dominance between the weights of those decision-makers that would work in the opposite direction.

5 Other classes of RDWAU orderings

The family of RDWAU orderings characterized in Theorem 1 is quite large and the family of RDWAU orderings characterized in Proposition 1 is even

larger. In this section, we explore some ways of restricting the rank-dependent weights of RDWAU orderings.

5.1 Optimism and pessimism

A possible way of restricting the weights is through the specification of the decision-maker's *optimism* with respect to ambiguous prospects. A natural definition of optimism (pessimism) for an RDWAU decision-maker is the requirement that the rank-dependent weights be decreasing (increasing) with respect to the outcome rank. This is compatible with the definition of optimism/pessimism given in the literature on rank-dependent expected utility models in terms of the superadditivity (or subadditivity) of the Choquet capacity (e.g. Dillenberger, Postlewaite, and Rozen, 2017; Wakker, 1990).

There is an easy ordinal test—and definition—of optimism (pessimism) in our context that leads precisely to this monotonicity of the rank-dependent weights.

Definition 3 *An ordering $\succsim$ on $\mathcal{P}(X)$ is optimistic if for all distinct $w, x, y, z \in X$ such that $(w, x) \Delta_e^\succsim (y, z)$ and $\{w\} \succ \{x\} \succsim \{y\} \succ \{z\}$ and every prospect $A \in \mathcal{P}(X)$ such that $\{x, w, y, z\} \cap A = \emptyset$, $r_x^{A \cup \{x, y\}} = r_w^{A \cup \{w, z\}}$ and $r_y^{A \cup \{x, y\}} = r_z^{A \cup \{w, z\}}$, we have*

$$A \cup \{w, z\} \succsim A \cup \{x, y\}. \quad (3)$$

In this definition, we consider the prospect $A \cup \{x, y\}$ in which we replace x and y by w and z . The benefit of replacing x by w is exactly the same as the cost of replacing y by z , because $(w, x) \Delta_e^\succsim (y, z)$. Yet, since the decision-maker is optimistic, she pays more attention to the replacement of x by w than to the replacement of y by z . Hence, $A \cup \{w, z\} \succsim A \cup \{x, y\}$. For pessimism, $\succsim$ is replaced by $\precsim$ in (3).

It is easy to show that, under the conditions of Theorem 1 or Proposition 1, the weights are non-increasing (non-decreasing) iff the decision-maker

is optimistic (pessimistic). If the weights are generated by a weighting function ψ , then ψ is concave (convex). This provides a connection between optimism (pessimism) and the ambiguity attitude discussed in the previous section. As noticed there, for a decision-maker with a given utility function, increasing ambiguity aversion is associated with a more convex ψ . Hence optimistic (pessimistic) decision-makers tend to be less (more) ambiguity averse, as one would intuitively expect.

Notice that a UEU ordering is simply an RDWAU ordering that is *both* optimistic and pessimistic.

5.2 Variation of the rank-dependent weights with the number of outcomes

The number of possible outcomes of a prospect may bear on the ambiguity the RDWAU decision-maker perceives in the prospect and may, for this reason, modify the rank-dependent weights used by such a decision-maker to evaluate those prospects. At least two different sets of reasons—running in opposite directions—may explain why the decision-maker’s attitude changes with the number of possible outcomes.

One, alluded to above in the Ellsberg experiment example, is that an increase in the number of possible outcomes may be associated with an increase in ambiguity. Intuitively, betting on the draw of a black ball from an urn that contains only two balls distributed in an unknown manner between black and white seems less ambiguous than the same bet on the draw of black ball from an urn containing 100 balls distributed in an unknown manner between the two colors.

However, there are also plausible circumstances where an increase in the number of outcomes of a prospect could be associated with an increase in the strength of evidence⁴ and, therefore, with a reduction in ambiguity. For an

⁴We are most grateful to Faruk Gul for suggesting this possibility.

ambiguity averse decision-maker, this could translate into a weighting scheme that becomes decreasingly convex (or increasingly concave) as the number of outcomes increases. In the medical treatment example, a new treatment tested on three samples with survival probabilities $1/5$, $1/2$ and $4/5$ would presumably be more appealing to a patient than a treatment tested on two samples with survival probabilities $1/5$ and $4/5$. The interpretation of the number of possible outcomes as reflecting the quantity of evidence could be even more plausible if we adopt a multiset perspective (e.g. Blizard, 1989) and accept that outcomes in a given set may appear more than once. In such a framework, the frequency of a given outcome in a multiset could be interpreted as reflecting its *a priori* likelihood. In Appendix C, we show that the main characterization result of this paper obtained for comparisons of sets of outcomes also applies *mutatis mutandis* to multisets.

Theorem 1 leads to weights generated by a weighting function. This does not leave much room for the weights to vary with the number of outcomes. At the other extreme, Proposition 1 leaves perhaps too much room. To find a middle way, we can appeal to a dominance axiom discussed by Barberà, Bossert, and Pattanaik (2004) and Vrijdags and Marchant (2015).

Axiom 6 *Dominance.* For any $A \in \mathcal{P}(X)$ and $x \in X$, $\{a\} \succsim \{x\}$ for every $a \in A \implies A \succsim A \cup \{x\}$ and $\{x\} \succsim \{a\}$ for every $a \in A \implies A \cup \{x\} \succsim A$.

This axiom is weaker than Strong Internality, but stronger than Internality. Replacing Strong Internality with Dominance in Theorem 1 would yield a characterization of a class of RDWAU models lying between those characterized by Theorem 1 and Proposition 1. The restrictions imposed on these intermediate models are given by Proposition 1 in Vrijdags and Marchant (2015).

6 Conclusion

We have axiomatically characterized the family of criteria for decision-making under objective ambiguity that result from comparing rank-dependent weighted average utilities of the prospects, for some utility function and some rank-dependent weighting scheme. We described prospects as finite sets of outcomes—either final consequences or lotteries over them. We have also provided in the appendix a proof of the main result that applies if prospects are modeled as multisets.

While the RDWAU criteria are evocative of the rank-dependent expected utility criteria *à la* Quiggin (1993) or Yaari (1987) considered in decision-making under risk (when prospects are described as probability distributions) or uncertainty (when prospects are described as Savagean acts), they are very different because they are not examined in the same decision context. In particular, there are no analogues—in rank-dependent expected utility models examined in the literature—of the possibility that the weights vary with the number of possible outcomes. Depending on the context, the number of possible outcomes may induce a higher ambiguity (caused by more ignorance about what will happen) or, on the contrary, a lower ambiguity (caused by more evidence). Our finite set-theoretic framework thus sets our results apart from existing ones.

We finally notice that the three main axioms used in the characterization—leaving aside Fixed Cardinality Continuity—are quite easily amenable to experimental testing. It is our hope that future work in the area will enable progress in this direction.

Appendices

A Proofs

Since the proofs will often involve the relation $\Delta^{\succsim}$, some preliminary observations are in order. First, Definition 1 does not preclude $A = B$ or $A = \emptyset$. Second, when viewed as a binary relation on $X \times X$, the relation $\Delta^{\succsim}$ need not be reflexive, complete or transitive. We also observe that $\Delta_s^{\succsim}$ is *not* the asymmetric factor of $\Delta^{\succsim}$, even though it is compatible with it. Somewhat dually, the symmetric factor of $\Delta^{\succsim}$ is compatible with $\Delta_e^{\succsim}$ but is not equivalent to it.

A.1 Theorem 1

The proof of Theorem 1 proceeds in several steps. The most important one consists in establishing Proposition 1. We first note for this purpose that Preference Strength Consistency implies the following condition.

Condition 1 *Weak monotonicity.* $D \cup \{x\} \succ D \cup \{y\}$ for every set $D \in \mathcal{P}(X)$, and every two outcomes x and $y \in X \setminus D$ such that $\{x\} \succ \{y\}$ and $r_x^{D \cup \{x\}} = r_y^{D \cup \{y\}} = i$ for some rank $i \in [\#D + 1]$ that is essential for prospects with $\#D + 1$ outcomes.

In other words, the attractiveness of a prospect is increased if an outcome is replaced by a preferable one of the same rank when that rank is essential. The following lemma establishes that this weak monotonicity condition is indeed implied by Preference Strength Consistency.

Lemma 1 *Let $\succsim$ be an ordering of $\mathcal{P}(X)$ that satisfies Preference Strength Consistency. Then $\succsim$ satisfies the Weak Monotonicity condition.*

Proof. Suppose by contradiction that Condition 1 is violated. Hence, there are two outcomes x and $y \in X$ for which both $\{x\} \succ \{y\}$ and

$\{x\} \cup D \precsim \{y\} \cup D$ hold for some set $D \in \mathcal{P}(X)$ such that $\{x, y\} \cap D = \emptyset$ and $r_x^{D \cup \{x\}} = r_y^{D \cup \{y\}}$ with $r_x^{D \cup \{x\}}$ (or $r_y^{D \cup \{y\}}$) being essential for prospects with $\#D + 1$ outcomes. From the statement $\{x\} \succ \{y\}$, we obtain, using the reflexivity of $\precsim$ restricted to singletons and the definition of $\Delta_s^\precsim$, that $(x, y) \Delta_s^\precsim (x, x)$. From $\{x\} \cup D \precsim \{y\} \cup D$, we conclude using this time the definition of $\Delta^\precsim$, that $(x, x) \Delta^\precsim (x, y)$, thus contradicting Preference Strength Consistency. $\square$

The proof of Proposition 1 then proceeds by establishing that Preference Strength Consistency and Fixed Cardinality Continuity—without Internality—characterize the RDWAU family of orderings of sets containing any number of outcomes when they are applied to an ordering of those sets. However, this first result does not say anything about comparisons of sets containing different numbers of outcomes. It does not even connect the numerical representation obtained for sets with m outcomes to the one obtained for sets with n outcomes. This first step of the proof is summarized in the following proposition.

Proposition 3 *Let X be a set of outcomes and $\precsim$ be a binary relation on $\mathcal{P}(X)$ and assume that X is connected in the order topology associated with $\precsim$ when restricted to singletons. Assume also that, for every integer n , there is at least one rank that is essential for sets with n outcomes. Then $\precsim$ is an ordering satisfying Preference Strength Consistency and Fixed Cardinality Continuity iff there exist w_i^n as in (1) and a continuous function $u^n : X \rightarrow \mathbb{R}$ such that, for all prospects $A, B \in \mathcal{P}(X)$ with the same cardinality n , one has*

$$A \precsim B \iff \sum_{i \in [n]} w_i^n u^n(a_i) \geq \sum_{i \in [n]} w_i^n u^n(b_i). \quad (4)$$

For any $n \geq 2$, if the prospects with n outcomes have at least two ranks that are essential, then the mapping u^n is unique up to a positive affine transformation. The weights w_i^n are also unique. The mapping u^n is unique

up to an increasing transformation if there is only one rank that is essential for prospects with n outcomes.

Proof. The result being true for $n = 1$ by Debreu's (1954) theorem (any continuous ordering on a connected and separable topological space can be numerically represented by a continuous utility function), consider any integer $n \geq 2$. Any prospect D with $n \geq 2$ ordered elements can be represented as an ordered vector in X^n . The set of all such vectors is a subset of X^n , denoted by $\mathcal{O}^n(X)$. Let us consider n disjoint connected subsets $\{Y_1, \dots, Y_n\}$ of X such that, for all $i \in [n-1]$ and for all $x \in Y_i, y \in Y_{i+1}$, we have $\{x\} \succsim \{y\}$. By construction, the Cartesian product $\Pi_{i=1}^n Y_i$ is a subset of $\mathcal{O}^n(X)$. The restriction of $\succsim$ to $\Pi_{i=1}^n Y_i$ satisfies Preference Strength Consistency and Continuity. We can therefore apply Theorem III.6.6 in Wakker (1989) (p. 70)—after noticing that Preference Strength Consistency implies the absence of contradictory trade-offs mentioned in this theorem—and conclude that there exist n continuous mappings $\{u_i^n\}_{i \in [n]}$ such that, for all $A, B \in \Pi_{i \in [n]} Y_i$

$$A \succsim B \iff \sum_{i \in [n]} u_i^n(a_i) \geq \sum_{i \in [n]} u_i^n(b_i). \quad (5)$$

Using then Observation III.6.6' in Wakker (1989) (p. 71), we conclude that the functions u_i^n are unique up to a positive affine transformation for all essential ranks if there are two or more such ranks (the functions associated with non-essential ranks being constant). If however there is only one such essential rank, then the function of this unique rank is unique up to an increasing transformation, and all the other functions are constant. Let $E(n) \subset [n]$ be the set of ranks that are essential for prospects with n possible outcomes and, since at least one rank is essential, let $\hat{n}$ be the integer defined by $0 < \hat{n} = \#E(n) \leq n$. Consider then the set $\mathcal{O}^{\hat{n}}(X)$ of all ordered vectors in $X^{\hat{n}}$ that correspond to the $\hat{n}$ ordered elements with essential rank in sets

with n outcomes for some such set D . Just like its cousin $\mathcal{O}^n(X)$, the set $\mathcal{O}^{\hat{n}}(X)$ can be written as the union of infinitely many Cartesian products of the form $\prod_{i \in [\hat{n}]} Y_i$ with the sets Y_i defined as above for ranks i that are all essential for sets with n outcomes. The set $\mathcal{O}^{\hat{n}}(X)$ satisfies Assumption 2.1 in Chateauneuf and Wakker (1993). In particular, all components of this subset of a Cartesian product are essential and, thanks to Lemma 1, the ordering of $\mathcal{O}^{\hat{n}}(X)$ induced by $\succsim$ satisfies the Chateauneuf and Wakker (1993) property of strong monotonicity required by their Assumption 2.1. By Theorem 2.2 in Chateauneuf and Wakker (1993), there is a continuous additive representation of $\succsim$ (restricted to sets of cardinality n). That is to say, Expression (5) provides a numerical representation of the ordering $\succsim$ not only on $\prod_{i \in [n]} Y_i$ but, also, on the whole set $\mathcal{O}^n(X)$, in which the functions u_i^n of Expression (5) can (and must) be taken to be constant for all ranks i that are not essential for sets with n outcomes.

Let $\hat{j} \in E(n)$ be an essential rank for sets with n outcomes such that $\hat{i} > \hat{j}$ for any other essential rank $\hat{i}$ (if any) in $E(n)$. Assume first that there are at least two essential ranks for sets with n outcomes and consider therefore any $\hat{i} \in E(n)$ such that $\hat{i} > \hat{j}$. Let A and B be any two sets of cardinality $n-1$ and x_1, x_2 and x_3 be outcomes in X not contained in either A or B such that $\{x_1\} \succsim \{x_2\} \succsim \{x_3\}$ and $r_{x_i}^{A \cup \{x_i\}} = r_{x_{i+1}}^{A \cup \{x_{i+1}\}} = r_{x_i}^{B \cup \{x_i\}} = r_{x_{i+1}}^{B \cup \{x_{i+1}\}} = \hat{i}$. Choose these outcomes in such a way that $A \cup \{x_{i+1}\} \sim B \cup \{x_i\}$ for $i \in [2]$. The existence of these three outcomes and two sets A and B having those features is secured by the continuity of the representation in (5) and the connectedness of X . Hence, using (5), $A \cup \{x_{i+1}\} \sim B \cup \{x_i\}$ for $i \in [2]$ can be written as

$$\begin{aligned} \sum_{g \in [\hat{i}-1]} u_g^n(a_g) + u_{\hat{i}}^n(x_3) + \sum_{g \in [\hat{i}+1, n]} u_g^n(a_{g-1}) \\ = \sum_{g \in [\hat{i}-1]} u_g^n(b_g) + u_{\hat{i}}^n(x_2) + \sum_{g \in [\hat{i}+1, n]} u_g^n(b_{g-1}) \quad (6) \end{aligned}$$

and

$$\begin{aligned} \sum_{g \in [i-1]} u_g^n(a_g) + u_i^n(x_2) + \sum_{g \in [i+1, n]} u_g^n(a_{g-1}) \\ = \sum_{g \in [i-1]} u_g^n(b_g) + u_i^n(x_1) + \sum_{g \in [i+1, n]} u_g^n(b_{g-1}). \end{aligned} \quad (7)$$

Subtracting (7) from (6) yields $u_i^n(x_2) - u_i^n(x_1) = u_i^n(x_3) - u_i^n(x_2)$. Let now C and D be sets of cardinality $n - 1$ not containing x_1, x_2 and x_3 such that the rank of x_1, x_2 and x_3 in $\{x_i\} \cup C$ and in $\{x_i\} \cup D$ (for $i \in [3]$) is $\hat{j}$. Again, the existence of these two sets C and D having those features is secured by the continuity of the representation in ((5)) and the connectedness of the set X . We observe that, thanks to Preference Strength Consistency, one must have $C \cup \{x_{i+1}\} \sim D \cup \{x_i\}$ for $i \in [2]$. The same reasoning as above therefore yields $u_j^n(x_2) - u_j^n(x_1) = u_j^n(x_3) - u_j^n(x_2)$. In other words, the images of x_1, x_2, x_3 under u_i^n are equally spaced and so are they under u_j^n . Let us say that x_1, x_2 and x_3 form a grid in X with a mesh of size 1. Thanks to the uniqueness remark made above with respect to the two essential ranks $\hat{j}$ and $\hat{i}$, we necessarily have that $u_i^n = \alpha_i + \beta_i u_j^n$ for some real numbers α_i and β_i with $\beta_i > 0$.

By continuity and connectedness, there is $x_{1 \circ 2}$ ‘halfway’ between x_1 and x_2 . More formally, there is $x_{1 \circ 2}$ such that $u_i^n(x_{1 \circ 2}) - u_i^n(x_1) = u_i^n(x_2) - u_i^n(x_{1 \circ 2})$. There is also $x_{2 \circ 3}$ such that $u_i^n(x_{2 \circ 3}) - u_i^n(x_2) = u_i^n(x_3) - u_i^n(x_{2 \circ 3})$. The images of $x_1, x_{1 \circ 2}, x_2, x_{2 \circ 3}$ and x_3 under u_i^n are thus equally spaced and so are they under u_j^n . The outcomes $x_1, x_{1 \circ 2}, x_2, x_{2 \circ 3}$ and x_3 thus form a grid in X with mesh of size $1/2$. We can again halve the mesh of this grid by adding the outcomes $x_{1 \circ (1 \circ 2)}, x_{(1 \circ 2) \circ 2}, x_{2 \circ (2 \circ 3)}$ and $x_{(2 \circ 3) \circ 3}$ to the grid. And we can make the mesh as fine as we want by repeating this process.

Let us denote the outcomes of the initial grid (mesh size = 1), by $g_1^1 = x_1, g_2^1 = x_2$ and $g_3^1 = x_3$. Similarly, we denote the elements of the second grid (mesh size = $1/2$) by $g_1^2 = x_1, g_2^2 = x_{1 \circ 2}, g_3^2 = x_2, g_4^2 = x_{2 \circ 3}$ and $g_5^2 = x_3$, those

of the third grid (mesh size = $1/2^2$, $g_1^3 = x_1$, $g_2^3 = x_{1 \circ (1 \circ 2)}$, $g_3^3 = x_{1 \circ 2}$, $\dots$, $g_8^3 = x_{(2 \circ 3) \circ 3}$ and $g_9^3 = x_3$ and so on.

If x_1 is not maximal for $\succsim$ restricted to singletons, we can try to extend the grid to the ‘left’ of $x_1 = g_1^1$, by looking for an element g_0^1 such that $u_j^n(g_1^1) - u_j^n(g_0^1) = u_j^n(g_2^1) - u_j^n(g_1^1)$. If such a g_0^1 does not exist, then there exists a mesh size s such that there is g_0^s satisfying $u_j^n(g_1^s) - u_j^n(g_0^s) = u_j^n(g_2^s) - u_j^n(g_1^s)$.

If g_0^s is not maximal for $\succsim$, we can again extend the grid to the ‘left’ of g_0^s (this may require using a finer mesh). By repeating this process, we can extend the grid to the ‘left’ of x_1 and go as close as we wish to the maximal prospects for $\succsim$ (if any).

We can also extend the grid to the ‘right’ of x_3 and go as close as we wish to the minimal singleton prospect for $\succsim$ (if any). Since the images of all elements of a grid are equally spaced in $\mathbb{R}$ under u_i^n and u_j^n , we necessarily have that $u_i^n(x) = \alpha_i + \beta_i u_j^n(x)$ for some real numbers α_i and β_i with $\beta_i > 0$ and for any element x of a grid of any mesh size.

Consider now an element x that does not belong to any grid. We have just seen above that we can refine or extend the initial grid in order to be as close as we wish to x . Continuity of u_i^n and u_j^n then imply that $u_i^n(x) = \alpha_i + \beta_i u_j^n(x)$ holds for any $x \in X$. The reasoning just made is valid for any essential rank $\hat{i} \in E(n)$ distinct from $\hat{j}$. Hence, if there is at least one such essential rank, we can write Equivalence (4) as

$$A \succsim B \iff \sum_{i \in E(n)} \beta_i u_j^n(a_i) \geq \sum_{i \in E(n)} \beta_i u_j^n(b_i)$$

after dropping all irrelevant constant terms (including the constant functions that represent non-essential ranks). Defining $u^n = u_j^n$ and $w_i^n = \beta_i / \sum_{j \in E(n)} \beta_j$ for all essential ranks i , and $w_i^n = 0$ for all non-essential ranks, one can equivalently write

$$A \succsim B \iff \sum_{i \in [n]} w_i^n u^n(a_i) \geq \sum_{i \in [n]} w_i^n u^n(b_i).$$

If however $\hat{j}$ is the only essential rank for prospects with n outcomes, then one sets $w_{\hat{j}}^n = 1$ and $w_i^n = 0$ for all $i \neq \hat{j}$ and $u^n = u_{\hat{j}}^n$. $\square$

The second step of the proof consists in showing that any of the functions u^n that appears in the numerical representation (4) of the ordering $\succsim$ restricted to prospects with n outcomes provides a numerical representation of the ordering $\succsim$ restricted to singletons. The required result for this step is the following lemma.

Lemma 2 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated with $\succsim$ when restricted to singletons. Assume also that, for every integer n , there is at least one rank that is essential for sets with n outcomes. If $\succsim$ satisfies Preference Strength Consistency and Fixed Cardinality Continuity, then, for any $n \in \mathbb{N}$, the function u^n in (4) also numerically represents the restriction of the ordering $\succsim$ to singletons. Formally, for any $x, y \in X$ and $n \in \mathbb{N}$, $x \succsim y$ iff $u^n(x) \geq u^n(y)$.*

Proof. Let x and y be two outcomes in X such that $\{x\} \succsim \{y\}$. Since for any n , there is at least one rank that is essential for prospects with n outcomes, this applies to $n = 1$ so that the ordering of singletons is not trivial. For any $n \geq 2$, consider a prospect D such that $\#D = n - 1$, $\{x, y\} \cap D = \emptyset$ and $r_x^{D \cup \{x\}} = r_y^{D \cup \{y\}} = i$ for some $i \in E(n)$. Again, the existence of such a prospect for any outcomes x and y such that $\{x\} \succsim \{y\}$ is guaranteed by the connectedness of X and the essentiality condition. By Preference Strength Consistency, one must have $D \cup \{x\} \succsim D \cup \{y\}$. Thanks to Proposition 3, this latter statement can equivalently be written as

$$\begin{aligned} \sum_{h \in [i-1]} w_h^n u^n(d_h) + w_i^n u^n(x) + \sum_{h \in [i+1, n]} w_h^n u^n(d_{h-1}) \\ \geq \sum_{h \in [i-1]} w_h^n u^n(d_h) + w_i^n u^n(y) + \sum_{h \in [i+1, n]} w_h^n u^n(d_{h-1}). \end{aligned}$$

Since $w_i^n > 0$ (because $i \in E(n)$), this is equivalent to $u^n(x) \geq u^n(y)$.

By a similar argument and the completeness of $\succsim$, one would obtain the strict inequality $u^n(y) > u^n(x)$ if one had assumed $\{x\} \prec \{y\}$ instead. Hence $\{x\} \succsim \{y\}$ if and only if $u^n(x) \geq u^n(y)$ for any $n \geq 1$. $\square$

Proposition 3 establishes the validity of the numerical representation (4) for the ordering of prospects with a given number of outcomes. However it does not connect together the functions u^n that appear in the definition of the numerical representation for prospects involving different numbers of outcome. With the help of Lemma 2, the next lemma establishes that all the functions u^n that appear in the numerical representations of the orderings of prospects containing n outcomes for variable n can all be taken to be the same (up to a positive affine transformation). The formal statement of this lemma is as follows.

Lemma 3 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated with $\succsim$ when restricted to singletons. Assume also that, for every integer n , there is at least one rank that is essential for sets with n outcomes. If $\succsim$ satisfies Preference Strength Consistency and Fixed Cardinality Continuity, then, for any $n \in \mathbb{N}$, and any prospects D and D' with n possible outcomes, one has*

$$D \succsim D' \iff \sum_{i \in [n]} w_i^n u(d_i) \geq \sum_{i \in [n]} w_i^n u(d'_i)$$

for some continuous function $u : X \rightarrow \mathbb{R}$ uniquely defined up to a positive affine transformation (if there are prospects with n outcomes with at least two essential ranks) or up to an increasing transformation (if only one rank is essential for all prospect sizes). Moreover, the weights w_i^n are unique.

We need a new definition before proving this lemma. For any $p \geq 3$, the list of outcomes $(a_i)_{i \in [p]}$ is a *standard sequence* if $(a_i, a_{i+1}) \Delta_e^{\succsim} (a_{i+1}, a_{i+2})$

for all $i \in [p-2]$. Put differently, any two pairs of adjacent outcomes in the sequence exhibit the same preference strength for their first outcome over their second.

Proof of Lemma 3. Thanks to Proposition 3, we know that, for any $n \in \mathbb{N}$, there exist non-negative weights w_i^n as in (1) and $u^n : X \rightarrow \mathbb{R}$ such that (4) holds. We consider two cases.

Case 1. Suppose there is at least one cardinality m such that two ranks are essential (one of them is rank $\hat{i}$). Suppose $(a_i)_{i \in [p]}$ is some standard sequence for the sets B and C of cardinality $m-1$ with $\{a_1, \dots, a_p\} \cap (B \cup C) = \emptyset$, $B \prec C$ and such that $r_{a_i}^{B \cup \{a_i\}} = r_{a_i}^{C \cup \{a_i\}} = \hat{i}$ for all $i \in [p]$. By Preference Strength Consistency, we must have $\{a_i\} \cup B \sim \{a_{i+1}\} \cup C$ for all $i \in [p-1]$ and, applying the numerical representation of Proposition 3 for $n = m$,

$$u^m(a_i) - u^m(a_{i-1}) = u^m(a_{i+1}) - u^m(a_i) \quad (8)$$

for all those i . Consider now any cardinality $m' \neq m$ with at least two essential ranks (if such an m' exists). Let $\hat{j}$ denote one of these essential ranks. One can find, thanks to continuity and connectedness of X , some sets D and E of cardinality $m'-1$ such that $\{a_1, \dots, a_p\} \cap (D \cup E) = \emptyset$ and, for all $i \in [p-1]$, $r_{a_i}^{D \cup \{a_i\}} = r_{a_{i+1}}^{E \cup \{a_{i+1}\}} = \hat{j}$. Thanks to Preference Strength Consistency, one will have $D \cup \{a_i\} \sim E \cup \{a_{i-1}\}$ for all $i \in [2, p-1]$ and, thanks to (4),

$$u^{m'}(a_i) - u^{m'}(a_{i-1}) = u^{m'}(a_{i+1}) - u^{m'}(a_i). \quad (9)$$

Thanks to the connectedness of X and continuity of $\succsim$ when restricted to singletons, we can choose the standard sequence $(a_i)_{i \in [p]}$ as fine as we want following the procedure described at the end of the proof of Proposition 3. Hence, the comparison of (8) and (9) shows that $u^m = \gamma^{m'} + \lambda^{m'} u^{m'}$ for any $m' \in [2, \cdot]$ with at least two essential ranks. We have seen in Proposition 3

that u^n is unique up to any positive affine transformation in the case of n with at least two essential ranks. Since u^m is a positive affine transformation of $u^{m'}$, we can use u^m instead of $u^{m'}$ in (4).

For any cardinality n with only one essential rank, (4) is a purely ordinal representation because only one weight w_i^n is non zero. We can therefore replace u^n by u^m because, as shown in Lemma 2, u^m is a strictly increasing transformation of u^n .

Since we can use u^m everywhere instead of u^n for every n (including $n = 1$), we can just define $u = u^m$ and this completes the proof for the case in which there is a cardinality m with at least two essential ranks.

Case 2. Suppose there is no cardinality such that two ranks are essential. Then, for every n , only one weight w_i^n is non zero and (4) is a purely ordinal representation. We can therefore replace u^n by u^1 because, as shown in Lemma 2, u^1 is a strictly increasing transformation of u^n . Setting $u = u^1$ completes the proof. $\square$

While Proposition 3 and Lemmas 2 and 3, which establish the validity of the numerical representation (1) for prospects with the same number of outcomes, make no use of the Internality axiom, the rest of the proof, which establishes the validity of that same representation for comparing sets with different number of outcomes, will use this axiom extensively. The last intermediate result that is required to prove Theorem 1 is the following lemma that establishes, under all the axioms, the existence of a certainty (or non-ambiguity) equivalent for any prospect.

Lemma 4 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated with $\succsim$ when restricted to singletons. Assume also that, for every integer n , there is at least one rank that is essential for sets with n outcomes. If $\succsim$ satisfies Preference Strength Consistency, Fixed Cardinality Continuity and Internality, then,*

for any prospect $D \in \mathcal{P}(X)$, there exists an outcome $\text{CE}(D) \in X$ such that $\{\text{CE}(D)\} \sim D$.

Proof. The result being trivially true for a prospect D with a single outcome, consider a prospect D with $n \geq 2$ outcomes. Let us write $D = \{d_1, \dots, d_n\}$. We define $\text{CE}(D)$ to be any element of the set $u^{-1}(\sum_{i \in [n]} w_i^n u(d_i))$ where the continuous function u and the weights w_i^n are those that define the numerical representation constructed in Lemma 3, which is also a numerical representation of the restriction of the ordering $\succsim$ to singletons thanks to Lemma 2. We need to show that $u^{-1}(\sum_{i \in [n]} w_i^n u(d_i)) \neq \emptyset$ and, also, that $\{\text{CE}(D)\} \sim D$.

The proof of the non-emptiness of $u^{-1}(\sum_{i \in [n]} w_i^n u(d_i))$ (and of the fact that $\{\text{CE}(D)\} \sim D$) being immediate if $u(d_i) = u(d_{i+1})$ for all $i \in [n-1]$, we consider the case where $u(d_1) > u(d_n)$ and, by Lemma 2, where $\{d_1\} \succ \{d_n\}$. We observe that since $w_i^n \geq 0$ for all i (with at least one strict inequality thanks to the assumption that at least one rank is essential for prospects with n outcomes) and $u(d_1) > u(d_n)$ by assumption, one must have (since $1 \geq w_i^n$) $u(d_1) \geq \sum_{i \in [n]} w_i^n u(d_i) \geq u(d_n)$ with at least one of the two inequalities strict. The continuity of u on its connected domain X implies that u^{-1} is non-empty for any real number in the non-degenerate interval $[u(d_n), u(d_1)]$ and thus, in particular, for the real number $\sum_{i \in [n]} w_i^n u(d_i)$. Hence $\text{CE}(D)$ exists.

Suppose now that $\text{CE}(D) \in X$ is such that $\{\text{CE}(D)\} \approx D$. Let us specifically assume that $\{\text{CE}(D)\} \succ D$. Let $i \in [n-1]$ be defined by $u(d_i) \geq u(\text{CE}(D))$ and $u(d_{i+1}) < u(\text{CE}(D))$. Consider then the sequence of prospects D^t (for $t \in \mathbb{N}$) defined by

$$D^t = \{\text{CE}(D)^{+\varepsilon_n^t}, \dots, \text{CE}(D)^{+\varepsilon_{i+1}^t}, \text{CE}(D)^{-\varepsilon_i^t}, \dots, \text{CE}(D)^{-\varepsilon_1^t}\}$$

for some list of n outcomes $\text{CE}(D)^{+\varepsilon_h^t}$ and $\text{CE}(D)^{-\varepsilon_h^t}$ such that $u(\text{CE}(D)^{-\varepsilon_h^t}) = u(\text{CE}(D)) - \varepsilon_h^t$, for $h \in [i+1, n]$, and $u(\text{CE}(D)^{+\varepsilon_h^t}) = u(\text{CE}(D)) + \varepsilon_h^t$, for

$h \in [i]$ for some sequence of suitably small positive real numbers ε_h^t (for $h \in [n]$) such that $\varepsilon_1^t < \varepsilon_2^t < \dots < \varepsilon_i^t, \varepsilon_n^t < \varepsilon_{n-1}^t < \dots < \varepsilon_{i+1}^t, \lim_{t \rightarrow \infty} \varepsilon_j^t = 0$ for all $j \in [n]$ and

$$\sum_{h \in [i]} w_h^n \varepsilon_h^t = \sum_{h \in [i+1, n]} w_h^n \varepsilon_h^t.$$

It is clear again that the existence of these $\text{CE}(D)^{+\varepsilon_h^t}$ and $\text{CE}(D)^{-\varepsilon_h^t}$ is secured by the continuity of u , the connectedness of X and the fact that $u(d_1) > \sum_{h \in [n]} w_h^n u(d_h) > u(d_n)$. We then have

$$\begin{aligned} \sum_{h \in [i]} w_h^n u\left(\text{CE}(D)^{-\varepsilon_h^t}\right) + \sum_{h \in [n]: h > i} w_h^n u\left(\text{CE}(D)^{+\varepsilon_h^t}\right) \\ = u\left(\text{CE}(D)\right) = \sum_{h \in [n]} w_h^n u(d_h). \end{aligned}$$

By Proposition 3, this implies $D \sim D^t$ for all t . By transitivity, we therefore have $\{\text{CE}(D)\} \succ D^t$ for all t . We observe also that the Internality axiom implies that

$$\{\text{CE}(D)^{+\varepsilon_n^t}\} \succsim D^t \succsim \{\text{CE}(D)^{-\varepsilon_1^t}\} \quad (10)$$

while the fact—established in Lemma 2—that u numerically represents the ordering $\succsim$ restricted to singletons implies that

$$\{\text{CE}(D)^{+\varepsilon_n^t}\} \succ \{\text{CE}(D)\} \succ \{\text{CE}(D)^{-\varepsilon_1^t}\}. \quad (11)$$

Since $\{\text{CE}(D)\} \succ D^t$ for all t , it follows from (10) and (11) and transitivity that

$$\{\text{CE}(D)^{+\varepsilon_n^t}\} \succ \{\text{CE}(D)\} \succ D^t \succsim \{\text{CE}(D)^{-\varepsilon_1^t}\}.$$

Yet both sequences of singletons $\{\text{CE}(D)^{+\varepsilon_n^t}\}$ and $\{\text{CE}(D)^{-\varepsilon_1^t}\}$ converge to $\{\text{CE}(D)\}$. Hence, having $\{\text{CE}(D)^{+\varepsilon_n^t}\} \succ \{\text{CE}(D)^{-\varepsilon_1^t}\}$ for all t is incompatible, in the limit, with the continuity of $\succsim$. The argument for the case where $D \succ \{\text{CE}(D)\}$ is similar. $\square$

The final lemma shows that $\succsim$ is trivial when prospects of some size have no essential rank.

Lemma 5 *If there is n such that no rank is essential for prospects of size n , then the relation $\succsim$ has a single equivalence class, that is, for all prospects $C, D \in \mathcal{P}(X)$, we have $C \sim D$.*

Proof. Suppose $n = 1$. Then all singletons are equivalent to each other. For any $m > 1$ and any prospect $D = \{a_1, \dots, a_m\}$, Internality imposes $\{a_1\} \succsim D \succsim \{a_m\}$. Since $\{a_1\} \sim \{a_m\}$, we conclude $D \sim \{a_1\}$. Hence, all prospects are equivalent to each other.

Suppose rank 1 is essential for prospects of size 1. There are therefore $a, d \in X$ such that $\{a\} \succ \{d\}$. By connectedness, there exists b, c such that $\{a\} \succ \{b\} \succ \{c\} \succ \{d\}$. By connectedness again, we can find two prospects $C = \{c_1, \dots, c_n\}$ and $D = \{d_1, \dots, d_n\}$ such that $\{c_n\} \sim \{b\}$ and $\{d_1\} \sim \{c\}$. By Internality, $C \succsim \{b\}$ and $\{c\} \succsim D$. Hence $C \succ D$ and this implies there is some essential rank for prospects of size n . $\square$

The proof of Proposition 1 is then completed by showing that the numerical representation (1) shown so far to represent the ordering $\succsim$ on any two prospects containing the same number of outcomes is also valid for comparing prospects with different numbers of outcomes.

Proof of Proposition 1. Assume first there is some n such that no rank is essential for prospects of size n . Then, by Lemma 5, $\succsim$ has a single equivalence class and there is an obvious representation with any constant u and arbitrary weights. For the rest of the proof, we assume there is at least one essential rank for prospects of any size.

We know from Proposition 3 and Lemma 3 that for any $n \in \mathbb{N}$, there is a continuous utility function $u : X \rightarrow \mathbb{R}$ and a set of weights w_i^n (for $i \in [n]$) for which (1) holds for any prospects D and D' having both n possible outcomes.

For any prospect D , let $U_D = \sum_{i \in [\#D]} w_i^{\#D} u(d_i)$. Consider two prospects $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ for some n and $m \in \mathbb{N}_{++}$ with $n \neq m$ such that $A \succsim B$. We need to show that $U_A \geq U_B$. By contradiction, suppose that $U_B > U_A$. Choose any suitably small strictly positive real number ε . Using Lemma 4, the continuity of u and the connectedness of X , define the sets C_n^j and C_m^j by

$$C_n^j = \{c_1^j, \dots, c_n^j\}$$

and

$$C_m^j = \{c_1^j, \dots, c_m^j\}$$

where, for any $i \in [m] \cup [n]$ and $j \in \mathbb{N}$, $c_i^j \in X$ is such that

$$c_i^j = u^{-1} \left(U_A + \frac{\varepsilon}{ij} \right).$$

By Internality, we have that $\{c_1^j\} \succsim C_n^j$. By the numerical representation, $C_n^j \succ A$. We observe also that, for sufficiently large j , $U_{C_m^j}$ can be made arbitrarily close to U_A . Take therefore a fixed j , say $\bar{j}$, sufficiently large for the assumption $U_B > U_A$ to imply $U_B > U_{C_m^{\bar{j}}} > U_A$. Since the numerical representation holds for sets of cardinality m , we must therefore have $B \succ C_m^{\bar{j}}$. By Internality, $C_m^{\bar{j}} \succsim \{c_m^{\bar{j}}\}$ and, by the numerical representation applied to singletons, $\{c_m^{\bar{j}}\} \succ \{c_m^j\}$ for all $j > \bar{j}$. Thanks to Lemma 4 and transitivity, we have

$$\{c_1^j\} \succsim C_n^j \succ A \sim \{\text{CE}(A)\} \succsim B \succ C_m^{\bar{j}} \succsim \{c_m^{\bar{j}}\} \succ \{c_m^j\},$$

for all $j > \bar{j}$. But having $\{c_1^j\} \succ \{\text{CE}(A)\} \succ \{c_m^{\bar{j}}\} \succ \{c_m^j\}$ for all $j > \bar{j}$ contradicts Fixed Cardinality Continuity (applied to singletons), since both sequences $\{c_1^j\}_{j \in \mathbb{N}}$ and $\{c_m^j\}_{j \in \mathbb{N}}$ converge to $u^{-1}(U_A) = u^{-1}(\text{CE}(A))$. The converse implication that $U_A \geq U_B$ implies $A \succsim B$ is proved in the same way. $\square$

Proof of Theorem 1. Since Strong Internality implies Internality, Proposition 1 applies and $\succsim$ has an RDWAU representation as in (1) with arbitrary non-negative weights w_i^n (for any integer n and any $i \in [n]$). Let q be any rational number in $[0, 1]_{\mathbb{Q}}$ and let us consider two pairs of integers (i, n) and (j, m) such that $i/n = j/m = q$. Let $A = \{a_1, \dots, a_n\}$ be a prospect such that, for all $h \in [n-1]$, we have $u(a_h) = u(a_{h+1}) + \delta$ for some positive real number δ . The existence of such a prospect poses no difficulty thanks to the connectedness of the set X and the continuity of u . For $h \in [n]$, let $B^h = \{b_1^h, \dots, b_{m-1}^h\}$ be a prospect with $m-1$ outcomes such that $u(b_k^h) = u(a_h) - \varepsilon\delta/k$ for all $k \in [m-1]$, with $0 < \varepsilon < 1$. By construction, $A \cap B^h = \emptyset = B^h \cap B^{h'}$ for every $h, h' \in [n]$ with $h \neq h'$ and $\{a_h\} \succ \{b_k^h\}$ for all $h \in [n]$ and $k \in [m-1]$. Hence, it follows from Strong Internality that

$$A \succsim A \cup B^1 \cup \dots \cup B^n. \quad (12)$$

Symmetrically, for $h \in [n]$, one can construct a prospect $C^h = \{c_1^h, \dots, c_{m-1}^h\}$ such that $u(c_k^h) = u(a_h) + \varepsilon\delta/k$ for all $k \in [m-1]$, with $0 < \varepsilon < 1$. By construction, $A \cap C^h = \emptyset = C^h \cap C^{h'}$ for every $h, h' \in [n]$ with $h \neq h'$ and $\{a_h\} \prec \{c_k^h\}$ for all $h \in [n]$ and $k \in [m-1]$. Applying again Strong Internality, we obtain

$$A \precsim A \cup C^1 \cup \dots \cup C^n. \quad (13)$$

When ε tends to zero, both $u(b_k^h)$ and $u(c_k^h)$ tend to $u(a_h)$. Hence, since (12) and (13) hold for all $0 < \varepsilon < 1$ and since the RDWAU representation is continuous, we have

$$w_h^n = \sum_{k=(h-1)m+1}^{hm} w_k^{nm}, \quad \forall h \in [n].$$

This implies, in particular,

$$\sum_{h=1}^i w_h^n = \sum_{h=1}^{im} w_h^{nm}.$$

A similar reasoning, starting from a set A with m elements and sets B^h, C^h with $n - 1$ elements, yields

$$\sum_{h=1}^j w_h^m = \sum_{h=1}^{jn} w_h^{nm}.$$

Since $im = jn$, we have

$$\sum_{h=1}^i w_h^n = \sum_{h=1}^j w_h^m. \quad (14)$$

For any $q \in [0, 1]_{\mathbb{Q}}$, let us therefore define $\psi(q) = \sum_{h=1}^i w_h^n$, using any i, n such that $i/n = q$. By (14), $\psi(q)$ does not depend on the particular choice of i and n . By construction, ψ is non-decreasing. $\square$

B Independence of the axioms

In the following four examples, we prove the independence of the four axioms of Theorem 1 by exhibiting binary relations that satisfy any three of the four axioms but not the remaining one.

Example 1 (Ordering) Assume $X = \mathbb{R}$ and define $\succsim^1$ on $\mathcal{P}(X)$ by $A \succsim^1 B \iff \sum_{i \in \#A} a_i \geq 1 + \sum_{i \in \#B} b_i$. This relation satisfies Fixed Cardinality Continuity, Strong Internality and Preference Strength Consistency. It violates the Ordering axiom because it is not complete. For instance, $\{1/2\} \not\succsim^1 \{0\}$ and $\{0\} \not\succsim^1 \{1/2\}$.

Example 2 (Strong Internality) Assume $X = \mathbb{R}$ and define $\succsim^{\text{add}}$ on $\mathcal{P}(X)$

by $A \succsim^{\text{add}} B \iff \sum_{i \in \#A} a_i \geq \sum_{i \in \#B} b_i$. This relation satisfies Ordering, Fixed Cardinality Continuity and Preference Strength Consistency. To see that it violates Internality (and therefore Strong Internality) simply observe that $\{1, 2\} \succ^{\text{add}} \{2\}$.

Example 3 (Preference Strength Consistency) Let $X = [0, 1]_{\mathbb{R}}$ and let $\succsim$ be the ordering defined by

$$A \succsim B \iff \frac{\sum_{a \in A} (1 + a - a^2)a}{\sum_{a \in A} (1 + a - a^2)} \geq \frac{\sum_{b \in B} (1 + b - b^2)b}{\sum_{b \in B} (1 + b - b^2)}.$$

This ordering satisfies Fixed Cardinality Continuity (since the two functions $a \mapsto 1 + a - a^2$ and $a \mapsto a$ involved in the numerical representation of this ordering are continuous). It also satisfies Strong Internality because $\frac{\sum_{a \in A} (1 + a - a^2)a}{\sum_{a \in A} (1 + a - a^2)}$ is a convex combination of the numbers $a \in A$ and always lies between the maximum and the minimum of those numbers. To see that it violates Preference Strength Consistency, consider the prospects $A = \{19/20, 1/80\}$ and $B = \{9/10, 1/15\}$. We have $\{19/20, 1/2, 1/80\} \succ \{9/10, 1/2, 1/15\}$ and $\{19/20, 1/5, 1/80\} \prec \{9/10, 1/5, 1/15\}$. This means that $(1/2, 1/2) \Delta_s^{\succsim} (1/5, 1/5)$, which indeed contradicts Preference Strength Consistency since $\{1/2\} \sim \{1/2\}$ and $\{1/5\} \sim \{1/5\}$ and, therefore, $(1/2, 1/2) \Delta_e^{\succsim} (1/5, 1/5)$.

Example 4 (Fixed Cardinality Continuity) Assume that $X = \mathbb{R}^2$ and, for any $x \in X$ and $i \in [2]$, let x^i denote the i^{th} component of x . Consider then the lexicographic version of the UEU ordering $\succsim^{\text{lex}}$ on $\mathcal{P}(X)$ defined, for any prospects A and B , by

$$A \sim^{\text{lex}} B \iff \sum_{i \in [\#A]} \frac{a_i^1}{\#A} = \sum_{i \in [\#A]} \frac{b_i^1}{\#A} \text{ and } \sum_{i \in [\#A]} \frac{a_i^2}{\#A} = \sum_{i \in [\#A]} \frac{b_i^2}{\#A}$$

and by $A \succ^{\text{lex}} B$ if either

$$\sum_{i \in [\#A]} \frac{a_i^1}{\#A} > \sum_{i \in [\#A]} \frac{b_i^1}{\#A}$$

or

$$\sum_{i \in [\#A]} \frac{a_i^1}{\#A} = \sum_{i \in [\#A]} \frac{b_i^1}{\#A} \text{ and } \sum_{i \in [\#A]} \frac{a_i^2}{\#A} > \sum_{i \in [\#A]} \frac{b_i^2}{\#A}.$$

Hence, the ordering $\succ^{\text{lex}}$ compares prospects on the basis of a lexicographic combination of the average of each component of the (two-dimensional) outcomes of those prospects. This ordering violates Fixed Cardinality Continuity. Indeed, for any outcome $(a^1, a^2) \in X$, the set of outcomes $(x^1, x^2) \in X$ such that $\{(x^1, x^2)\} \succ^{\text{lex}} \{(a^1, a^2)\}$ is not closed in X .

To see that $\succ^{\text{lex}}$ satisfies Strong Internality, consider the prospect $D = \{d_1, \dots, d_n\}$ for some $n \in \mathbb{N}$. Since $\{d_n\} \preceq^{\text{lex}} \{d_i\}$ for all $i \in [n]$, we have either

$$d_i^1 > d_n^1 \tag{15}$$

or

$$d_i^1 = d_n^1 \text{ and } d_i^2 \geq d_n^2 \tag{16}$$

for all $i \in [n]$. Summing over i the inequalities or equalities (15) and (16) yields either:

$$\begin{aligned} \sum_{i \in [n]} d_i^1 > n d_n^1 &\iff \frac{\sum_{i \in [n]} d_i^1}{n} > d_n^1 \quad \text{if (15) holds for some } i \text{ or} \\ \frac{\sum_{i \in [n]} d_i^1}{n} = d_n^1 \text{ and } \frac{\sum_{i \in [n]} d_i^2}{n} &\geq d_n^2 \quad \text{if (16) holds for all } i. \end{aligned}$$

Hence, one has $\{d_1, \dots, d_n\} \succ^{\text{lex}} \{d_n\}$ as required by the Internality axiom. A similar reasoning yields $\{d_1\} \succ^{\text{lex}} \{d_1, \dots, d_n\}$. One can extend this reasoning to any sequence of m disjoint sets $D^j = \{d_1^j, \dots, d_n^j\}$ and conclude,

after some verification, that,

$$\bigcup_{j \in [m]} \{d_1^j\} \succsim^{\text{lex}} \bigcup_{j \in [m]} D^j \succsim^{\text{lex}} \bigcup_{j \in [m]} \{d_n^j\}$$

as required by Strong Internality. To show that Preference Strength Consistency is satisfied by $\succsim^{\text{lex}}$, suppose by contradiction that it is not. This means that there exist prospects A, A', B and B' in $\mathcal{P}(X)$ satisfying $\#A = \#A' = n$ and $\#B = \#B' = m$ for some $m, n \in \mathbb{N}$ and distinct outcomes $x, y, x', y' \in X$ such that $\{x, y, x', y'\} \cap (A \cup A' \cup B \cup B') = \emptyset$ for which one has

$$\{a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n\} \succsim^{\text{lex}} \{a'_1, \dots, a'_{i-1}, y, a'_{i+1}, \dots, a'_n\}, \quad (17)$$

$$\{a_1, \dots, a_{i-1}, x', a_{i+1}, \dots, a_n\} \succsim^{\text{lex}} \{a'_1, \dots, a'_{i-1}, y', a'_{i+1}, \dots, a'_n\}, \quad (18)$$

$$\{b_1, \dots, b_{j-1}, x, b_{j+1}, \dots, b_m\} \prec^{\text{lex}} \{b'_1, \dots, b'_{j-1}, y, b'_{j+1}, \dots, b'_m\}, \quad (19)$$

$$\{b_1, \dots, b_{j-1}, x', b_{j+1}, \dots, b_m\} \succsim^{\text{lex}} \{b'_1, \dots, b'_{j-1}, y', b'_{j+1}, \dots, b'_m\}, \quad (20)$$

for some $i \in [n]$ and $j \in [m]$ or, possibly, with the comparison (19) weak and the comparison (20) strict. Yet, we focus on (19) strict and (20) weak in the following sketch. From (17) we conclude

$$\sum_{h=1}^{i-1} a_h^1 + x^1 + \sum_{h=i+1}^n a_h^1 > \sum_{h=1}^{i-1} a_h'^1 + y^1 + \sum_{h=i+1}^n a_h'^1 \quad (21)$$

or

$$\begin{aligned} \sum_{h=1}^{i-1} a_h^1 + x^1 + \sum_{h=i+1}^n a_h^1 &= \sum_{h=1}^{i-1} a_h'^1 + y^1 + \sum_{h=i+1}^n a_h'^1 \text{ and} \\ \sum_{h=1}^{i-1} a_h^2 + x^2 + \sum_{h=i+1}^n a_h^2 &\geq \sum_{h=1}^{i-1} a_h'^2 + y^2 + \sum_{h=i+1}^n a_h'^2. \end{aligned} \quad (22)$$

Similarly, we obtain from (18)

$$\sum_{h=1}^{i-1} a_h^1 + x'^1 + \sum_{h=i+1}^n a_h^1 < \sum_{h=1}^{i-1} a_h'^1 + y'^1 + \sum_{h=i+1}^n a_h'^1 \quad (23)$$

or

$$\begin{aligned} \sum_{h=1}^{i-1} a_h^1 + x'^1 + \sum_{h=i+1}^n a_h^1 &= \sum_{h=1}^{i-1} a_h'^1 + y'^1 + \sum_{h=i+1}^n a_h'^1 \text{ and} \\ \sum_{h=1}^{i-1} a_h^2 + x'^2 + \sum_{h=i+1}^n a_h^2 &\leq \sum_{h=1}^{i-1} a_h'^2 + y'^2 + \sum_{h=i+1}^n a_h'^2. \end{aligned} \quad (24)$$

Four cases need to be considered:

1. [label=(0)]

2. (21) and (23) imply $x^1 - x'^1 > y^1 - y'^1$;

3. (21) and (24) imply $x^1 - x'^1 > y^1 - y'^1$ and

$$\sum_{h=1}^{i-1} a_h^2 + x'^2 + \sum_{h=i+1}^n a_h^2 \leq \sum_{h=1}^{i-1} a_h'^2 + y'^2 + \sum_{h=i+1}^n a_h'^2;$$

4. (22) and (23) imply $x^1 - x'^1 > y^1 - y'^1$ and

$$\sum_{h=1}^{i-1} a_h^2 + x^2 + \sum_{h=i+1}^n a_h^2 \geq \sum_{h=1}^{i-1} a_h'^2 + y^2 + \sum_{h=i+1}^n a_h'^2;$$

5. (22) and (24) imply $x^1 - x'^1 = y^1 - y'^1$ and $x^2 - x'^2 \geq y^2 - y'^2$.

Similarly, we can derive from (19) that

$$\sum_{h=1}^{j-1} b_h^1 + x^1 + \sum_{h=j+1}^m b_h^1 < \sum_{h=1}^{j-1} b_h'^1 + y^1 + \sum_{h=j+1}^m b_h'^1 \quad (25)$$

or

$$\begin{aligned} \sum_{h=1}^{j-1} b_h^1 + x^1 + \sum_{h=j+1}^m b_h^1 &= \sum_{h=1}^{j-1} b_h'^1 + y^1 + \sum_{h=j+1}^m b_h'^1 \text{ and} \\ \sum_{h=1}^{j-1} b_h^2 + x^2 + \sum_{h=j+1}^m b_h^2 &\leq \sum_{h=1}^{j-1} b_h'^2 + y^2 + \sum_{h=j+1}^m b_h'^2, \end{aligned} \quad (26)$$

while (20) leads to

$$\sum_{h=1}^{j-1} b_h^1 + x'^1 + \sum_{h=j+1}^m b_h^1 > \sum_{h=1}^{j-1} b_h'^1 + y'^1 + \sum_{h=j+1}^m b_h'^1 \quad (27)$$

or

$$\begin{aligned} \sum_{h=1}^{j-1} b_h^1 + x'^1 + \sum_{h=j+1}^m b_h^1 &= \sum_{h=1}^{j-1} b_h'^1 + y'^1 + \sum_{h=j+1}^m b_h'^1 \text{ and} \\ \sum_{h=1}^{j-1} b_h^2 + x'^2 + \sum_{h=j+1}^m b_h^2 &\geq \sum_{h=1}^{j-1} b_h'^2 + y'^2 + \sum_{h=j+1}^m b_h'^2. \end{aligned} \quad (28)$$

The four implications resulting from all the possible combinations of these expressions are

1. [label=()]
2. (25) and (27) yield $x^1 - x'^1 < y^1 - y'^1$;

3. (25) and (28) yield $x^1 - x'^1 < y^1 - y'^1$ and

$$\sum_{h=1}^{j-1} b_h^2 + x'^2 + \sum_{h=j+1}^m b_h^2 \geq \sum_{h=1}^{j-1} b_h'^2 + y'^2 + \sum_{h=j+1}^m b_h^2;$$

4. (26) and (27) yield $x^1 - x'^1 < y^1 - y'^1$ and

$$\sum_{h=1}^{j-1} b_h^2 + x^2 + \sum_{h=j+1}^m b_h^2 \leq \sum_{h=1}^{j-1} b_h'^2 + y^2 + \sum_{h=j+1}^m b_h^2;$$

5. (26) and (28) yield $x^1 - x'^1 = y^1 - y'^1$ and $x^2 - x'^2 > y^2 - y'^2$.

Since any combination of one of the cases (1)–(4) with one of the cases (a)–(d) leads to a contradiction, this shows that the ordering $\succsim^{\text{lex}}$ does indeed satisfy Preference Strength Consistency.

C A characterization of RDWAU orderings for multisets

In this appendix⁵, we show that a suitably adapted version of the most general Proposition 1 holds if the compared ambiguous prospects are multisets—or bags—rather than sets. Multisets may provide a plausible description of ambiguous prospects in situations where some a priori information is available to the decision-maker about the likelihood of the various possible outcomes. Unlike sets, multisets allow their elements to appear more than once.

We shall formally describe a multiset as a mapping $m : X \rightarrow \mathbb{N} \cup \{0\}$. The mapping m assigns to every conceivable outcome x of the universe an integer $m(x)$, interpreted as the number of occurrences of that element in m . We denote the cardinality of m by $\#m = \sum_{x \in X} m(x)$. The support of a

⁵We are grateful to Faruk Gul for suggesting multisets and for inspiring this appendix.

multiset m is denoted by $\underline{m}$ and is defined by $\underline{m} = \{x \in X : m(x) > 0\}$. In this appendix, we refer to any multiset with finite cardinality and non-empty support as an m-prospect. The set of all such m-prospects is denoted by $\mathcal{M}$. For any $x \in X$, we denote by $\hat{x}$ the m-prospect such that $\hat{x}(x) = 1$ and $\#\hat{x} = 1$. The m-prospects in $\mathcal{M}$ are compared by a binary relation $\succsim$. Given $\succsim$, we associate with every m-prospect $m \in \mathcal{M}$ the vector $m^\downarrow \in \underline{m}^{\#m}$ in such a way that

- for all $x \in \underline{m}$, $\#\{i \in [\#m] : m_i^\downarrow = x\} = m(x)$,
- for all $i, j \in [\#m]$ such that $i < j$, $m_i^\downarrow \not\prec m_j^\downarrow$,
- for all $i < j < k$, $m_i^\downarrow = m_k^\downarrow \implies m_i^\downarrow = m_j^\downarrow$.

For example, if $\hat{x} \succ \hat{y}$ and $m = 2\hat{x} + 3\hat{y}$, then $m^\downarrow = (x, x, y, y, y)$. There may of course be several such vector representations of m-prospects, in which case one of them is chosen once and for all. For example, if $\hat{x} \succ \hat{y} \sim \hat{z}$ and $m = 2\hat{x} + 3\hat{y} + \hat{z}$, then $m^\downarrow = (x, x, y, y, y, z)$ or $m^\downarrow = (x, x, z, y, y, y)$. For any m-prospect m , if $m(x) = 1$, we denote by r_x^m the rank of x in m defined by $r_x^m = i$ if $m_i^\downarrow = x$. In the above example, with $m^\downarrow = (x, x, z, y, y, y)$, we have $r_z^m = 3$. For any $n \in \mathbb{N}$, we say that a rank $i \in [n]$ is essential for m-prospects with n outcomes if there are two distinct outcomes x and $y \in X$ and an m-prospect $m \in \mathcal{M}$ such that $\#m = n - 1$, $x, y \notin \underline{m}$, $r_x^{m+\hat{x}} = r_y^{m+\hat{y}} = i$ and $m + \hat{x} \succ m + \hat{y}$.

We are interested in RDWAU orderings of $\mathcal{M}$ for which there exist a continuous function $u : X \rightarrow \mathbb{R}$ and, for any $n \in \mathbb{N}$, n non-negative real numbers w_i^n satisfying $\sum_{i \in [n]} w_i^n = 1$ such that, for all $m, m' \in \mathcal{M}$,

$$m \succsim m' \iff \sum_{i \in [\#m]} w_i^{\#m} u(m_i^\downarrow) \geq \sum_{i \in [\#m']} w_i^{\#m'} u(m_i'^\downarrow). \quad (29)$$

Hence the definition of an RDWAU ordering is literally unaffected when applied to m-prospects. We now reformulate some definitions for m-prospects.

Definition 4 Let x, x', y, y' be outcomes in X . The binary relation $\succsim$ on $\mathcal{M}$ is said to reveal a weakly larger preference strength for x over y than for x' over y' , which we write formally as $(x, y) \triangle^{\succsim} (x', y')$ if there are two m -prospects m and $m' \in \mathcal{M}$ satisfying $\#m = \#m' = n - 1$ for some integer n , $x, x' \notin \underline{m}$, $y, y' \notin \underline{m'}$ and $r_x^{m+\hat{x}} = r_y^{m'+\hat{y}} = r_{x'}^{m+\hat{x}'} = r_{y'}^{m'+\hat{y}'} = i$ for some $i \in [n]$ that is essential for m -prospects with cardinality n such that

$$m + \hat{x} \succsim m' + \hat{y} \quad \text{and} \quad m + \hat{x}' \precsim m' + \hat{y}'. \quad (30)$$

If at least one of $\succsim$ and $\precsim$ in (30) is strict, we then say that the preference strength for x over y is strictly larger than that for x' over y' , which we write formally as $(x, y) \triangle_s^{\succ} (x', y')$.

We now reformulate some axioms for m -prospects.

Axiom 7 *m-Fixed Cardinality Continuity.* For any m -prospect $m \in \mathcal{M}$, the sets

$$\{m'^{\downarrow} \in X^{\#m} : m' \succsim m\} \quad \text{and} \quad \{m'^{\downarrow} \in X^{\#m} : m' \precsim m\}$$

are closed in the product topology.

Axiom 8 *m-Internality.* For every m -prospect m , one has $\widehat{m}_1^{\downarrow} \not\prec m \not\prec \widehat{m}_{\#m}^{\downarrow}$.

We can then establish the following analogue to Proposition 1 (we could also prove an analogue to Theorem 1 without difficulties).

Proposition 4 Assume X is a connected topological space and $\succsim$ is a binary relation on $\mathcal{M}$. Then $\succsim$ is an ordering that satisfies Preference Strength Consistency, m -Fixed Cardinality Continuity and m -Internality iff $\succsim$ is an RDWAU ordering as in (29). The weights w_i^n ($i \in [n]$) are unique for any n .

The function u is unique up to a positive affine transformation if there is an m -prospect size for which at least two ranks are essential. Otherwise u is unique up to an increasing transformation.

Proof. (sketch) Let $X' = X \times \mathbb{N}$. With each m-prospect m in $\mathcal{M}$, we associate the prospect $A^m = \{(m_i^\downarrow, i)\}_{i \in [\#m]}$ in $\mathcal{P}(X')$. Let us define the ordering $\succsim'$ on $\mathcal{P}(X')$ by

- for all $m, m' \in \mathcal{M}$, if $m \succsim m'$, then $A^m \succsim^{m'} A^{m'}$;
- for all $n \in \mathbb{N}$, $A = \{(a_i, j_i)\}_{i \in [n]}$ and $B = \{(b_i, k_i)\}_{i \in [n]}$, if $a_i = b_i$ for all $i \in [n]$, then $A \sim' B$.

It is easy to see that X' and $\succsim'$ satisfy all conditions for the existence of an RDWAU representation, with u' and w' . The representation is such that $u'((x, i)) = u'((x, j))$ for all $i, j \in \mathbb{N}$. Let us define $u(x) = u'((x, 1))$ and $w = w'$. Then u and w form the required representation for $\mathcal{M}$ and $\succsim$. $\square$

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