The probability of ties with scoring methods: some results

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Abstract

The main flaw of the Condorcet method is that a Condorcet winner does not always exist. Therefore, a society or committee using the Condorcet method risks to face an indeterminacy. An indeterminacy can also happen when using other methods: for example the Borda method, plurality voting or any scoring method; but the origin of the indeterminacy is completely different. It happens when all candidates are tied. We study the probability that all candidates are tied when using a scoring method. We show that it is equivalent to some random walk problems. Some analytical and numerical results show that, under the assumptions underlying our study, the probability is very small and decreases when the number of voters or candidates increases.

Keywords: Borda, plurality, approval, scoring method, probability of ties

1 Introduction

One of the weaknesses of the Condorcet method lies in the fact that it doesn't verify the universal domain condition. That is, for some profiles of complete orders, the Condorcet method fails in pointing out a winner, a best candidate. This is not the only critique that can be addressed to this method [Fishburn 77], but it is probably the main one.

If we consider the Borda method, the situation is very different: in any case, the method points out a set of winners. This set contains one or more candidates. And in some cases, it contains them all. We call this situation k-ties, k

being the number of candidates. Let us have a closer look at these cases. From a mathematical viewpoint, it means that all candidates are socially equivalent and anyone can be picked. But, practically, unless all voters individually consider all candidates as equivalent, such a situation is rather perceived as an indeterminacy and is very problematic. Hence, the possibility for the Borda method to lead to such situations must be considered as a drawback of the method. In fact, it is a drawback of many methods: e.g. all scoring methods can lead to such a situation. To determine if this drawback is severe or not, we propose in this paper to compute the probability of a k-ties. Let us call $N(n, k)$ the number of possible profiles with n voters and k candidates and $N_k(n, k)$ the number of profiles yielding a k-ties. We define the probability of a k-ties as

$$P(k\text{-ties}, n, k) = N_k(n, k)/N(n, k).$$

In section 2, we will present some notations and basic results about $P(k\text{-ties}, n, k)$. Section 3 will be devoted to the presentation of a geometrical representation of scoring methods. On the basis of this geometrical representation, we will show an equivalence between our problem and random walks on some lattices. Interesting results based on this equivalence are obtained, mostly for the asymptotic behaviour of the probability of k-ties when n approaches infinity. In section 4, we will consider some variants of our initial problem: profiles of votes which are not equally likely, profiles such that j ($j < k$) candidates belong to the winners set (this situation, although less severe than k-ties, can be seen as an indeterminacy) and profiles in which the voters are not independent. The most interesting result in this section tells us that single-peakedness, combined with some minimal assumption, rules out the possibility of any k-ties. Numerical results will be presented in section 5. Some of them are obtained analytically. Some others are obtained using a Monte-Carlo technique. They all indicate that the probability of k-ties with scoring methods is very low.

2 Notations and basic results

First, we are going to precisely define what a scoring method is. Let $X = \{1, 2, \ldots, k\}$ be the set of the candidates and $V = \{v_1, v_2, \ldots, v_n\}$ the set of the voters. A scoring vector is a vector in $\Re^X$, denoted by $s = (s_1, s_2, \ldots, s_k)$. To vote, a voter chooses one scoring vector in the set of all scoring vectors corresponding to the scoring method used in that election. E.g., when the Borda method is used, the scoring vectors set is the set of all permutations of $(1, 2, \ldots, k)$. The scoring vectors set for plurality voting consists of all distinct permutations of $(1, 0, \ldots, 0)$. For approval voting, it consists of all vectors such that $s_r \in \{0, 1\}, \forall r$ and $\exists r, s : s_r = 0$ and $s_s = 1$. For the sake of brievity, we will denote the scoring vectors set by SVS . Note that, in general, we do not impose that, in any scoring vector s, not all scores be equal. A voter might want to choose a scoring vector where all candidates have the same score. This can happen, for example, when a voter is indifferent between all candidates.

A profile is a vector in SVS^V, denoted by $p = (s^1, s^2, \ldots, s^n)$, where s^l is the scoring vector choosen by voter v_l. The score of a candidate r in a profile p, $S_r(p)$, is the sum over all voters l of s^l_r, where s^l_r is the r-th component of s^l. The winners of the election are the candidates with highest score. Hence, a particular scoring method is completely determined if we know SVS.

2.1 The Borda method

Throughout this paper, the Borda method will be considered as a method used for the aggregation of rankings without ties. A quite common generalisation of the Borda method allows for the aggregation of any kind of binary relations (e.g. rankings with ties): we will call it generalised Borda method. Note that a still more generalised Borda method has been described and axiomatized by Marchant [Marchant 96]. It allows for the aggregation of fuzzy relations.

Let us first remark that for some values of n and k, $P(k\text{-ties}, n, k) = 0$ for the Borda method. If we add the score of all candidates in a profile p, we obtain:

$$\sum_{i=1}^{k} S_i(p) = \frac{k(k+1)}{2} n. \tag{1}$$

For a k-ties to happen, all candidates must have identical scores. Thus, using (1), we obtain:

$$S_i(p) = \frac{(k+1)}{2} n, \qquad i = 1 \ldots k. \tag{2}$$

As the score is always integer (all components of any Borda scoring vector are integer), (2) is possible only if $(k+1)n$ is even, i.e. k odd or n even. This leads us to our first proposition.

Proposition 1 *With the Borda method, $P(k\text{-ties}, n, k) = 0$ for k odd and n even.*

Let us now set the value of n to 2. The scoring vector of the first voter can be any of the $k!$ permutations of $(1, 2, \ldots, k)$. But as soon as the scoring vector of the first voter is choosen the scoring vector of the second one is fixed, if we want to observe a k-ties. The second scoring vector must be the inverse of the first one, i.e. $s^2_r = s^1_{k+1-r}$. Hence, $N_k(2, k) = k!$ and, as $N(2, k) = (k!)^2$, we can state the following result.

Proposition 2 *With the Borda method, $P(k\text{-ties}, 2, k) = 1/k!$.*

Note that $P(k\text{-ties}, 2, k)$ is decreasing with respect to k and its limit is 0 when k goes to infinity.

Let us now set the value of k to 2. If n is even, we have a k-ties iff the number of voters voting $(2,1)$ is equal to the number of voters voting $(1,2)$, i.e. $n/2$. Hence $N_k(n,2) = \binom{n}{n/2}$, for n even.

Proposition 3 *With the Borda method, $P(k\text{-ties}, n, 2) = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{3}{4}\frac{1}{2}$ for n even.*

Proof. For n even, $P(k\text{-ties}, n, 2)$ is given by

$$\begin{aligned}
\frac{\frac{n!}{(n/2)!(n/2)!}}{2^n} &= \frac{n!}{2^n((n/2)!)^2} \\
&= \frac{n!}{2^n 2^{-n}(n(n-2)(n-4)\dots 2)^2} \\
&= \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{3}{4}\frac{1}{2}. \qquad \square
\end{aligned}$$

Note that $P(k\text{-ties}, n, 2)$ is decreasing with respect to n and its limit is 0 when n goes to infinity.

2.2 The Generalised Borda method with weak orders

As mentionned above, a very classic and straightforward generalisation of the Borda method allows for the aggregation of any kind of binary relations. The score of a candidate for a given voter is just the number of candidates to which he is preferred minus the number of candidates that are preferred to him. For example, if there are three candidates and a voter prefers r to s and t but is indifferent between s and t, then the score of r (resp. s and t) is 2 (resp. -1 and -1).

Hence, for 3 candidates, as we restrict ourselves to weak orders, SVS is $\{$ all permutations of $(2,0,-2)\} \cup \{$ all permutations of $(2,-1,-1)\} \cup \{(0,0,0)\}$. For 2 candidates, $SVS = \{(1,-1),(-1,1),(0,0)\}$. It is obvious, in this case, that the number of voters choosing (1,-1) must be equal to the number of voters choosing (-1,1). Therefore,

$$N_k(n,2) = \begin{cases} \sum_{i=0}^{n/2} \binom{2i}{i}\binom{n}{2i} & \text{for } n \text{ even,} \\ \sum_{i=0}^{(n-1)/2} \binom{2i}{i}\binom{n}{2i} & \text{for } n \text{ odd} \end{cases} \tag{3}$$

and proposition 4 easily follows.

Proposition 4 *With the generalised Borda method and weak orders,*

$$(a) \quad P(k\text{-}ties, n, 2) = \begin{cases} \frac{1}{3^n} \sum_{i=0}^{n/2} \frac{n!}{(n-2i)!(i!)^2} & \textit{for } n \textit{ even,} \\ \\ \frac{1}{3^n} \sum_{i=0}^{(n-1)/2} \frac{n!}{(n-2i)!(i!)^2} & \textit{for } n \textit{ odd.} \end{cases}$$

$$(b) \qquad \lim_{n=\infty} P(k\text{-}ties, n, 2) \leq \frac{3^{3/2}}{4\pi}.$$

Proof. We only prove (b) and in the case of n even. We are going to use Stirling's approximation for factorials: $i! = \sqrt{2\pi i} \exp^{-i} i^i$, for i large. But some terms of the sum in (a) contain factorials of small numbers. For example, when $i = 0, 1, 2, \ldots$ or $i = n/2, n/2-1, \ldots$ For this reason, we rewrite (a) to handle these terms separately.

$$\begin{aligned} P(k\text{-ties}, n, 2) &= \sum_{i=0}^{a} \frac{n!}{3^n (n-2i)!(i!)^2} \\ &+ \sum_{i=a+1}^{b-1} \frac{n!}{3^n (n-2i)!(i!)^2} \\ &+ \sum_{i=b}^{n/2} \frac{n!}{3^n (n-2i)!(i!)^2}, \end{aligned} \tag{4}$$

where a and b are finite and can be chosen arbitrarily large so that all factorials appearing in the second sum are correctly approximated by Stirling's formula. The limit of each term in the first sum of (4) is zero as proved below (remember that i is finite).

$$\begin{aligned} \lim_{n=\infty} \frac{n!}{3^n (n-2i)!(i!)^2} &= \lim_{n=\infty} \frac{\sqrt{2\pi n} \exp^{-n} n^n}{3^n \sqrt{2\pi(n-2i)} \exp^{-n+2i} (n-2i)^{n-2i} (i!)^2} \\ &= \lim_{n=\infty} \frac{n^{n+1/2}}{3^n (n-2i)^{n-2i+1/2} (i!)^2} \\ &= \lim_{n=\infty} \frac{n^{2i}}{3^n (i!)^2} \\ &= 0 \end{aligned}$$

As the limit of each term is 0 and we have a finite number of these terms in the first sum, the first sum is equal to 0. Similarly, the third sum in (4) can be proved to be equal to 0. Let us now consider the second sum of (4), where $n-2i$ and i are as large as we want (by choosing a and b). Proving that each term is zero is not sufficient because there are infinitely many terms in the second sum. Let $\widehat{i}$ be the smallest integer such that $\widehat{i} \geq n/3$. It is not hard to see that the largest term in the sum is obtained for $i = \widehat{i}$. As n is growing to ∞, $\widehat{i}$ becomes equal to $n/3$. So, we can use it to bound our sum.

$$
\begin{aligned}
\lim_{n=\infty} \sum_{i=a+1}^{b-1} \frac{n!}{3^n (n-2i)!(i!)^2} &\leq \lim_{n=\infty} \sum_{i=a+1}^{b-1} \frac{n!}{3^n (n/3)!((n/3)!)^2} \\
&\leq \lim_{n=\infty} \frac{n}{2} \frac{n!}{3^n ((n/3)!)^3} \\
&\leq \lim_{n=\infty} \frac{n}{2} \frac{\sqrt{2\pi n} \exp^{-n} n^n}{3^n (\sqrt{2\pi (n/3)} \exp^{-(n/3)} (n/3)^{(n/3)})^3} \\
&\leq \lim_{n=\infty} \frac{n^{n+3/2}}{3^n 4\pi (n/3)^{n+3/2}} \\
&\leq \frac{3^{3/2}}{4\pi}
\end{aligned}
$$

□

2.3 Plurality voting

With plurality voting, there are k different scoring vectors in SVS . In order to observe a k-ties, the number of voters casting each of the k different scoring vectors must be equal, i.e. n/k. As n/k must be integer (it is a number of voters), a k-ties can happen only if n is a multiple of k. In such a case,

$$N_k(n,k) = \frac{n!}{((n/k)!)^k}.$$

Hence we obtain the following result.

Proposition 5 *With plurality voting,*

$$P(k\text{-}ties, n, k) = \begin{cases} \frac{n!}{(n(n-k)(n-2k)\dots k)^k} & \textit{for } n \textit{ multiple of } k \\ 0 & \textit{otherwise.} \end{cases} \tag{5}$$

Proof. For n multiple of k, $P(k\text{-ties}, n, k)$ is given by

$$
\begin{aligned}
\frac{\frac{n!}{((n/k)!)^k}}{k^n} &= \frac{n!}{k^n k^{-n} (n(n-k)(n-2k)\dots k)^k} \\
&= \frac{n!}{(n(n-k)(n-2k)\dots k)^k}.
\end{aligned}
$$

□

Note that $P(k\text{-ties}, n, k)$ is decreasing with respect to n and k and its limit is 0 when n or k goes to infinity. When $k = 2$, $P(k\text{-ties}, n, k)$ is identical for plurality voting and the Borda method. This is normal as both methods are identical when there are only two candidates.

An approximate expression for $n \to \infty$ is easily obtained using Stirling's formula:

$$\lim_{n\to\infty} P(k\text{-ties}, n, k) = \sqrt{\frac{k^k}{(2\pi n)^{k-1}}} \text{ for } n \text{ multiple of } k. \tag{6}$$

2.4 Approval voting

When there are only two candidates, there are just two scoring vectors: $(1, 0)$ and $(0, 1)$. Hence approval voting, in this case, is equivalent to plurality voting and the Borda method and we do not need to compute the probability of a k-ties.

When there are only two voters, once the first voter's scoring vector is fixed, the second voter can force a k-tie by voting for the candidates that were not approved by the first voter and this is the only possible way. The number of possible scoring vectors is $2^k - 2$ (all vectors of 0 and 1 except $(0, 0, 0, \ldots, 0)$ and $(1, 1, 1, \ldots, 1)$). The first voter has thus $2^k - 2$ possibilities and the second voter has no choice. Hence, $N_k(n, k) = 2^k - 2$.

Proposition 6 *With approval voting,* $P(k\text{-ties}, 2, k) = \frac{1}{2^k-2}$.

Note that $P(k\text{-ties}, 2, k)$ is decreasing with respect to k and its limit is 0 when k goes to infinity.

3 Geometrical representations

Given our definition of a scoring method, there is a quite obvious geometrical representation in k dimensions. Each scoring vector can be represented by a vector in $\Re^X$. Let us represent any point in this space by his coordinates $(x_1, x_2, \ldots, x_k)$. Let us consider the vector obtained by summing all scoring vectors of the profile p. We call it $S(p)$. The coordinates of this vector are the numbers $S_r(p)$, i.e. the scores of each candidate. A very similar geometrical representation can be found in Saari [Saari 94]. It is easy to see that the extremity of $S(p)$ is on the line $x_1 = x_2 = \ldots = x_k$ iff all candidates have the same score, i.e. if we observe a k-ties. Our problem is thus to find the probability that the sum of n vectors choosen in SVS falls on the line $x_1 = x_2 = \ldots = x_k$.

To find this probability, we can use the following function ϕ. Let

$$\phi = \sum_{s \in SVS} x_1^{s_1} x_2^{s_2} \ldots x_k^{s_k}. \tag{7}$$

Let us now consider the polynomials ϕ^n whose coefficients will help us to count all possible profiles. If $n = 1$, each term of this polynomial in k variables obviously represents a different profile, consisting of only 1 scoring vector. Let $C = \#SVS$. Our polynomial has C terms. If $n = 2$, then our polynomial has C^2 terms. But as multiplication of scalars is commutative, some terms may be identical and can be grouped. As vector addition is commutative as well, it is clear that terms that have been grouped represent different profiles $(p, p', p'', \ldots)$ such that $S(p) = S(p') = S(p'') = \ldots$ Thus the coefficient of each term (after grouping) is the number of profiles of two voters with a particular vector $S(\cdot)$. If we proceed with higher values of n, we obtain C^n terms (before grouping). After grouping, the coefficient of each term is the number of profiles of n voters with a particular vector $S(\cdot)$. Clearly the terms where all variables are raised to the same power represent the profiles such that the extremity of $S(\cdot)$ is on the line $x_1 = x_2 = \ldots = x_k$. Hence

Proposition 7 *$N_k(n, k)$ is the sum of the coefficients of all terms of ϕ^n that can be written under the form $x_1^z x_2^z \ldots x_k^z$ for any z.*

Hence, our problem boils down to the computation of some coefficients of a polynomial. Unfortunately, computing these coefficients is a tough task when k or n grows.

3.1 Geometrical representation in $k - 1$ dimensions

Let us consider the representation in k dimensions that we presented in previous section. We are going to project it on the hyperplane perpendicular to the line $x_1 = x_2 = \ldots = x_k$. From now onward, we will call this hyperplane H. It is clear that a profile p yields a k-ties iff the projection of $S(p)$ on H appears as a point at the origin. Hence, this representation perfectly serves our purpose. Let us now "draw" a profile on H in the following way. Draw the vector corresponding to the first voter. At the end of this vector, draw the scoring vector of the second voter, and so on until the n-th voter. A profile appears as a succession of segments of straight lines. If we pick a profile at random, the process of drawing this profile can be seen as a random walk. We start from the origin and move in one of the C possible directions (corresponding to the C scoring vectors). At the end of each move, there are C possible new moves.

For the set of possible moves is always the same, the walk is constrained to follow edges of a lattice. By lattice, we mean a regular disposition of edges and nodes in the space. This acceptation of the word lattice is classic in cristallography and generally in physics. It is different from the graph theory meaning. To make our point clear, we present two examples of lattices: one for the Borda method and one for approval voting.

3.1.1 The Borda method with with three candidates

When there are three candidates, SVS contains just 6 vectors: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$ and $(3, 2, 1)$. If we project these 6 scoring vectors on

H, we obtain six vectors with equal length and the angle between each pair of adjacent vector is $\pi/3$ as depicted in fig 1. If we reproduce the pattern of

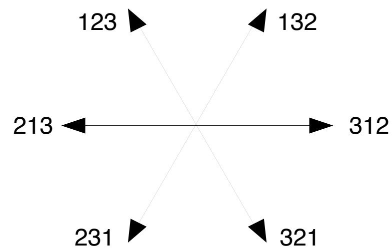

Figure 1: Projection of the scoring vectors

fig. 1 at the extremity of each vector in a recursive way, we obtain the lattice illustrated by fig. 2. As all nodes of this lattice are equivalent, we do not need

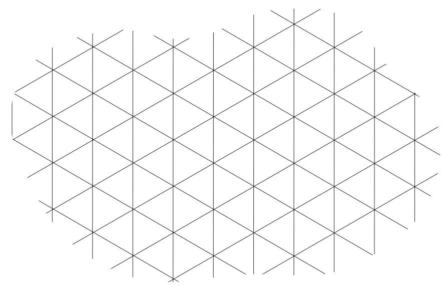

Figure 2: Lattice of the Borda method for $k = 3$

to put a special label on the node corresponding to the origin. Any node can be considered as the origin. A profile can be represented in this lattice by a path starting from any node and travelling along the edges. If the last edge of a path ends at the origin, then the profile yields a k-tie.

Based on this representation, Marchant [Marchant 98] proved the following result.

Proposition 8 *With the Borda method, $P(k\text{-ties}, n, 3)$ is given by*

$$\frac{n!}{6^n} \sum_{2s+3t=n} 2^{[t>0]} \sum_{q=0}^{s} \sum_{r=0}^{s-q} \frac{1}{q!r!(s-q-r)!(q+t)!(r+t)!(s-q-r+t)!},$$

where $[t > 0]$ equals 1 if $t > 0$ and 0 otherwise.

In July 98, shortly after we proved proposition 8, Cyril Domb (personal communication) found two equivalent expressions which are in fact corrections to a result that he presented in [Domb 60].

Proposition 9 (Domb) *With the Borda method,* $P(k\text{-ties}, n, 3)$ *is given by*

$$\frac{1}{6^n}\sum_{s,t}\frac{n!}{s!t!}\sum_{q=0}^{s}2^{s-q}\frac{(t+q)!}{((\frac{t+q}{2})!)^2}\frac{1}{q!(s-q)!},$$

$s,t = 0,1,\ldots n$, $2s+t=n$, $(t+q)$ *even. It is equivalent to*

$$\frac{1}{6^n}\sum_{s}\frac{n!}{(s!)^2(n-2s)!}\sum_{q=0}^{s}2^{s-q}\frac{(n-2s+q)!}{((\frac{n-2s+q}{2})!)^2}\frac{s!}{q!(s-q)!},$$

$n-2s+q$ *even.*

3.1.2 Approval voting with three candidates

When there are three candidates, SVS contains just 6 vectors: $(1,1,0)$, $(1,0,1)$, $(1,0,0)$, $(0,1,0)$, $(0,1,1)$ and $(0,0,1)$. If we project these 6 scoring vectors on H, we obtain six vectors with equal length and the angle between each pair of adjacent vector is $\pi/3$ as depicted in fig 3. Compared to te Borda method (fig.

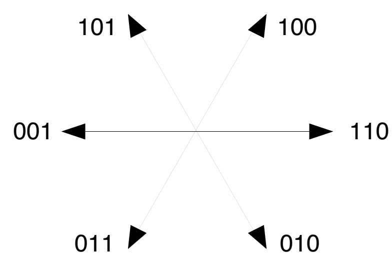

Figure 3: Projection of the scoring vectors

1), the only difference is that the vectors are longer and they have been rotated around the origin. Therefore, the lattice will have exactly the same structure as the lattice of the Borda method for 3 candidates. As a consequence, the Borda method and approval voting, with three candidates, will behave in the same way as long as we are interested by the probability of being on the line $x_1 = x_2 = \ldots = x_k$ and, hence, we can restrict ourselves to the projection on H.

Proposition 10 *With approval voting,* $P(k\text{-ties}, n, 3)$ *is given by*

$$\frac{n!}{6^n}\sum_{2s+3t=n}2^{[t>0]}\sum_{q=0}^{s}\sum_{r=0}^{s-q}\frac{1}{q!r!(s-q-r)!(q+t)!(r+t)!(s-q-r+t)!},$$

where $[t>0]$ *equals 1 if* $t>0$ *and 0 otherwise.*

Of course, the expressions provided by Domb (prop. 9) are valid in this case as well.

When the number of candidates is strictly larger than 3, the similarity between approval voting and the Borda method vanishes. And there is no hope to recover it for some k. Indeed, for $k > 3, 2^k - 2 < k!$.

3.2 Generating functions and Green functions

3.2.1 Basic concepts

This section is based on some classic works about random walks. The interested reader will find a lot of valuable informations in [Hugues 95]. Let us consider a lattice in d dimensions. Let $P_n(\mathbf{l})$ denote the probability that a random walk ends after n steps at a node whose position, relatively to the starting node, is described by the d-dimensional vector $\mathbf{l}$. We will often speak about the node $\mathbf{l}$ instead of the node whose position is at the extremity of $\mathbf{l}$. Let $p(\mathbf{l}-\mathbf{l}')$ be the probability of a move from $\mathbf{l}'$ to $\mathbf{l}$. A very simple recurrence relation can be written:

$$P_{n+1}(\mathbf{l}) = \sum_{\mathbf{l}'} p(\mathbf{l}-\mathbf{l}')P_n(\mathbf{l}'), \tag{8}$$

the sum being taken over all nodes $\mathbf{l}'$ of the lattice.

To solve expression (8), the discrete Fourier transform is introduced:

$$\tilde{P}_n(\mathbf{k}) = \sum_{\mathbf{l}} \exp(i\mathbf{l}\cdot\mathbf{k})P_n(\mathbf{l}), \tag{9}$$

as well as the *structure function* of the lattice:

$$\lambda(\mathbf{k}) = \sum_{\mathbf{l}} \exp(i\mathbf{l}\cdot\mathbf{k})p(\mathbf{l}). \tag{10}$$

With the initial condition $\tilde{P}_0(\mathbf{k}) = 1$, it follows that

$$\tilde{P}_n(\mathbf{k}) = \lambda(\mathbf{k})^n. \tag{11}$$

The discrete Fourier transform can be inverted by the formula

$$P_n(\mathbf{l}) = \frac{1}{(2\pi)^d}\int\cdots\int_B \exp(-i\mathbf{l}\cdot\mathbf{k})\tilde{P}_n(\mathbf{k})d^d\mathbf{k} \tag{12}$$

where B is the *first Brillouin zone* ($B = [-\pi,\pi]^d$). Substituting (11) into (12), we obtain

$$P_n(\mathbf{l}) = \frac{1}{(2\pi)^d}\int\cdots\int_B \exp(-i\mathbf{l}\cdot\mathbf{k})\lambda(\mathbf{k})^n d^d\mathbf{k}. \tag{13}$$

Let us now define the generating function

$$P(\mathbf{l},\xi) = \sum_{n=0}^{\infty} P_n(\mathbf{l})\xi^n \tag{14}$$

and replace $P_n(\mathbf{l})$ by (13). We find after some manipulations

$$P(\mathbf{l},\xi) = \frac{1}{(2\pi)^d}\int \cdots \int_B \frac{\exp(-i\mathbf{l}\cdot\mathbf{k})d^d\mathbf{k}}{1-\xi\lambda(\mathbf{k})}, \tag{15}$$

which is the *Green function* of our lattice.

3.2.2 The Borda method with 3 candidates

We will use the lattice presented in fig. 2. But in order to simplify the calculations, we will distort it slightly so that each node of the lattice has integer coordinates. We obtain the lattice presented in fig. 4. The structure function

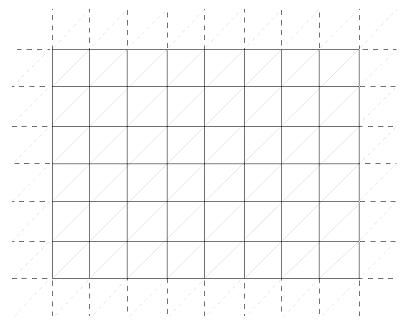

Figure 4: Distorted lattice of the Borda method for $k = 3$

for this lattice is

$$\lambda(\mathbf{k}) = \frac{1}{3}(\cos k_1 + \cos k_2 + \cos(k_1 + k_2)). \tag{16}$$

If we substitute (16) in (15), it can be proved (see [Zumofen and Blumen 82]) that

$$P(\mathbf{0},\xi) = \frac{6}{\pi\xi\sqrt{a+1}\sqrt{b-1}} K\left(\sqrt{\frac{2(b-a)}{(a+1)(b-1)}}\right), \tag{17}$$

where $a = \frac{3}{\xi} + 1 - \sqrt{3 + \frac{6}{\xi}}$ and $b = \frac{3}{\xi} + 1 + \sqrt{3 + \frac{6}{\xi}}$. In the vicinity of $\xi = 1$, expression (17) is equal to

$$P(\mathbf{0},\xi) = \frac{\sqrt{3}}{2\pi}\log(\frac{12}{1-\xi})(1 + O(1-\xi)) \tag{18}$$

and can be expanded into the following serie

$$P(\mathbf{0}, \xi) = \left(\frac{\sqrt{3}}{2\pi} \log 12 + \sum_{i=1}^{\infty} \frac{\sqrt{3}}{2\pi} \frac{\xi^i}{i} \right) (1 + O(1 - \xi)). \tag{19}$$

If we compare this expression to (14), we deduce that, for n sufficiently close to ∞,

$$P_n(\mathbf{0}) \approx \frac{\sqrt{3}}{2\pi n}. \tag{20}$$

Numerically, it turns out that this expression for $P_n(\mathbf{0})$ is quite good from $n \geq 10$. Note that $P_n(\mathbf{0})$ is nothing but $P(k\text{-ties}, n, 3)$. Hence, for the Borda method with 3 candidates, we know that the probability of a k-ties decreases smoothly with n as in (20) and is equal to 0 when n is infinite.

Proposition 11 *With the Borda method,* $P(k\text{-}ties, n, 3) \approx \frac{\sqrt{3}}{2\pi n}$ *and* $\lim_{n=\infty} P(k\text{-}ties, n, 3) = 0$.

3.2.3 The Borda method with 2 candidates

When there are only two candidates, the lattice in $k-1$ dimensions is quite simple. All nodes are equally spaced on a single line. The structure function is $\cos k_1$. Let us substitute it in (15), set $\mathbf{l} = \mathbf{0}$ and we obtain

$$P(0, \xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_1}{1 - \xi \cos k_1}. \tag{21}$$

If we impose that $|\xi| < 1$,

$$P(0, \xi) = \frac{1}{\sqrt{1 - \xi^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!n!} (pq\xi^2)^n. \tag{22}$$

Comparing this expression to (14), we come to the conclusion that, for n even,

$$P_n(0) = \frac{1}{4} \frac{(2n)!}{n!n!},$$

while it is zero otherwise. This result was already presented in proposition 3 but was obtained in a purely combinatorial way.

The method presented in the previous sections could in principle be applied to the Borda method with any number of candidates. Unfortunately, the integration of the Green function becomes very difficult. We were not able to integrate it for $k > 3$.

3.2.4 The generalised Borda method with weak orders

When there are two candidates, the lattice is the same as in the previous section except that there is a loop at each node. This loop corresponds to the scoring vector (3/2,3/2) which itself corresponds to the weak order such that the voter is indifferent between both candidates. The structure function for this lattice is

$$\lambda(\mathbf{k}) = \frac{1}{3} + \frac{2}{3}\cos k_1. \tag{23}$$

Let us substitute it into (13) and set $\mathbf{l}$ to 0. We obtain

$$P_n(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\frac{1}{3} + \frac{2}{3}\cos k_1)^n dk_1. \tag{24}$$

Let us expand the *cosine* and make the substitution $k_1 = q/\sqrt{n}$. We find that

$$P_n(0) = \frac{1}{2\pi\sqrt{n}}\int_{-\infty}^{\infty}(1 - \frac{q^2}{3n} + \ldots)^n dq. \tag{25}$$

When n is large enough, we obtain the approximation

$$P_n(0) \approx \frac{1}{2\pi\sqrt{n}}\int_{-\infty}^{\infty}(1 - \frac{q^2}{3n})^n dq = \frac{1}{2\pi\sqrt{n}}\int_{-\infty}^{\infty}\exp(-\frac{q^2}{3})dq = \frac{\sqrt{3}}{2\sqrt{\pi n}}. \tag{26}$$

Numerically, it turns out that this expression for $P_n(0)$ is quite good from $n \geq 10$. Note that $P_n(0)$ is nothing but $P(k\text{-ties}, n, 2)$. Hence, for the generalised Borda method with 2 candidates, we know that the probability of a k-ties decreases smoothly with n as in (26) and is equal to 0 when n is infinite. Remember that, in proposition 4, using only combinatorial arguments, we couldn't find the limit for $P(k\text{-ties}, n, 2)$. We only found an upper bound. Now, thanks to (26), we state a much better result.

Proposition 12 *For the generalised Borda method with weak orders,*
$P(k\text{-}ties, n, 2) \approx \frac{\sqrt{3}}{2\sqrt{\pi n}}$ *and*
$\lim_{n=\infty} P(k\text{-}ties, n, 2) = 0.$

When there are three candidates, the lattice is represented in figure 5. As the lattice is very dense, we only represent the vectors that generate it so that all node coordinates are integer.

The structure function for this lattice, $\lambda(\mathbf{k})$, is given by

$$\frac{2}{13}(\cos k_1+\cos k_2+\cos(k_1+k_2)+\cos(k_1-k_2)+\cos(2k_1+k_2)+\cos(k_1+2k_2))+\frac{1}{13}. \tag{27}$$

Unfortunately, we are not able to integrate the Green function with this structure function.

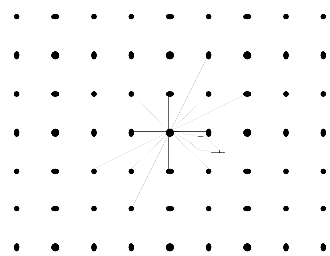

Figure 5: Distorted lattice of the generalised Borda method for $k = 3$

3.2.5 A general result about $P(k\text{-ties}, \infty, k)$

The Green function approach can be, at least in principle, applied to any scoring method. Unfortunately, the integration is often very difficult. Nevertheless, a very general result can be obtained.

Definition 1 Central symmetry. *A scoring method satisfies the property of central symmetry iff for each vector s in* SVS *there exists a vector s' in* SVS *such that $||s|| = ||s'||$ and $s + s'$ is a vector that has all coordinates identical.*

Many scoring methods verify central symmetry (approval voting, Borda, ...) but not all of them (e.g. plurality voting).

Proposition 13 *For any scoring method satisfying central symmetry,*
$\lim_{n=\infty} P(k\text{-ties}, n, k) = 0$ *and*
$P(k\text{-ties}, n, k)$ *is proportional to $n^{-(k-1)/2}$, for n large and k fixed.*

Proof. Let us project the vectors of SVS on H. Due to central symmetry, to any vector $\mathbf{l}$ corresponds a vector $-\mathbf{l}$. And if several vectors of SVS project on $\mathbf{l}$ the same number project on $-\mathbf{l}$. The probability to be assigned to $\mathbf{l}$ in the structure function is $1/C$ times the number of vectors projecting on $\mathbf{l}$. Note that if $\mathbf{l}$ is the null vector, then $\mathbf{l}$=$-\mathbf{l}$. It can also happen that several vectors project onto the null vector. Let us consider any pair of vectors $\mathbf{l}$ and $-\mathbf{l}$ and write the corresponding terms in te structure function. We have

$$p(\mathbf{l})[\cos(\mathbf{l}\cdot\mathbf{k}) + i\sin(\mathbf{l}\cdot\mathbf{k})] + p(-\mathbf{l})[\cos(-\mathbf{l}\cdot\mathbf{k}) + i\sin(-\mathbf{l}\cdot\mathbf{k})].$$

Thanks to central symmetry, we can rewrite it as

$$p(\mathbf{l})[\cos(\mathbf{l}\cdot\mathbf{k}) + i\sin(\mathbf{l}\cdot\mathbf{k}) + \cos(\mathbf{l}\cdot\mathbf{k}) - i\sin(\mathbf{l}\cdot\mathbf{k})]$$

$$= 2p(\mathbf{l})(\cos(\mathbf{l}\cdot\mathbf{k})).$$

The only vector that doesn't come in a pair is, eventually, the null vector. For that one, the corresponding term in the structure function is just 1. Let L

denote the set of vectors $\mathbf{l}$ such that $L \cup \{\mathbf{l} : -\mathbf{l} \in L\} \cup \{\mathbf{0}\}$ is equal to the set of all projections of SVS . Then, the general form of the structure function is

$$\lambda(\mathbf{k}) = p(\mathbf{0}) + \sum_{\mathbf{l}\in L} 2p(\mathbf{l})\cos(\mathbf{l}\cdot\mathbf{k}) = p(\mathbf{0}) + \sum_{\mathbf{l}\in L} 2p(\mathbf{l})(1 - (\mathbf{l}\cdot\mathbf{k})^2 + \ldots). \quad (28)$$

Let us make the substitution $\mathbf{q} = \mathbf{k}/\sqrt{n}$ and we find

$$\lambda(\mathbf{k}) = p(\mathbf{0}) + \sum_{\mathbf{l}\in L} 2p(\mathbf{l})(1 - \frac{1}{n}(\mathbf{l}\cdot\mathbf{q})^2 + \ldots). \quad (29)$$

Because all probabilities sum to 1,

$$\lambda(\mathbf{k}) = 1 - \frac{1}{n}\sum_{\mathbf{l}\in L} 2p(\mathbf{l})((\mathbf{l}\cdot\mathbf{q})^2 - \ldots). \quad (30)$$

Using the well-known limit

$$(1 + x/n)^n \to e^x \text{as } n \to \infty,$$

we conclude that

$$(\lambda(\mathbf{k}))^n \to \exp(\sum_{\mathbf{l}\in L} 2p(\mathbf{l})(\mathbf{l}\cdot\mathbf{q})^2)\text{as } n \to \infty. \quad (31)$$

Let us substitute this expression in (13), with $d = k - 1$.

$$\lim_{n=\infty} P_n(\mathbf{0}) = \frac{1}{(2\pi\sqrt{n})^d}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(\sum_{\mathbf{l}\in L} 2p(\mathbf{l})(\mathbf{l}\cdot\mathbf{q})^2)d^d\mathbf{q}. \quad (32)$$

We integrate from $-\infty$ until ∞ because we changed the variables $\mathbf{k}$ into $\mathbf{q}$. For this same reason, a factor $1/n^{d/2}$ has appeared. As the argument of the exponential is necessarily strictly positive, the integral converges. Therefore, because of the factor $1/n^{d/2}$, it is clear that

$$\lim_{n=\infty} P_n(\mathbf{0}) = 0$$

and for sufficiently large n , $P_n(\mathbf{0})$ is proportional to $1/n^{d/2}$. □

When SVS does not satisfy the central symmetry condition, the structure function is more complex. The imaginary terms do not cancel each other and the integral becomes much less tractable.

Note that the approximation given in proposition 13 is just an approximation and furthermore it is certainly not a good one when n is such that $P(k\text{-ties}, n, k) = 0$. This happens when the lattice contains no path of length n coming back to the origin. For example, if $SVS = \{(0,1,2),(2,1,0),(1,0,0),(0,1,1)\}$, the corresponding lattice can be distorted so as to take the form of a square lattice. And it is clear that paths of length n, where n is odd, cannot return to the origin. Another example is given by the Borda method with k even: in that case, no path of length n (n odd) returns to the origin.

3.3 Degenerated scoring methods

For some scoring methods, the structure of SVS can be such tat the corresponding lattice is actually living in $d < k-1$ dimensions. Such scoring methods are called degenerated. An example of such a scoring method is given by $SVS = \{(2,1,0),(0,1,2),(1,1,1)\}$. The corresponding lattice can be represented on a single line. For these methods, $P(k\text{-ties}, n, k)$ decreases with $n^{-d/2}$ and not $n^{-(k-1)/2}$. Sufficient but not necessary conditions to avoid degenerated scoring methods are presented below.

Definition 2 Neutral scoring vector. *A scoring vector s is called neutral iff for any r, s in X, $s_r = s_s$.*

Let σ be a permutation on X. A permutation of a scoring vector s is a scoring vector $\sigma(s)$ such that $\sigma(s)_r = s_{\sigma(r)}$ for all r in X.

Definition 3 Stability under permutation. *The set* SVS *is stable under permutation iff, for any s in* SVS *, $\sigma(s)$ is also in* SVS *.*

As far as we know, in all scoring methods actually in use, SVS is stable under permutation.

Proposition 14 *If* SVS *contains at least one non neutral scoring vector and if* SVS *is stable under permutation, then the corresponding scoring method is not degenerated*

The proof is left to the reader.

4 Some variants of our initial problem

4.1 Taking into account scoring vectors with different probabilities

Our definition of the probability $P(k\text{-ties}, n, k)$ is based on three assumptions.

1. The probability that a scoring vector be chosen by a given voter is the same for all vectors.

2. The probability that a voter chooses a given scoring vector is the same for all voters.

3. Each voter chooses a vector independently of the other voters.

These assumptions should and have been questionned [Gehrlein 83]. The combination of them has been called *Impartial Culture Condition* by Gehrlein [Gehrlein 83]. There is an easy way to get rid of the first one. We just have to introduce some weights in the definition of the function ϕ as follows (remember that C is the cardinal of SVS).

$$\phi(w_1, w_2, \ldots, w_C) = \sum_{s \in SVS} w_s x_1^{s_1} x_2^{s_2} \ldots x_k^{s_k}, \qquad (33)$$

where the w_s are real numbers such that their sum is equal to 1 and such that w_s is the probability of the scoring vector s. After raising $\phi(w_1, w_2, \ldots, w_C)$ to the n-th power, the coefficients that we obtain are the probabilities of all different profiles. To find the probabilty of a k-ties, noted $P(k\text{-ties}, n, k, w_1, w_2, \ldots, w_C)$, we can proceed as previously.

Proposition 15 *$P(k\text{-ties}, n, k, w_1, w_2, \ldots, w_C)$ is the sum of the coefficients of all terms of $\phi(w_1, w_2, \ldots, w_C)^n$ that can be written under the form $x_1^z x_2^z \ldots x_k^z$ for any z.*

Similar changes can also be introduced in the Green function (15) via a ponderation of the structure function (10). Alas, the integration becomes much more difficult.

4.2 About single-peakedness

The third assumption that we mentionned in section 4.1 is the independence of the voters (in a probabilistic way). If we abandon it, we can no longer use equations (7) and (33) because the probability of a profile is no more the product of the probabilities of the scoring vectors in that profile. Nevertheless, some results can be obtained.

Single-peakedness is a well known condition that has been introduced by Black [Black 58]. Under this condition, it can be shown that the Condorcet method never fails in pointing out a winner. Many variants of single-peakedness have been proposed in the literature: e.g. single-cavedness, single-peakedness over triples, ... As we are dealing with scoring vectors, unlike Black (he was dealing with complete orders), we need to define our own single-peakedness. We will define it over triples because it is less demanding.

Definition 4 Single-peakedness over triples. *A profile of scoring vectors is single-peaked over triples iff, for every $r, s, t \in X$ there exists a complete order T_{rst} on $\{r, s, t\}$ (let us say $rT_{rst}sT_{rst}t$, without loss of generality) such that, for every voter v_l,*
$s_r^l > s_s^l \Rightarrow s_s^l \geq s_t^l$
and
$s_s^l < s_t^l \Rightarrow s_r^l \leq s_s^l$.

This condition is not sufficient to prevent k-ties to happen. Consider the following 2-voter profile, on candidates set $X = \{1, 2, 3\}$, with scoring vectors $s^1 = (1, 2, 3)$ and $s^2 = (3, 2, 1)$. This profile could occur with the Borda method. If we choose T_{123} as $1T_{123}2T_{123}3$, we see that this profile is single-peaked over triples. Yet, it leads to a k-tie. Therefore, we need another condition.

Definition 5 Degeneracy. *Let p be a single-peaked over triples profile with T_{rst} the corresponding complete order on each triple $r, s, t \in X$. The profile p is degenerated iff, for any triple r, s, t and any voter l,*

$$rT_{rst}sT_{rst}t \Rightarrow \begin{cases} s^l_r > s^l_s > s^l_t \\ or \\ s^l_r < s^l_s < s^l_t. \end{cases}$$

And we obtain our first result about single-peakedness.

Proposition 16 *Let P be a set of single-peaked over triples and non degenerated profiles. With any scoring method, if only profiles from P are considered and $k > 2$, $P(k\text{-ties}, n, k) = 0$*

Proof. Let us consider a single-peaked over triples profile p. Because of non degeneracy, we can find a triple $\{r, s, t\}$ violating the conditions stated in definition 5. Because of single-peakedness, candidate s never (in no scoring vector of p) has a score strictly lower than the score of the two other ones. For this triple violates non degeneracy, we know that for at least one voter, s has a score strictly larger than the score of r or t. Therefore, when we sum up the scores, s necessarily has an overall score larger than that of r or t. □

Loosely speaking, single-peakedness can be seen as a property holding when all candidates and all voters can be put on a line (e.g. from left to right) such that the score of a candidate for a given voter is a monotonic function of the distance between the voter and the candidate. Degeneracy happens when the voters can be partitioned in two groups: one group lying on the left of the most leftist candidate and one group lying on the right of the most rightist candidate.

The condition of non degeneracy might be considered as too restrictive because, among others, it doesn't allow all voters to be indifferent between all candidates. But it is well worth to note that, if all voters are indifferent between all candidates, the k-tie should not be seen as an indeterminacy resulting from the use of a particular method. In such a case, the indeterminacy lies on the side of the voters, not on the side of the voting method and any method that would not lead to a k-tie would be highly dubious. Nevertheless, we formulate a second proposition, without non triviality.

Proposition 17 *Let* SVS *be stable under permutation and contain at least one non neutral scoring vector (with weight $\neq 0$). If only single-peaked profiles over triples are considered and $k > 2$, $P(k\text{-ties}, \infty, k) = 0$.*

Proof. We imposed that SVS contains at least one non neutral scoring vector. Let $\tilde{s}$ be this scoring vector. Because of stability under permutation, there is at least k non neutral scoring vectors (if only one component of $\tilde{s}$ is different from the other ones). Let $\{r, s, t\}$ be a triple in X such that at least one of $\{\tilde{s}_r, \tilde{s}_s, \tilde{s}_t\}$ is different from the other two. Such a triple necessarily exists as $\tilde{s}$ is not neutral.

Suppose that T_{rst} is a complete order on $\{r, s, t\}$, with $rT_{rst}sT_{rst}t$. Any neutral scoring vector is compatible (in the sense of single-peakedness over triples) with T_{rst}. Consider a non neutral scoring vector; say $\tilde{s}$.

(a) If $\tilde{s}$ is not compatible with T_{rst}, then $\tilde{s}_s < \tilde{s}_r$ and $\tilde{s}_s < \tilde{s}_t$. Let us consider $\sigma_1(\tilde{s})$ the permutation of $\tilde{s}$ such that the scores of r and s are exchanged. The vector $\sigma_1(\tilde{s})$ is compatible with T_{rst}. Let $\sigma_2(\tilde{s})$ be a permutation of $\sigma_1(\tilde{s})$ such that the scores of s and t are exchanged. This vector is also compatible with T_{rst}, in the sense of single-peakedness.

Let p be a single-peaked over triples profile with T_{rst} being the corresponding complete order for the triple $\{r, s, t\}$. Let p_1 be the same profile as p except that one voter, l, has been added, such that $s^l = \sigma_1(\tilde{s})$. Let p_2 be the same profile as p_1 with $s^l = \sigma_2(\tilde{s})$. If the scores of s and t are equal in $\sigma_1(\tilde{s})$, then p_1 is not degenerated and so is p_2. If the scores of s and t are not equal in $\sigma_1(\tilde{s})$, then the score of s is larger than that of t in $\sigma_1(\tilde{s})$ or in $\sigma_2(\tilde{s})$. In the first case, p_1 is not degenerated. In the second case, p_2 is not degenerated.

(b) If $\tilde{s}$ is compatible with T_{rst}, then we build p_3 and p_4, using $\tilde{s}$ and one of its permutations. And we show in the same way that at least one of p_3 and p_4 is not degenerated.

This proves that, in any single-peaked over triples profile, a non neutral scoring vector can be added so that the resulting profile is not degenerated.

When there are infinitely many voters, the probability that a profile contains only neutral scoring vectors is zero. Hence, any profile that will be formed will be non degenerated and proposition 16 can be applied. □

In the context of proposition 17, degenerated profiles are allowed. But they are very unlikely to occur when n is large. That is why proposition 17 holds. Propositions 16 and 17 can be easily rephrased by using single-cavedness instead of single-peakedness.

Definition 6 Single-cavedness over triples. *A profile of scoring vectors is single-caved over triples iff, for every $r, j, t \in X$ there exists a complete order T_{rst} on $\{r, s, t\}$ (let us say $rT_{rst}sT_{rst}t$, without loss of generality) such that, for every voter v_l,*

$s_r^l < s_s^l \Rightarrow s_s^l \leq s_t^l$
and
$s_s^l > s_t^l \Rightarrow s_r^l \geq s_s^l$.

4.3 Probability of other events

A k-ties is not the only case of indeterminacy that can happen. If most of the candidates are tied and only some of them have lower scores, a high degree of indeterminacy remains. This situation such that j ($j < k$) candidates belong to the winners set is called j-ties. Results about j-ties can easily obtained using the function ϕ. Let $N_j(n,k)$ be the number of profiles yielding a j-ties. Let $r_1, r_2, \ldots r_j$ be the j candidates that belong to the winners set and $s_1, s_2, \ldots s_{k-j}$ those that do not belong to the winners set.

Proposition 18 *$N_j(n,k)$ is the sum of the coefficients of all terms of ϕ^n that can be written under the form $x_{r_1}^z x_{r_2}^z \ldots x_{r_j}^z x_{s_1}^{z_1} x_{s_2}^{z_2} \ldots x_{s_{k-j}}^{z_{k-j}}$ for any $z > z_1, z_2, \ldots, z_{k-j}$.*

If SVS is stable under permutation, some simplifications can be performed in Proposition 18 as will be shown below.

Proposition 19 *If SVS is stable under permutation, $N_j(n,k)$ is equal to $\binom{k}{j}$ multiplied by the sum of the coefficients of all terms of ϕ^n that can be written under the form $x_1^z x_2^z \ldots x_j^z x_{j+1}^{z_1} x_{j+2}^{z_2} \ldots x_k^{z_{k-j}}$ for any $z > z_1, z_2, \ldots, z_{k-j}$.*

Sometimes, scoring methods are used as ranking (or ordering) functions and not as choice functions. In such instances, one could be interested by the probability that the produced ranking be without ties, i.e. that all candidates have different scores. This can also be obtained using ϕ.

Proposition 20 *If SVS is stable under permutation, the number of profiles such that all candidates have different scores is equal to $k!$ multiplied by the sum of the coefficients of all terms of ϕ^n that can be written under the form $x_1^{z_1} x_2^{z_2} \ldots x_k^{z_k}$ for any $z_1 > z_2 > \ldots > z_k$.*

This proposition can also be formulated without stability under permutation (left to the reader).

The probability of j-ties can also, at least in principle, be computed using the Green function approach.

5 Numerical results

The goal of this section is to illustrate some analytical results that we obtained in the previous sections. We also display some numerical results for problems where we do not have analytical results. In all this section, our strategy is the following. When an explicit expression exists, we use it. When it becomes intractable, we turn to a Monte-Carlo technique. When there is no explicit expression, we use the ϕ function and identify the desired coefficients. When this becomes intractable, we turn to a Monte-Carlo technique.

The Monte-Carlo technique that we used is straightforwardly based on the impartial culture condition (see section 4.1). The number of generated profiles was 300000 for the high probability events (larger than 0.5), 1200000 for the low probability events (smaller than 0.0001) and somewhere in the middle for other events. We are aware that these numbers of profiles might be small but we are not interested by precise numerical results. We are rather looking at the general shape of the probabilities.

5.1 Central symmetry

Table 1 presents the probability of k-ties for the Borda method. For any k, the probability decreases very fast when n increases. And the rate of decrease increases with k.

	n													
k	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0.5	0	0.375	0	0.3125	0	0.27344	0	0.24609	0	0.22559	0	0.20947	0
3	0.16667	0.05556	0.06944	0.04630	0.04372	0.03601	0.03263	0.02884	0.02627	0.02392	0.02204	0.02040	0.01900	0.01777
4	0.04167	0	0.00750	0	0.00364	0	0.00238	0	*0.00148*	0	*0.00121*	0	*0.00088*	0
5	0.00833	0.00042	*0.00059*	*0.00029*	*0.00020*	*0.00012*	*0.00010*	*0.00011*	*0.00009*	*0.00008*	*0.00007*	*0.00004*	*0.00004*	*0.00003*
6	0.00139	0	*0.00005*	0	*0.00002*	0	*0.00001*	0	*0.00000*	0	*0.00000*	0		0
7	0.00020	0.00000	*0.00000*	*0.00000*										
8	0.00002	0		0		0		0		0		0		0
9	0.00000													
10	0.00000	0		0		0		0		0		0		0

	n													
k	16	17	18	19	20	30	40	50	60	70	80	90	100	infinity
2	0.19638	0	0.18547	0	0.17620	0.14446	0.12537	0.11228	0.10258	0.09503	0.08893	0.08387	0.07959	0
3	0.01670	0.01574	0.01489	0.01413	0.01344	0.00904	0.00681	0.00546	0.00456	0.00391	0.00342	0.00305	0.00274	0
4	*0.00092*	0	*0.00072*	0	*0.00061*	*0.00038*	*0.00024*	*0.00022*	*0.00015*	*0.00010*	*0.00006*	*0.00008*	*0.00008*	0
5	*0.00006*	*0.00005*												0
6		0		0										0

Table 1: Probabilities of k-ties with the Borda method. Five decimal places entries are truncated numbers. Others are exact. Italic entries denote probabilities estimated by a Monte-Carlo method.

In table 2, we show the probability of k-ties for the generalised Borda method with weak orders. For any k, the probability decreases very fast when n increases. And the rate of decrease increases with k. The probabilities are everywhere (where we computed them) lower than the probabilities of the Borda method (except when the probability for the Borda method is 0).

	n													
k	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0.33333	0.25926	0.23457	0.20988	0.19342	0.1797	0.16872	0.15948	0.15162	0.14481	0.13885	0.13356	0.12884	0.12458
3	0.07690	0.02777	0.02230	0.01600	0.01347	0.01148	0.01011	0.00902	0.00816	*0.00674*	*0.00631*	*0.00608*	*0.00553*	*0.00545*

	n													
k	16	17	18	19	20	30	40	50	60	70	80	90	100	infinity
2	0.12072	0.1172	0.11397	0.11099	0.10823	0.08865	0.07689	0.06884	0.06288	0.05824	0.05450	0.05140	0.04877	0
3	*0.00483*	*0.00486*	*0.00461*	*0.00448*	*0.00426*	*0.00292*	*0.00210*	*0.00163*	*0.00136*	*0.00123*	*0.00105*	*0.00100*	*0.00091*	0

Table 2: Probabilities of k-ties with the generalised Borda method with weak orders. Five decimal places entries are truncated numbers. Others are exact. Italic entries denote probabilities estimated by a Monte-Carlo method.

The probability of k-ties for the plurality method are presented in table 3. Once more, for any k, the probability decreases very fast when n increases. And the rate of decrease increases with k. The probabilities are almost everywhere 0. When they are not 0, they are larger than the probabilities of the Borda method (where we computed them).

These results as well as others that we do not reproduce here seem to indicate that the rate of decrease of the probability in n and in k is a related somehow to the richness of SVS . But this is just a vague conjecture.

We also present two plots with the probabilities of k-ties for the Borda method, approval voting and plurality voting with three candidates (figure 6) and the Borda method and plurality voting with five candidates (figure 7). In addition, on these plots, we added the probability that there is no Condorcet winner. Note that the absence of Condorcet winner is not the only reason for indeterminacy in the Condorcet method. When the number of voters is even, k-ties can also happen. We have not taken these probabilities into account.

5.2 No central symmetry

We were not able to prove the equivalent of proposition 13 in the case of methods that do not have the central symmetry. But the proposition might be true.

Let us look at plurality voting. It doesn't satisfy the condition of central symmetry. Nevertheless, if we consider expression (6), we see that the probability of k-ties is proportional to $n^{(k-1)/2}$ as predicted by proposition 13. To check if this is true for other methods violating the central symmetry condition, we computed the probabilties of k-ties for the scoring method defined by the following expression.

k \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0.5	0	0.375	0	0.3125	0	0.27344	0	0.24609	0	0.22559	0	0.20947	0
3	0	0.22222	0	0	0.12346	0	0	0.08535	0	0	0.06520	0	0	0.05274
4	0	0	0.09375	0	0	0	0.03845	0	0	0	0.02203	0	0	0
5	0	0	0	0.03840	0	0	0	0	0.01161	0	0	0	0	0.00551
6	0	0	0	0	0.01543	0	0	0	0	0	0.00344	0	0	0
7	0	0	0	0	0	0.00612	0	0	0	0	0	0	0.00100	0
8	0	0	0	0	0	0	0.00240	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0.00094	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0.00036	0	0	0	0	0

k \ n	16	17	18	19	20	30	40	50	60	70	80	90	100	infinity
2	0.19638	0	0.18547	0	0.17620	0.14446	0.12537	0.11228	0.10258	0.09503	0.08893	0.08387	0.07959	0
3	0	0	0.04428	0	0	0.02696	0	0	0.01363	0	0	0.00912	0	0
4	0.01468	0	0	0	0.01067	0	0.00389	0	0.00214	0	0.00140	0.00000	0.00100	0
5	0	0	0	0	0.00320	0.00147	0.00084	0.00054	0.00038	0.00028	0.00022	0.00017	0.00014	0
6	0	0	0.00135	0	0	0.00040	0	0	0.00007	0	0	0.00003	0	0
7	0	0	0	0	0	0	0	0	0	0.00001	0	0.00000	0	0
8	0.00029	0	0	0	0	0	0.00001	0	0	0	0	0.00000	0	0
9	0	0	0.00008	0	0	0	0	0	0	0	0	0.00000	0	0
10	0	0	0	0	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0

Table 3: Probabilities of k-ties with plurality voting. Five decimal places entries are truncated numbers. Others are exact. Italic entries denote probabilities estimated by a Monte-Carlo method.

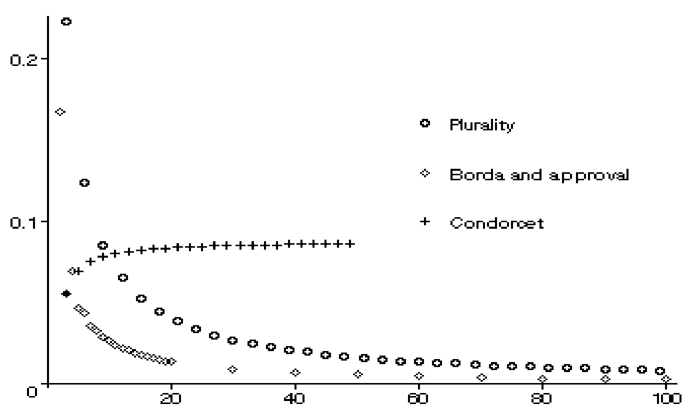

Figure 6: Probabilities of indeterminacy with different methods in the case of 3 candidates. Condorcet values are taken from Gehrlein, 1983

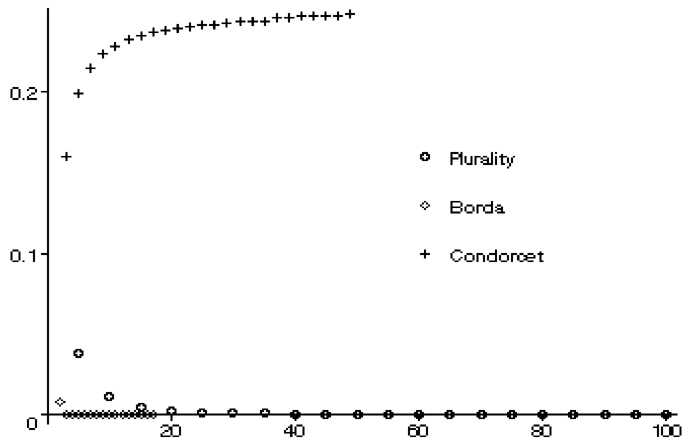

Figure 7: Probabilities of indeterminacy with different methods in the case of 5 candidates. Condorcet values are taken from Gehrlein, 1983

$$SVS = \{(1,3,4),(2,0,1),(4,4,2),(2,2,4)\} \tag{34}$$

The results are presented in figure 8. Instead of plotting the probabilities themselves, we plotted $P(k\text{-ties}, n, 3)/n$. If proposition 13 is true for this scoring method, then we should observe all points lying close to an horizontal line. It is not at all the case. Hence, we can conclude that, in general, proposition 13 is false when central symmetry doesn't hold .

6 Conclusion

Under the hypothesis of impartial culture, from all the results that we collected, analytically and numerically, we can quite safely conclude that the probability of k-ties with the Borda method is very low and certainly much lower than the probability of non existence of a Condorcet winner (except in the case of three voters and three candidates). The same holds for a large class of scoring methods (those satisfying central symmetry) and the plurality method. Even for those not satisfying central symmetry, the probability seems to be very low.

Which scoring method or which class of scoring methods has the lowest probabilities? It seems that, for given n and k, the larger SVS , the smaller the probability. But in all the instances where we computed numerical probabilities, they are so small that comparing different scoring methods on the basis of probabilities seems futile.

We explored in a very limited way the impact on probabilities that conditions on a profile structure can have. Under single-peakedness, the possibility of a

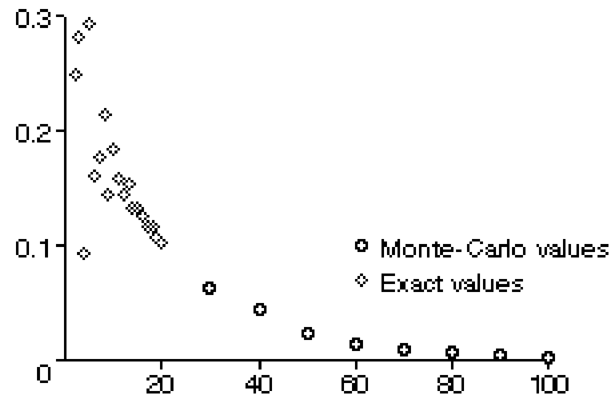

Figure 8: $P(k\text{-ties}, n, 3)/n$ as a function of n for the scoring method defined by expression (34)

k-ties almost vanishes. But other conditions might be more realistic. Under these conditions, our conclusions could be confirmed or rejected. They should be explored.

Many interesting problems remain open: explicit expressions for many scoring methods, asymptotic results for classes of methods that do not satisfy the central symmetry condition, asymptotic results for particular methods, ...

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