Cardinality and the Borda score.

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Abstract :

In most theoretical papers about the Borda rule, the Borda score appears as a number whose only purpose is to allow the derivation of a ranking. So, only his ordinal properties are considered; not his cardinal ones (if any). In some practical papers, the Borda score is implicitly considered to have cardinal properties. In this paper we present characterizations of the Borda rule where differences of Borda scores play an important role.

Key words :

Multi criteria analysis, Borda, fuzzy relation.

1. Introduction

Whether it is in multicriteria decision aid or in social choice, when one has to aggregate a profile of preferences expressed under the form of crisp binary relations, it is very common to use the Borda method, yielding a weak order (ranking method) or a choice set (choice method). The Borda method can easily be generalized in order to allow also the aggregation of fuzzy relations instead of crisp ones (Marchant, 1996).

Young (1974) characterized the Borda method as a choice method when the preferences are expressed by means of linear orders (crisp, asymmetric, complete and transitive relations). Nitzan and Rubinstein (1981) axiomatized it as a ranking method aggregating asymmetric and connected relations. Debord (1987) generalized Young's result when he presented a characterization of the Borda method as a choice method when the preferences are modelled by crisp binary relations belonging to a family D, that must contain all linear orders. He also characterized the Borda method as a ranking method for profiles of weak orders. Marchant (1996) generalized the previous results by characterizing the Borda method as a ranking (or choice) method when the set of the allowed preferences is any set of fuzzy relations, provided that this set contains all crisp weak orders and satisfies some weak technical conditions.

In each characterization, the axioms are almost the same : neutrality, cancellation, consistency and faithfulness (monotonicity for Nitzan and Rubinstein). We don't present an exact formulation of these axioms because they are slightly different for each author, according to the context, but the idea is the following :

- *Neutrality* expresses the fact that the result of the method does not depend on the names or labels given to the alternatives or items to be compared.
- *Cancellation* : if for any pair of alternatives there are as much voters in favour of the first alternative as in favour of the second one, then all alternatives tie.
- If the method is applied to two groups of voters and if the result is the same for both, *consistency* implies that the method applied to a group of voters made of the two previous groups must yield the same result.
- When there is only one voter, if the relation that he uses to express his preferences is so simple that one result seems the only reasonable one, the result of the method must be that one. This is what *faithfulness* says.
- If one voter improves the position of one alternative in the relation expressing his preferences, then *monotonicity* implies that the position of the alternative in the result of the method cannot worsen. Faithfulness and monotonicity are very much related : Debord (1987) showed that under some conditions, strict monotonicity and faithfulness are equivalent.

In all these studies, no attention is paid to the cardinal properties of the Borda score. The Borda score is only considered as an ordinal number allowing to derive a ranking. This is probably because the relations that are aggregated are usually considered as ordinal information and it would be strange to obtain cardinal (thus richer) information after the aggregation process. Information would seem to be created from nothing.

However, in some practice oriented papers, it appears that the Borda score is considered as cardinal :

- In the PROMCALC software, implementation of the PROMETHEE method described in Brans-Mareschal (1994), the net flow of the alternatives (which is a Borda score computed on fuzzy relations) is diplayed in the following form :

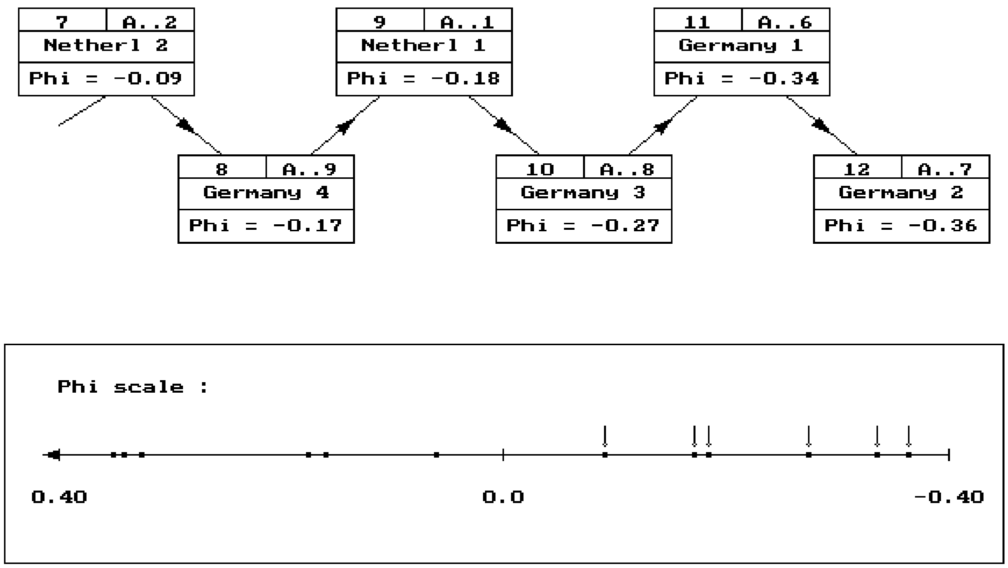

fig.1.: PROMCALC software

The upper part of the screen displays the tail of a ranking where each rectangle represents an alternative. The lower part displays an axis on which the little arrows indicate the Borda score of each alternative. This representation clearly suggests that the net flow has cardinal properties.

- In the same software, in the PROMETHEE V method, the Borda scores are used as coefficients of binary variables in the objective function of a linear program.
- In 1984, Brans-Mareschal-Vincke derive an interval order from the net flow of the PROMETHEE method.
- In 1993, Barberis derives interval orders, semiorders and pseudo orders from the net flow of the PROMETHEE method.
- In many committees the Borda rule is implicitly used and, according to our experience, it is not rare that, when some candidtates have slightly different Borda scores, a second tour is organized to break the « ties ».

Thus it seems to us that the formalism used up to now to study the Borda rule is not well suited to what the Borda rule acually is, at least in some cases. That's why we decided to suggest another formalism and to study the cardinal properties of the Borda score.

Definitions and notations are presented in section II. A generalization of the Borda method to the case of fuzzy relations is introduced. Section III is devoted to characterizations of the Borda method taking into account cardinal properties. Our results are discussed in the last section.

2. Definitions.

Let X be a finite set of alternatives of cardinality n. The alternatives will be denoted x, y, z, ... or x_1, x_2, ... Let $V = \{v, w, ...\}$ or $\{v_1, v_2, ...\}$ denote the set of the voters or criteria. We will consider that the set of voters is not fixed a priori. V can vary in size.

A fuzzy relation S is a mapping : $X^2 \rightarrow [0,1] : (x,y) \rightarrow S(x,y) = S_{xy}$. We shall write $\overline{S}$ a fuzzy relation such that $\forall\, x, y \in X, \overline{S}_{xy} = S_{yx}$. A relation S is rational iff $\forall\, x, y \in X$, S_{xy} is a rational number. A set D of fuzzy relations is said stable by transposition iff for any relation $S \in D$, $\overline{S} \in D$. Let σ be a permutation on the alternative set X. The expression $\sigma(S)$ will denote the fuzzy relation obtained from S by relabelling the elements of X according to σ. Thus $\forall\, x, y \in X$ $\sigma(S)_{xy} = S_{\sigma(x)\sigma(y)}$. A set D of fuzzy relations is said stable by permutation iff for any relation $S \in D$ and for any σ, $\sigma(S) \in D$. Obviously most sets D encountered in practice are stable by transposition and permutation. Let $H = \{h_1, h_2, ...\}$ be the set of all orders i.e. antisymmetric, strongly complete and transitive binary relations (Roubens and Vincke, 1985). A profile is a mapping p from V to a set D stable by transposition and permutation. $P_D = \{p, q, ...\}$ or $\{p_1, p_2, ...\}$ denotes the set of all profiles, given a set D of fuzzy relations.

Let M be the set of all real valued mappings on X. We will consider a numerical ordering function (NOF) as a mapping $U : P_D \rightarrow M : p \rightarrow U(p)$. And $U_x(p)$ denotes the value of $U(p)$ which is associated to x.

$p(v)_{xy}$ denotes the value associated with the arc (x,y) in the graph that represents the fuzzy relation of voter v in profile p.

Let $\pi_{xy}(p) = \sum_{v \in V} p(v)_{xy}$. If the relations in D are crisp (i.e. all arcs have a value belonging to $\{0,1\}$), then $\pi_{xy}(p)$ is the number of voters for which x is preferred to y.

Let p be a profile. We shall denote by $\overline{p}$ a profile such that $\forall\, x, y \in X, v \in V, p(v)_{xy} = \overline{p}(v)_{xy}$. Thus, $\overline{p}$ denotes a profile where each voter has reversed his preferences. Let σ be a permutation of the alternative set X. $\sigma(p)$ will denote the profile obtained from p by relabelling the elements of X according to σ.

3. Characterizations of the Borda rule.

In this section, we are going to consider the Borda rule as a NOF. The rule is then the following:

a) for each alternative x compute the generalized Borda score which is given by $B_x(p) = \sum_{y \in X} \pi_{xy}(p) - \sum_{y \in X} \pi_{yx}(p)$.

b) build $U(p)$ by means of the rule: $U_x(p) = B_x(p)$.

Let us now define the axioms that we will use to characterize the Borda rule.

A1. Neutrality : let σ be a permutation on the alternative set X. The expression $\sigma(U(p))$ will denote the function on X obtained from $U(p)$ by relabelling the elements of X according to σ. The mapping U is neutral if $U(\sigma(p)) = \sigma(U(p))$.

Neutrality implies that the method cannot depend on the names of the alternatives.

A2. Additivity : let S_1 and S_2 be a partition of V. To subset S_1 (resp. S_2) corresponds the profile p_1 (resp. p_2). To V corresponds the profile p also noted p_1+p_2. When $p_1 = p_2 = p$, we will write : $p_1+p_2 = 2p$. Consistency means that:
$\forall\, x \in X$, $U_x(p_1) + U_x(p_2) = U_x(p)$.

A3. Faithfulness : if $|V|= 1$ and the relation S used by the only voter is a crisp linear order then $U(p)$ is a real valued mapping on X respecting the relation S.

A4. Cancellation : $\forall\, x, y \in X$, $\pi_{xy}(p) = \pi_{yx}(p) \Rightarrow \forall\, x, y\ U_x(p) = U_y(p)$.

A5. Separability : let p and q be two profiles corresponding to two disjoint subsets of V, as in additivity (A2). For all x, y belonging to X , $U_x(p) - U_y(p) > U_y(q) - U_x(q) \Rightarrow U_x(p+q) > U_y(p+q)$.

Obviously, additivity implies separability.

If we interpret $U_x(p) - U_y(p)$ as a measure of the overal strength of preference of x over y in profile p, then, Separability can be interpreted as follows. If the strength of preference of x over y in profile p is larger than the strength of preference of y over x in profile q, then, x is preferred to y in profile p+q.

A6. Homogeneity : $\forall\, x, y \in X$, $\forall\, \alpha \in \mathrm{N}$ and such that $\alpha > 1$, $U_x(\alpha p) - U_y(\alpha p) = \alpha.\gamma.[U_x(p) - U_y(p)]$, where $\gamma \in \mathrm{R}^+$.

More intuitively, if we add to profile p a copy of itself, we obtain profile $2p$. Homogeneity implies that, for all pair of alternatives, the difference of their values in $2p$ is proportional to the difference in p.

Now we can propose two characterisations of the Borda method, whose proofs are somewhat similar to the proof given by Debord.

Theorem 1. Up to an affine transform, the Borda method is the only neutral (A1), additive (A2), and faithful (A3) numerical ordering function that satisfies cancellation (A4), when the set D of fuzzy relations contains H, the set of all orders.

Before proving this theorem, we shall go through seven lemmas.

Let $\mathcal{P}_x$ be the set of all permutations σ on X such that $\sigma(x) = x$.

Let's define $p_x = \sum_{\sigma \in \mathcal{P}_x} \sigma(p)$. It is a profile consisting of the juxtaposition of the profiles obtained by all the permutations σ on X leaving x unchanged. By construction, we have :

$\forall\ y, z \neq x$, $\pi_{yz}(p_x) - \pi_{zy}(p_x) = 0$ (because z and y have symmetrical positions in p_x) and

$\forall\ z \neq x$, $\pi_{xz}(p_x) - \pi_{zx}(p_x) = (n\text{-}1)!\ B_x(p)$.

Lemma 1. If U satisfies cancellation (A4) and $B_x(p) = 0$, then $\forall\ y, z \in X\ U_y(p_x) = U_z(p_x)$.

Proof. If $B_x(p) = 0$, then $\forall\ y, z \in X$, $\pi_{yz}(p_x) - \pi_{zy}(p_x) = 0$, and by cancellation, for all y and z belonging to X $U_y(p_x) = U_z(p_x)$. ■

Lemma 2. If U satisfies neutrality (A1) and separability (A5), then the following holds: $[\forall y \in X\ U_x(p) \geq U_y(p)$ and $\exists\ w, z \in X : U_w(p) \geq U_z(p)\ ] \Rightarrow U_x(p_x) > U_y(p_x)$ for all y belonging to X.

Proof. By neutrality, $\forall y \in X, \forall \sigma \in \mathcal{P}_x\ U_x(\sigma(p)) \geq U_y(\sigma(p))$.

$\exists\ w, z \in X : U_w(p) \geq U_z(p) \Rightarrow \forall y \in X, \exists\ \sigma \in \mathcal{P}_x : U_x(\sigma(p)) > U_y(\sigma(p))$. By separability, $\forall y \in X\ U_x(p_x) > U_y(p_x)$. ■

Lemma 3. If f satisfies neutrality (A1), separability (A5), cancellation (A4) and for all x belonging to X, $B_x(p) = 0$, then $\forall\ x, y\ U_x(p) = U_y(p)$.

Proof. Suppose that $\exists\ w, z \in X : U_w(p) \geq U_z(p)$. Then, by lemma 2, $\forall y \in X$, $\exists\ x : U_x(p_x) > U_y(p_x)$. But by lemma 1 we know that $\forall x, y, z \in X\ U_y(p_x) > U_z(p_x)$. Contradiction. ■

Lemma 4. If f satisfies cancellation (A4) and separability (A5), then $U_x(p) - U_y(p) = U_y(\overline{p}) - U_x(\overline{p})$.

Proof. Suppose that $U_x(p) - U_y(p) \neq U_y(\overline{p}) - U_x(\overline{p})$. By separability, we find that $U_x(p+\overline{p}) - U_y(p+\overline{p}) \neq 0$. But $\forall\ y, z \in X\ \pi_{yz}(p+\overline{p}) = \pi_{zy}(p+\overline{p})$. Thus, by cancellation, $U_x(p+\overline{p}) - U_y(p+\overline{p}) = 0$. Contradiction. ■

Lemma 5. if f satisfies neutrality (A1), separability (A5) and cancellation (A4), then $\forall x \in X\ B_x(p) = B_x(q) \Rightarrow \forall\ x, y \in X,\ U_x(p) - U_y(p) = U_x(q) - U_y(q)$.

Proof. $\forall x \in X\ B_x(p + \overline{q}) = 0$. By lemma 3, $\forall\ x, y \in X,\ U_x(p + \overline{q}) = U_y(p + \overline{q})$. Suppose that $U_x(p) - U_y(p) \neq U_x(q) - U_y(q)$. By lemma 4, $U_x(p) - U_y(p) \neq U_y(\overline{q}) - U_x(\overline{q})$ and by separability, $U_x(p + \overline{q}) - U_y(p + \overline{q}) \neq 0$. Contradiction. ■

Lemma 6. if f satisfies neutrality (A1), additivity (A2), faithfulness (A3) and cancellation (A4) and if q is a profile consisting of the only following order $x_1 > x_2 > x_3 > \ldots > x_n$ then $U_{x_1}(q) = a$, $U_{x_{i+1}}(q) = U_{x_i}(q) + b$ where a is a constant from R and b from R^+.

Proof. Let p_1 and p_2 be the following profiles, each consisting of one order : $x_1 > x_2 > x_3 > \ldots > x_n$ and $x_n > x_1 > x_2 > \ldots > x_{n-1}$ and let $p = p_1 + p_2$. Let us sort the elements of X according to their generalized Borda score and rename them as follows: y_1 for the element with the highest generalized Borda score, y_2 for the next one, ... and y_n for the element with the lowest generalized Borda score. This is licit since U is neutral. Let $\delta_i(p) = B_{y_i}(p) - B_{y_{i+1}}(p)$. Let p^{**} be the profile $y_n > y_{n-1} > \ldots > y_2 > y_1$ and p_i^* the profile $y_{i+1} > y_{i+2} > \ldots > y_n > y_1 > y_2 \ldots > y_i$. And finally, let p* be the profile $2np + \sum_i \delta_i(p)\ (p^{**} + p_i^*)$.

It is easy to verify that $\forall i\ B_{y_i}(p^*) = 0$. We show it just for y_1. $B_{y_1}(p) = 2n\text{-}4$, $B_{y_1}(p^{**}) = 1\text{-}n$, $B_{y_1}(p_1^*) = 1\text{-}n$, $B_{y_1}(p_2^*) = 3\text{-}n$, ..., $B_{y_1}(p_{n-1}^*) = n\text{-}3$. $B_{y_1}(p^*) = 2n\, B_{y_1}(p) + (4n\text{-}8)\, B_{y_1}(p^{**}) + \delta_1(p)\, (B_{y_1}(p_1^*)) + \delta_2(p)\, B_{y_1}(p_2^*) + \ldots + \delta_{n-1}(p)\, B_{y_1}(p_{n-1}^*)$ for $\sum_i \delta_i(p) = 4n\text{-}8$.

Thus $B_{y_1}(p^*) = 2n\, B_{y_1}(p) + (4n\text{-}8)\, B_{y_1}(p^{**}) + (B_{y_1}(p) - B_{y_2}(p))\,(1\text{-}n) + (B_{y_2}(p) - B_{y_3}(p))\,(3\text{-}n) + \ldots + (B_{y_{n-1}}(p) - B_{y_n}(p))\,(n\text{-}3) = 2n\, B_{y_1}(p) + (4n\text{-}8)\, B_{y_1}(p^{**}) + B_{y_1}(p)\,(1\text{-}n) + 2B_{y_2}(p) + 2B_{y_3}(p) + \ldots + 2B_{y_{n-1}}(p) + (3\text{-}n)B_{y_n}(p) = 2n\, B_{y_1}(p) + (4n\text{-}8)\, B_{y_1}(p^{**}) + B_{y_1}(p)\,(1\text{-}n) - 2B_{y_1}(p)$

$+ B_{y_n}(p)$ (1-n) for $\sum_i B_{y_i}(p) = 0$. Finally, we obtain $B_{y_1}(p^*) = (n\text{-}1)\, B_{y_1}(p) + (4n\text{-}8)\, B_{y_1}(p^{**})$ $+ B_{y_n}(p)\,(1\text{-}n) = 0$.

According to lemma 3, we can now state that $\forall\, x, y \in X\ U_x(p^*) = U_y(p^*)$. Since we supposed additivity, we can write $U_{x_i}(p^*) = 2n\ (\ U_{x_i}(p_1) + U_{x_i}(p_2)\ ) + \sum_i \delta_i(p)(U_{x_i}(p^{**}) + U_{x_i}(p_i^*))$. Each profile intervening in this expression can be obtained from q by a permutation of the set X. Thus by neutrality, $U_{x_i}(p^*)$ can be expressed as a linear function of $U_{x_i}(q)$, whose exact expression is not important for us. All $U_{x_i}(p^*)$ being equal, we can write $U_{x_1}(p^*) = U_{x_2}(p^*)$, $U_{x_1}(p^*) = U_{x_3}(p^*)$, ... , $U_{x_1}(p^*) = U_{x_n}(p^*)$. So, we obtain an homogeneous system of n-1 linear equations in n unknowns which are $U_{x_1}(q), \ldots , U_{x_n}(q)$.

We know that if we choose $U_{x_i}(q) = B_{x_i}(q)$, i.e. $U_{x_1}(q) = n\text{-}1$, $U_{x_{i+1}}(q) = U_{x_i}(q) + 2$, we obtain a solution of our system. Hence, it is obvious that [$U_{x_1}(q) = a$, $U_{x_{i+1}}(q) = U_{x_i}(q) + b$ where a and b are constants from R] is also a solution. We now have to prove that there is no other solution to our system, i.e. to show that any n-2 equations of our system are independent. This can be done by reasoning on the profile p^*. Suppose that we construct the profile p^* step by step, fixing the position of x_1 first, then x_2 and so on until we fix the position of x_n. After the second step, we have one equation of our system, after the third step, we have two equations, ... , after the n-2, we have n-3, after the n-1, we have n-2 equations but we may derive the last one because when n-1 alternatives are placed, there is no more choice for the nth. However, after step n-2, it is not possible to derive any other equation from the n-3 previous one, because we still have choice to fix the position of our remaining alternative. This proves that there is no other solution.

It is obvious that b must be a positive constant if the procedure is faithful. ■

Lemma 7. if U satisfies neutrality (A1), additivity (A2) and cancellation (A4), then $\forall x \in X$, $\forall \alpha \in \mathrm{N}\ B_x(p) = \alpha\, B_x(q) \Rightarrow \forall\, x, y \in X,\ U_x(p) - U_y(p) = \alpha\,(U_x(q) - U_y(q)\,)$.

Proof. $\forall x \in X\ B_x(\alpha\, q) = \alpha B_x(q) = B_x(p)$. By lemma 5, $\forall\, x, y \in X,\ U_x(\alpha\, q) - U_y(\alpha\, q) = U_x(p) - U_y(p)$ and by additivity, $\alpha\,(U_x(q) - U_y(q)\,) = U_x(p) - U_y(p)$. ■

Proof of theorem 1. Let us sort the elements of X according to their generalized Borda score and rename them as follows: x_1 for the element with the highest generalized Borda score, x_2 for the next one, ... and x_n for the element with the lowest generalized Borda score.

Let p_i* be a profile consisting of only two crisp orders : $x_1 > ... > x_i > x_{i+1} > ... > x_n$ and $x_i > ... > x_1 > x_n > ... > x_{i+1}$. Let p* be a profile defined by $p^* = \delta_1(p)\, p_1^* + \delta_2(p)\, p_2^* + ... + \delta_{n-1}(p)\, p_{n-1}^*$. It is easy to verify (see lemma 6) that $\forall i\ B_{x_i}(p^*) = 2n\ B_{x_i}(p)$. Thus by lemma 7, $U_x(p) - U_y(p) = (U_x(p^*) - U_y(p^*)) / 2n$. And p* consisting only of crisp orders, we know by lemma 6 that, up to an affine transform, $U_x(p^*)$ is equal to $B_x(p^*)$. ■

Theorem 2. Up to an affine transform, the Borda method is the only neutral (A1), separable (A5), homogeneous (A6) and faithful (A3) numerical ordering function that satisfies cancellation (A4), when the set D of fuzzy relations contains only rational relations.

Before proving this theorem, we shall go through two lemmas.

Lemma 8. Under conditions of theorem 2, for any p, the differences $U_x(p) - U_y(p)$ are proportional to the differences $B_x(p) - B_y(p)$.

Proof. Let us consider three alternatives x, y, z such that$B_x(p) - B_y(p)$ is equal to $\varepsilon.[B_y(p) - B_z(p)]$ with ε positive. If it is not possible to find three such alternatives, it means that there are at most two different Borda scores and the proof of the lemma is trivial.

For ε is rational and positive, there exists natural numbers α and β such that α and β are greater than 1 and $\alpha/\beta = \varepsilon$. We obtain $B_x(\beta p + \alpha \bar{p}_{xyz}) = B_y(\beta p + \alpha \bar{p}_{xyz})$ where p_{xyz} is a profile obtained from p by the following permutation: $\sigma(x) = z$, $\sigma(y) = x$, $\sigma(z) = y$ and $\forall\ w \neq x, y, z, \sigma(w) = w$. According to lemma 5 and neutrality,

$U_x(\beta p + \alpha \bar{p}_{xyz}) = U_y(\beta p + \alpha \bar{p}_{xyz})$. Let us suppose that

$U_x(p) - U_y(p) \neq \varepsilon.[\ U_y(p) - U_z(p)]$.

By homogeneity,

$U_x(\beta p) - U_y(\beta p) = \beta.\gamma.[\ U_x(p) - U_y(p)]$

and

$$
\begin{aligned}
U_x(\alpha \bar{p}_{xyz}) - U_y(\alpha \bar{p}_{xyz}) &= \alpha.\gamma.[\ U_x(\bar{p}_{xyz}) - U_y(\bar{p}_{xyz})] \\
&= \alpha.\gamma.[\ U_z(p) - U_y(p)] \\
&\neq \alpha.\gamma.[\ U_y(p) - U_x(p)]/\varepsilon = \beta.\gamma.[\ U_y(p) - U_x(p)].
\end{aligned}
$$

Using separability, we can conclude that $U_x(\beta p + \alpha \bar{p}_{xyz}) \neq U_y(\beta p + \alpha \bar{p}_{xyz})$, which is a contradiction. Thus the differences $U_x(p) - U_y(p)$ are proportionnal to the differences $B_x(p) - B_y(p)$ or to their opposite, for any x and y. Finally, we see without any difficulty that faithfulness imposes that $U_x(p) - U_y(p)$ be proportional to $B_x(p) - B_y(p)$. ■

Lemma 9. Let p be a profile such that $B_x(p) - B_y(p) = \tau$. Let q be a profile such that, for at least one pair (w, z), $B_w(q) - B_z(q) = \tau$. Under conditions of theorem 2, the proportionality coefficient between $U_x(p) - U_y(p)$ and $B_x(p) - B_y(p)$ is the same as between $U_x(q) - U_y(q)$ and $B_x(q) - B_y(q)$.

Proof. Let p and q be two profiles as described in lemma 9. If we permute y and w as well as x and z, it is easy to build a profile q' such that $B_y(q') - B_x(q') = \tau$. Hence,

$B_y(p+q') - B_x(p+q') = 0$ and $U_y(p+q') = U_x(p+q')$.

Let us suppose that the proportionality coefficients in p and q are different; then

$U_x(p) - U_y(p) \neq U_w(q) - U_z(q)$.

By neutrality, $U_y(q') - U_x(q') = U_w(q) - U_z(q)$. By separability,

$U_y(p+q') \neq U_x(p+q')$. Contradiction. ■

Proof of theorem 2. Let p and q be two profiles such that there are no x, y, z, w verifying $B_x(p) - B_y(p) = B_w(q) - B_z(q)$. It is always possible to build a profile q'' such that $B_x(p) - B_y(p) = B_{x_1}(q'') - B_{x_2}(q'')$ and $B_{x_3}(q'') - B_{x_4}(q'') = B_w(q) - B_z(q)$. It follows that the proportionality coefficient is the same in p and q. This, along with lemma 8 and 9, completes the proof of theorem 2. ■

4. Discussion.

Finiteness of X. (This paragraph has been inspired by Bouyssou, 1997). Our sets of axioms characterize the Borda method, up to an affine transformation. Is such a strong result necessary ? If our goal is to compare $U_x(p) - U_y(p)$ to $U_x(p) - U_z(p)$ and to be able to state

(i) $U_x(p) - U_y(p) = \alpha\,(U_x(p) - U_z(p))$,

with α being real, then, the answer is yes, because the answer would remain yes even after an affine transformation. Our statement would be meaningful, in the sense of measurement theory (Roberts, 1979). If our goal is just to be able to state

(ii) $U_x(p) - U_y(p) > U_x(p) - U_z(p)$,

then, the answer is no. A weaker result would work as well. It would be sufficient to characterize a method equivalent to the Borda method, up to a transformation preserving the weak order on the differences. And there exist such transformations which are not affine because X is finite. The axioms to impose might then be weaker and more satisfying.

The question raised by Barberis (1993) and Brans-Mareschal-Vincke (1984) corresponds to (ii). But in PROMETHEE V, (i) is needed. When a user sees a representation as in figure 1, does he implicitly use statements of type (i) or (ii) ? It probably depends on many variables. That is why we think that both approach are interesting and should be followed. This paper studied (i). The (ii) approach is still open.

Fuzzy relations. The results presented in theorem 1 and 2 characterize the Borda method when applied to a profile of fuzzy relations. The fuzzy relations must belong to a set D containing (1) the total orders or (2) only rational relations and (1 & 2) stable by permutation and transposition. The restrictions on the set D are very weak. D can be for example the set of all weak orders, the set of all semi-orders, the set of tournaments or ... (It was already so in Debord, 1987, for the Borda method as a choice method). It can be also the set of all fuzzy (rational) relations or the set of all fuzzy (rational) relations equivalent to a homogeneous family of semi-orders (for definition, see Roubens and Vincke, 1985) or ... Hence our results place very few restrictions on D and can be applied in many different cases. For more details, see Marchant (1996).

Additivity and separability in multicriteria decision aid. (This paragraph has been inspired by Pirlot, 1996). The criteria in MCDA, unlike the voters in social choice, are often treated differently. And the differences are usually modelled by weights or importance coefficients. Up to now, we didn't introduce any weights in the Borda method but it can easily be done by redefining $\pi_{xy}(p)$:

$$\pi_{xy}(p) = \sum_{v \in V} w_v \, p(v)_{xy} \,,$$

where w_v is the weight of criterion v (as in PROMETHEE). If the weights are rational (and we can assume they are rational in real life problems) the introduction of weights is equivalent to letting each criterion be represented by a number of voters proportional to its weight. The interesting point is that our characterizations use additivity and separability which allow the size of the voters'set to vary. So it could seem that our characterizations are valid not only in social choice but also in multicriteria decision aid. But, in our proof, we use additivity and separability in a way which is not equivalent to letting the weights vary. In fact, we combine many profiles which are very different. It amounts to adding many new criteria. But we know that the number of criteria used in a decision process can be changed by a few units, not by huge numbers ($|\mathcal{P}_x|$) as we do in lemma 1. So these characterisations are not fully satisfactory from a MCDA point of view.

Additivity, separability and homogeneity. In theorem 2, the less appealing axiom of theorem 1 (additivity) has been replaced by separability, which is weaker and much more satisfying. But homogeneity has been added during the same operation. Homogeneity imposes that the ratios between the differences $U_x(\alpha p)$ - $U_y(\alpha p)$ and

$U_w(\alpha p)$ - $U_z(\alpha p)$ be independent of α. This is in our opinion not very strong and quite reasonable. It implies that if we multiply all weights by the same factor (we do not impose that the weights add up to unit), almost nothing changes, which is what we expect. At the same time homogeneity imposes that the differences $U_x(\alpha p)$ - $U_y(\alpha p)$ vary linearly with α. This is probably a less satisfying aspect of homogeneity.

Cancellation. In this axiom, we use $\pi_{xy}(p)$ which is a sum of valuations $p(v)_{xy}$. Thus, our characterizations are valid only if the valuations have cardinal properties. Otherwise, the cancellation property is meaningless. To know if the $p(v)_{xy}$ have cardinal properties (in many cases, they don't), the preference modelling phase must be studied thoroughly.

5. Appendix .

In this section, we show that the four axioms used in theorem 1 are independent. In order to do so, we build four numerical ordering functions, each one satisfying all axioms but one. We failed in proving the same for theorem 2. Thus it might be possible to prove theorem 2 with a smaller set of axioms.

- Let $U_x^*(p) = \sum_{v \in V} \sum_{y \in X} p(v)_{xy}$.

  This numerical ordering function is neutral, separable, additive, faithful, homogeneous but doesn't satisfy cancellation.

- Let $U_x^{**}(p) = -B_x(p)$. This numerical ordering function is neutral, separable, additive, homogeneous and satisfies cancellation but not faithfulness.

- Let ψ, bijective : $X \rightarrow \{1,2,..,|X|\}$ and $U_x^{***}(p) = \sum_{y \in X} \psi(y)(\pi_{xy}(p) - \pi_{yx}(p))$. This numerical ordering function is separable, additive, faithful, homogeneous and satisfies cancellation but not neutrality.

- Let $U_x^{****}(p) = \sum_{y \in X} (\pi_{xy}(p) - \pi_{yx}(p))/|V|$. This numerical ordering function is neutral, faithful, homogeneous and satisfies cancellation but neither separability nor a fortiori. In order to check that it is not separable, consider the following profiles:

$$p_1 : x > y > z \text{ et } p_2 : \begin{cases} y > z > x \\ y > z > x \\ y > z > x. \end{cases}$$

According to separability, we should find $x > z$, but we obtain $z > x$.

- homogeneity: as earlier mentioned, we didn't find any numerical ordering function satisfying neutrality, separability, faithfulness and cancellation but not homogeneity.

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