Conditional Expected Utility Criteria for Decision Making under Ignorance or Objective Ambiguity*

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June 21st 2018

Keywords: Ignorance, Ambiguity, Conditional Probabilities, Expected Utility, Ranking Sets, axioms

JEL classification numbers: D80, D81.

Abstract

We provide an axiomatic characterization of a family of criteria for ranking *completely uncertain* and/or *ambiguous* decisions. A completely uncertain decision is described by the set of all its consequences (assumed to be finite). An ambiguous decision is described as a set of possible probability distributions over a set of prizes. Every criterion in the family compares sets on the basis of their *conditional expected utility*, for some "likelihood" function taking strictly positive values and some utility function both having the universe of alternatives as their domain.

1 Introduction

Suppose that a public decision maker examines the possibility of adopting an economically costly regulation that would limit carbon emissions in the next 50

*With the usual disclaiming qualification, we are grateful to the associate editor and two anonymous referees for their helpful suggestions. We are also indebted to Bhavook Bhardwaj, Komal Malik and Kriti Manocha for their reading of the manuscript. The authors of the paper are listed in alphabetical order and deserve therefore equal blame and/or credit for its content.

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years with the aim of preventing global warming. The decision maker is *uncertain* about the impact of carbon emission on the average Earth temperature and tries to get evidence from the best scientists and available models about this. For instance, the decision maker could obtain, in Meinshausen, Meinshausen, Hare, Raper, Frieler, Knutti, Frame, and Allen (2009), the collection of estimated distributions of increase in Earth temperature (above the pre-industrial level) that would result from doubling the amount of carbon in the atmosphere depicted in Figure 1, taken from Millner, Dietz, and Heal (2013). The decision maker could possibly obtain similar collections of distributions of increases in the temperature of the Earth for alternative scenarios of variations of carbon emissions, and base the regulation policy on the information provided by those collections. This is an example of decision making under *objective ambiguity.*

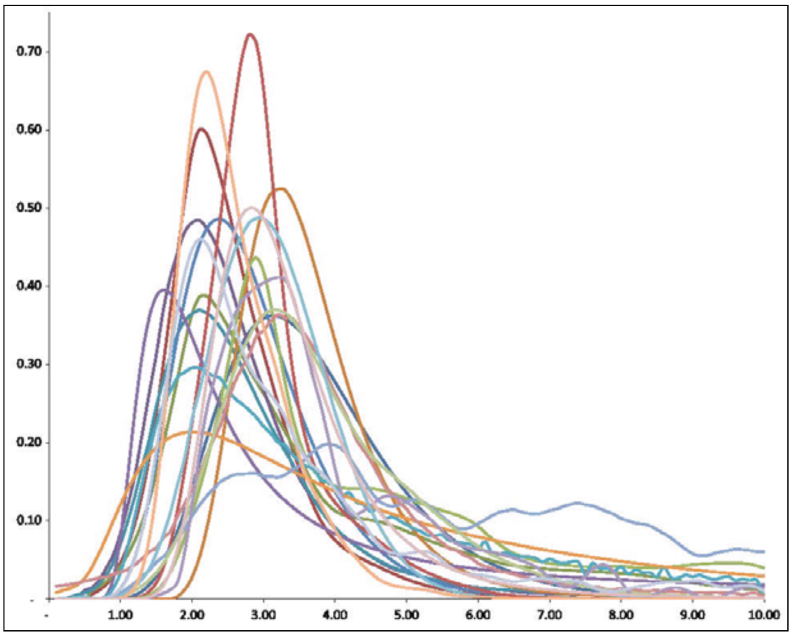

Figure 1: Estimated distributions of increase in the Earth temperature by 2050 under a "business as usual" scenario

There is *ambiguity* because the (probabilistic) knowledge required to take the decision is *not* unique. As shown in Figure 1, there are several estimates of the distributions of increase of the temperature of the Earth. The decision maker has no additional *a priori* knowledge that would enable a further discrimination between these different estimates. The ambiguity is, however, *objective* in the sense that these probability distributions are given to the decision maker by credible - here scientific - sources that he/she has all the reasons to believe. Other examples of decisions involving objective ambiguity include those involved in the well-known Ellsberg (1961) experiment and the excellent one, provided by Ahn (2008), of a gravely ill patient having to choose between two medical treatments associated with ambiguous evidence on their probability of survival.

From a formal point of view, deciding under objective ambiguity involves comparing *sets* of possible probability distributions such as that described in Figure 1. It differs to that extent from decision making under (subjective) ambiguity studied in an important literature (see e.g. Epstein and Zhang (2001), Ghirartado and Marinacci (2002), Ghirartado, Maccheroni, and Marinacci (2004), Klibanoff, Marinacci, and Mukerji (2005), Segal (1987) or Segal (1990)) that describes decisions as Savagian acts. Recall that the latter are functions from a set of (mutually exclusive) states of nature - that can be enriched to lotteries *à la* Anscombe and Aumann (1963) - into a set of consequences. Describing decisions as Savagian acts imposes a mathematical structure that may not always be present in actual decision making processes. For instance, the public decision maker who is given the probability distributions of Figure 1 is unlikely to have clear ideas - if any at all - on the "states of nature" that have generated these probability distributions. On the other hand, such a decision maker can very well understand that a given global warming policy be associated with a collection of different probability distributions concerning a consequence of interest - for instance the average Earth temperature. Additional justifications for describing decision making under objective ambiguity in terms of sets ranking can be found in Ahn (2008) or Olszewski (2007).

Ranking sets of objects also describes decision making processes in situations of *ignorance* or *radical uncertainty*—as these are sometimes called. In such situations, an element of a set is interpreted as a "certain" consequence that the decision associated to that set can have. A significant literature, surveyed for instance in Barberà, Bossert, and Pattanaik (2004), has analyzed such situations. With the notable exceptions of Baigent and Xu (2004) and Nitzan and Pattanaik (1984), most of the decision making criteria studied in this literature are based on the *best* and the *worst* consequences of the decisions or on associated lexicographic extensions. There are two obvious limitations of such "extremist" rankings. The first is that it is natural to believe (in line with various "expected utility" hypotheses) that decision makers are concerned with "averages" rather than "extremes". A second drawback of "extremist" rankings is that they do *not* allow for much diversity of attitudes toward ignorance across decision makers. In situations where decisions have only monetary consequences, all decision makers who prefer more money to less will rank all decisions in the same way under "extremist" rules such as maximin, maximax, leximin and so on. This is unsatisfactory since the fact that two decision makers have the same preference over certain outcomes should not imply that they have the same attitude toward ignorance.

In this paper, we pursue the line of inquiry of Gravel, Marchant, and Sen (2012) by characterizing of a family of rankings of sets that apply to decision making under either objective ambiguity - if the elements in the sets are probability distributions - or ignorance - if the elements are ultimate consequences. Contrary to the "extremist" criteria considered in the literature on ignorance, those examined herein can all be thought of as "smooth" averages of values attached by the decision maker to the elements of the sets. In Gravel, Marchant, and Sen (2012), we characterize the family of set rankings that can be thought of as resulting from the following two-step procedure. In the first step, all conceivable probability distributions (objective ambiguity) or consequences (ignorance) are evaluated by some (utility) function. In the second step, decisions are compared on the basis of their expected utility (given the function chosen in the first step) under the (uniform) assumption that all elements of a given set are equally likely. We call "Uniform Expected Utility" (UEU) any such ranking of sets. This uniform assumption about the occurrence of the various elements in the set is restrictive. In the example above, why would a public decision maker consider equally "credible" the different scientific studies that have given rise to the distributions of Figure 1 ?

The criteria characterized in this paper avoid this limitation, while keeping the "smoothness" associated with the evaluation of a decision on the basis of some average value. Specifically, any criterion characterized herein can be thought of as resulting from the following two-step procedure. In the *first step*, the decision maker assigns to every conceivable distribution of the temperature of the Earth, say, *two* different numerical valuations. One such valuation is interpreted, just as in the UEU case, as reflecting the "utility" associated to the distribution. Again, this "utility" can, but does not need to, be an "expected utility". The other valuation, restricted by our characterization to be strictly positive, is interpreted as reflecting the *a priori* "likelihood" attached by the decision maker to every conceivable distribution of the temperature of the Earth. For instance, the decision maker may believe that a sure increase of the temperature of the Earth by 3° is more likely than the unpredictable distribution of that increase associated with some of the distributions of Figure 1. In the *second step*, the decision maker compares alternative sets of distributions of the temperature of the Earth on the basis of their "expected utility", with expectations taken with respect to the likelihood function determined in the first step *conditional* on the fact that the distributions of the temperature of the Earth are present in the set. We refer to any such criterion as to a Conditional Expected Utility (CEU) criterion. Any UEU criterion is a member of this family that assumes, in the first step, that all distributions of the temperature of the Earth are equally likely. Hence the CEU family of criteria is a (significant) generalization of the UEU family that enables the decision maker to differently weigh the different estimates of the distributions of the temperature of the Earth in terms of their likelihood.

The CEU family of rankings of finite sets of objects characterized in this paper is a discrete and general version of the family of rankings of *atomless* measurable sets of probability distributions characterized by Ahn (2008) (and before him by Bolker (1966), Bolker (1967) and Jeffrey (1965)). Atomless measurable sets contain a continuum of elements and can not be finite like the set underlying Figure 1, the urns considered in Ellsberg's experiments or the choice of a medical treatment discussed in Ahn (2008). The fact that we consider only finite sets makes our setting very different from that of Ahn (2008). As indicated in Gravel, Marchant, and Sen (2012), we believe that our finite subsets framework is an important one conceptually, at least from the viewpoint of practical implementability and testability, and descriptive faithfulness.

The characterization of the CEU family of rankings of finite sets obtained in this paper uses three axioms, and assumes that the objects are taken from a "rich" environment that may (or may not) be endowed with a topological structure. Two of our axioms are common with those of Ahn (2008), and one of the two, called Averaging, was also used in the characterization of the UEU family of finite sets. Ahn (2008) obtains his characterization by combining the two axioms with two continuity conditions, and by exploiting the properties of his atomless set-theoretic structure. We obtain ours by combining the two axioms with an Archimedean condition, and by exploiting the assumed "richness" of the universe from which the finite sets are taken. We show that our three axioms are all independent, a result that is, to the best of our knowledge, not available in Ahn (2008) and in the Bolker (1966), Bolker (1967) and Jeffrey (1965) papers. However, unlike in our characterization of UEU in Gravel, Marchant, and Sen (2012), we are not able to provide a version of our characterization result that relies explicitly on a topological structure imposed on the universe of objects, one that replaces the richness condition and the Archimedean axiom by an appropriate continuity condition. Moreover the richness condition that we use is quite strong. For instance, it rules out any UEU criterion that uses a continuous utility function when applied to finite sets of objects taken from a topological space.

The organization of the remainder of this paper is as follows. The next section introduces the formal setting and discusses the axioms and the family of criteria characterized. The results are presented in section 3, and discussed, along with examples showing the independence of the axioms, in Section 4. Section 5 provides some conclusions.

2 The Model

2.1 Notation

The sets of (non-negative) integers, strictly positive integers, real numbers, non-negative real numbers and strictly positive real numbers are respectively denoted by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$. If v is a vector in $\mathbb{R}^k$ for some strictly positive integer k and α is a real number, we denote by $\alpha.v$ the scalar product of α and v. Our notation for vectors inequalities is $\geqq$ (weak inequality in every component), $\geq$ (weak inequality in every component with at least one of them strict) and $>$ (strict inequality in all components). Given a collection $\mathcal{C}$ of sets, a function $f : \mathcal{C} \to \mathbb{R}$ is *disjoint-set-additive* if it satisfies $f(A \cup B) = f(A) + f(B)$ for any two disjoint sets A and B in $\mathcal{C}$ when $A \cup B$ belongs to $\mathcal{C}$. By a *binary relation* $\succsim$ on a set Ω, we mean a subset of $\Omega \times \Omega$. Following the convention in economics, we write $x \succsim y$ instead of $(x, y) \in \succsim$. Given a binary relation $\succsim$, we define its *symmetric factor* $\sim$ by $x \sim y \iff x \succsim y$ and $y \succsim x$ and its *asymmetric* factor $\succ$ by $x \succ y \iff x \succsim y$ and not $(y \succsim x)$. A binary relation $\succsim$ on Ω is *reflexive* if the statement $x \succsim x$ holds for every x in Ω, is *transitive* if $x \succsim z$ always follows $x \succsim y$ and $y \succsim z$ for any $x, y, z \in \Omega$ and is *complete* if $x \succsim y$ or $y \succsim x$ holds for every distinct x and y in Ω. An equivalence class C of a binary relation $\succsim$ on Ω is a subset of Ω such that $c \sim c'$ for all $c, c' \in C$ and it is not the case that $c \sim c'$ if $c \in C$ and $c' \in \Omega \backslash C$. A reflexive, transitive and complete binary relation is called an *ordering*. An ordering is *trivial* if it has only one equivalence class.

2.2 Basic concepts

Let X (with $\#X \geq 3$) be an arbitrary universe of objects that we will refer to as "consequences". But keeping in mind the objective ambiguity context discussed in the preceding section, we could as well interpret X as the set of all conceivable probability distributions over a more fundamental set of "prizes" (perhaps different levels of the average Earth temperature). While we do not make any specific assumptions on X, it will subsequently become clear that the axioms that we propose make this set infinite and "rich".

We denote by $\mathcal{P}(X)$ the set of all non-empty *finite* subsets of X (with generic elements A, B, C, etc.). Any such subset is interpreted as a description of all consequences (probability distributions) of a particular decision under ignorance (ambiguity). Accordingly, any such set is called a *decision*. A *certain* (non-ambiguous) decision with the unique consequence $x \in X$ is identified by the singleton $\{x\}$.

Let $\succsim$ (with asymmetric and symmetric factors $\succ$ and $\sim$ respectively) be an *ordering* on $\mathcal{P}(X)$. We interpret the statement $A \succsim B$ as meaning "decision A is weakly preferred to decision B". A similar interpretation is given to $A \succ B$ ("strictly preferred to") and $A \sim B$ ("indifferent to").

We also find useful to define the (possibly empty) sets $m(X)$ and $M(X)$ of minimal and maximal decisions in X by $m(X) = \{A \in \mathcal{P}(X) : B \succsim A \ \forall B \in \mathcal{P}(X)\}$ and $M(X) = \{A \in \mathcal{P}(X) : A \succsim B \ \forall B \in \mathcal{P}(X)\}$. We then define the set $\mathcal{P}_*(X)$ by $\mathcal{P}_*(X) = \mathcal{P}(X) \setminus (m(X) \cup M(X))$. Hence, $\mathcal{P}_*(X)$ contains all finite subsets of X that are *not* maximal or minimal with respect to the ordering $\succsim$. One may, of course, have $\mathcal{P}_*(X) = \mathcal{P}(X)$ if no decisions are maximal or minimal for $\succsim$. Yet there are many plausible contexts where the set $\mathcal{P}_*(X)$ would differ from $\mathcal{P}(X)$. One of them is when the universe X is the $k-1$ dimensional simplex interpreted as the set of all lotteries on a finite set of k prizes. In such a setting, it seems natural that there is a "best" prize (say the certainty that no increase in the Earth temperature will take place in the next 50 years) and a "worst" prize (say the certainty that the Earth temperature will increase by 10° C in the next 50 years). If this is the case, the singleton that gives unambiguously the lottery that assigns a probability 1 to the best (resp. to the worst) prize could be maximal (resp. minimal) in $\mathcal{P}(X)$.

We want to identify the properties (axioms) of $\succsim$ that are necessary and sufficient for the existence of a function $u : X \to \mathbb{R}$ and a function $p : X \to \mathbb{R}_{++}$

such that, for every A and B in $\mathcal{P}(X)$:

$$A \succsim B \iff \frac{\sum_{a\in A} p(a)u(a)}{\sum_{a\in A} p(a)} \geq \frac{\sum_{b\in B} p(b)u(b)}{\sum_{b\in B} p(b)}. \tag{1}$$

We refer to an ordering numerically represented as per (1) for some functions p and u as a Conditional Expected Utility (CEU) criterion. Indeed, the function p is naturally interpreted as assigning to every consequence a number that reflects its *a priori* "plausibility" or"likelihood", while u is interpreted as a utility function that evaluates the "desirability" of every consequence from the decision maker's view point. Hence an ordering represented by (1) can be seen as comparing decisions on the basis of the expected utility of their consequences *conditional* upon the fact that these consequences will materialize. The requirement that $p(x) > 0$ for every $x \in X$ guarantees indeed that the "event" on which the conditioning is performed is well-defined.

We notice also that the family of UEU criteria characterized in Gravel, Marchant, and Sen (2012) is, *a priori*, a subclass of CEU family in which p is any constant function. However, it happens that the characterization provided below of this family is not fully complete and notably excludes UEU criteria. One reason for this is that we characterize this family by assuming that both the universe X and the ordering $\succsim$ satisfies the following "richness" condition that is somewhat different from the condition with the same name used in Gravel, Marchant, and Sen (2012).

Condition 1 Richness. *For all sets A, B, C, $\underline{D}$ and $\overline{D}$ in $\mathcal{P}^*(X)$ such that $\overline{D} \sim C \succ B \succ A \sim \underline{D}$, there exist sets $\underline{E}$ and $\overline{E}$ in $\mathcal{P}(X)$ satisfying $\underline{E} \cap (A \cup C \cup \underline{D}) = \varnothing = \overline{E} \cap (A \cup C \cup \overline{D})$ such that $\overline{E} \sim C$, $\underline{E} \sim A$ and $\overline{E} \cup A \sim B \sim \underline{E} \cup C$.*

This condition requires the domain to be sufficiently rich, and the ordering $\succsim$ to be sufficiently "smooth", for opening up the possibility of "matching" - in terms of indifference - any decision that is neither maximal nor minimal by appropriate combinations of other decisions that are strictly better, and strictly worse, than that decision. We emphasize that this condition restricts *both* the universe from which the objects are taken *and* the ordering $\succsim$. For instance, an "extremist" ordering like, say, the Maximin one that would compare sets on the basis of their "worst" - as per the ordering $\succsim$ restricted to singletons - elements would violate this condition. Another example of an ordering that violates the Richness condition is the ranking of all finite subsets of X based on their cardinality. It is difficult to fully appraise the strength of this richness condition. On the one hand, it may be considered to be weak because its asserted existence of specific sets $\underline{E}$ and $\overline{E}$ is contingent upon the existence of sets A, B, C, $\underline{D}$ and $\overline{D}$ having the properties indicated in the antecedent of the condition. On the other hand, it may be considered to be strong because, in addition to eliminating cardinality and Maximin types of criteria - also ruled out by the Averaging axiom introduced below - it also excludes from the class of rankings represented by (1) those for which the function p is constant. From a technical point of view, the characterization of the CEU criteria requires one to identify two objects: a ranking of sets based on the "probability of occurrence of their consequences" and a ranking of sets based on their (average) utility. A key role played by the richness condition for this endeavour is to guarantee the existence of sets with *any* strictly positive "probability of occurrence". Requiring the existence of sets of arbitrarily large probability is not terribly demanding. Indeed, such an increase in probability can be achieved by arbitrarily enlarging the size of the set. However, the converse requirement of existence of sets with arbitrarily small (but positive) probability of occurrence entailed by richness is more demanding. It should also be noticed that the richness condition is *not* necessary for an ordering to be represented by (1).

Before turning to the three necessary (and sufficient) axioms for an ordering on $\mathcal{P}(X)$ satisfying the richness condition to be a CEU criterion, we find useful to compare our framework to that of Ahn (2008), in which X is explicitly taken to be the $k-1$ dimensional simplex $S^{k-1} : \{x \in \mathbb{R}^k_+ : x_j \in [0,1]$ for all $j = 1, ..., k$ and $\sum_{j=1}^{k} x_j = 1\}$, interpreted as the set of all conceivable lotteries on some finite set of k prizes. Instead of considering finite subsets of S^{k-1}, Ahn (2008) applies his analysis to subsets of S^{k-1} that are equal to the closure of their interior (using the topology of the Euclidean distance) *and* to singletons. This means that non-singleton sets considered in Ahn (2008) all contain continuously many elements. Ahn (2008) characterizes all orderings $\succsim$ of the subsets of S^{k-1} that are equal to the closure of their interior - along with singletons - that can be written as:

$$A \succsim B \iff \frac{\int_A u(a)d\mu}{\mu(A)} \geq \frac{\int_B u(a)d\mu}{\mu(B)}. \tag{2}$$

for some continuous function $u : S^{k-1} \longrightarrow \mathbb{R}$ and some probability measure μ on the Borel subsets of S^{k-1}. Orderings that can be represented as per (2) have also been characterized by Bolker (1966), Bolker (1967) and Jeffrey (1965) (see e.g. Broome (1990) for a nice discussion of the Bolker-Jeffrey theory). One can view the representation (1) as a finite version of the representation (2) in which the measure μ is defined, for any finite set A, by:

$$\mu(A) = \sum_{a \in A} p(a) \tag{3}$$

It is worth observing that the atomless setting of Ahn (2008), Bolker (1966), Bolker (1967) and Jeffrey (1965) also entails, when combined with the two continuity conditions, the existence of sets with arbitrarily small probability of occurrence.

The first two axioms that characterize CEU family of orderings of $\mathcal{P}(X)$ are also used by Ahn (2008). The first of them has been called "Averaging" by Fishburn (1972) and Broome (1990). It also contributes to the characterization of the UEU family of criteria provided by Gravel, Marchant, and Sen (2012). The formal statement of this axiom is as follows.

Axiom 1 Averaging. *Suppose A and $B \in \mathcal{P}(X)$ are disjoint. Then $A \succsim B \iff A \cup B \succsim B \iff A \succsim A \cup B$.*

This axiom is stated somewhat differently in Ahn (2008) under the name "disjoint set betweenness". Ahn (2008) formal statement of the axiom is equivalent to that considered here when applied to a complete binary relation. The averaging axiom requires that enlarging the outcomes of a decision A to those of a (disjoint) decision B is worth doing (resp. not worth doing) *if and only if* the set B of added consequences is better (resp. worse) than the set A to which it is added. It captures an intuitive property satisfied by calculations of "averages" in various settings (e.g. adding a student to a class will increase the average of the class if and only if the grade of the added student is larger than the average of the class). The "only if" part of the axiom is strong since it asserts that the *only* reason for ranking a set B above (resp. below) a set A is when the addition of B to A is considered a good (resp. bad) thing. A weaker version of Averaging (that only requires the "if" part in its statement) is used in Olszewski (2007). A very similar axiom is also used in Gul and Pesendorfer (2001) for ranking menus of alternatives in a way that reflects "temptation" and "self-control". While *a priori* appealing, it has been reported by Vridags and Marchant (2015) to be violated by the choice behavior of a majority of subjects in experimental settings.

The second axiom is called Balancedness by Ahn (2008). It is stated as follows.

Axiom 2 Balancedness. *Suppose A and B are two sets in $\mathcal{P}(X)$ such that $A \sim B$. Suppose there exists a set $C \in \mathcal{P}(X)$ satisfying (i) $(A \cup B) \cap C = \varnothing$ (ii) $A \sim B \succ C$ and (iii) $A \cup C \succsim B \cup C$. Then one must have $A \cup D \succsim B \cup D$ for all sets $D \in \mathcal{P}(X)$ such that $(A \cup B) \cap D = \varnothing$ and $A \sim B \succ D$.*

This axiom can be viewed as a separability condition. It plays a key role in guaranteeing that the measure of finite sets provided by the finite summation of their elements by the function p in the representation (1) is well-defined. One of the difficulty indeed in characterizing CEU criteria is to disentangle the role played by the two functions in expression (1). The function u measures the utility of a consequence, while the function p evaluates the *a priori* "likelihood" of that same consequence. When do we have evidence that a (finite) collection of outcomes of a decision is "more likely" than another ? One such evidence - put forth by the balancedness axiom - is provided when the decision maker is indifferent between two decisions A and B but is not anymore indifferent between them if the set of possible outcomes to which they may lead is enlarged to outcomes of another decision C that is worse than both A and B. Suppose specifically that a decision leading to $A \cup C$ is deemed better than a decision leading to either $B \cup C$. Such a preference can only come from the fact that the good outcomes in $A \cup C$ (that are in A) are more likely than the good outcomes (in B) in $B \cup C$. The balancedness axiom guarantees that the definition of what it means for A to be more likely than B does not depend upon which particular bad set C is chosen.

The last axiom used in our characterization is the following "Archimedean" one.

Axiom 3 Archimedean. *Let* A, B, C, D *and* E *be sets in* $\mathcal{P}(X)$ *such that* $A \sim B \sim C \sim D \succ E$, $A \cup E \succ B \cup E$ *and* $E \cap (C \cup D) = \emptyset$. *Then, if there are two infinite sequences of sets* $A_0, A_1, \ldots, A_i, \ldots$ *and* $B_0, B_1, \ldots, B_i, \ldots$, *satisfying* $A_i \cap (E \cup C \cup A_j) = \emptyset = B_i \cap (E \cup D \cup B_j)$, $A_i \sim A$, $B_i \sim B$, $A_i \cup E \sim A \cup E$ *and* $B_i \cup E \sim B \cup E$ *for all* $i \neq j \in \mathbb{N}$, *there must be some* $n \in \mathbb{N}$ *for which* $C \cup E \bigcup_{i=0}^{n} A_i \succsim D \cup E \bigcup_{i=0}^{n} B_i$ *holds.*

The Archimedean axiom is cumbersome to express but has a simple interpretation in our context : no decision is infinitely more likely than any other. As stated here, the axiom applies to decisions A and B between which the decision maker is indifferent. Suppose that the uncertainty relating to these two decisions is "enlarged" to some set E that is worse than both A and B. Suppose also that the decision leading to the enlarged consequences in $A \cup E$ is strictly better than a decision with consequences in $B \cup E$. As discussed above, this ranking suggests that decision A is more likely than decision B. Consider then replacing decisions A and B by a sequence of decisions A_j and B_j (respectively, for $j = 1, \ldots,$) that are, again respectively, disjoint from A and from B. Suppose that this replacement is a matter of indifference for the decision maker. Intuitively then, the sets A_0, $A_1, \ldots, A_i, \ldots$ (resp. $B_0, B_1, \ldots, B_i, \ldots$) can all be considered to be "clones" of A (resp. B) relative to E in the sense that the decision maker is totally indifferent between a decision with consequences in either any of these sets or E and a decision with consequences in A (resp. B) or in E. The Archimedean axiom says that replacing, in this enlargement to E, decision A by an indifferent decision C and replacing, in the very same enlargement to E, decision B by an equivalent decision D can not reverse the ranking of $C \cup E$ *vis-à-vis* $D \cup E$ to such an extent that the reversal - if any - can not be outweighed by adding to $C \cup E$ and to $D \cup E$ a suitably long sequence of clones of A and B respectively. That is, decision C cannot be "infinitely more likely" than D.

Although this axiom is technical, it is required in the characterization, as shown in example 1 of section 4. Ahn (2008) does not use an Archimedean axiom. He uses, instead, two continuity axioms that can not be defined in the abstract universe considered here that is not endowed with a topological structure.

3 Main result

The main result proved in this paper is the following.

Theorem 1 *Assume that* $\succsim$ *is an ordering of* $\mathcal{P}(X)$ *that satisfies Richness. Then* $\succsim$ *satisfies Balancedness, Averaging and the Archimedean axiom if and only if there are two functions* $u : X \to \mathbb{R}$ *and* $p : X \to \mathbb{R}_{++}$ *such that (1) holds.*

The proof of this result proceeds in several steps. We first prove the result on the set $\mathcal{P}_*(X)$. Once having obtained the numerical representation as per (1) on $\mathcal{P}_*(X)$, we show that the representation can be extended to the whole set $\mathcal{P}(X)$. The proof is based on several auxiliary results that we now present. We relegate all proofs to the Appendix.

The first result is simple, but very important. It says that if $\succsim$ is not trivial and satisfies Averaging, and A and B are two decisions that are respectively minimal and maximal in $\mathcal{P}(X)$, then A and B must be disjoint. We formally state this result as follows.

Lemma 1 *Let $\succsim$ be a non-trivial ordering of $\mathcal{P}(X)$ satisfying Averaging. Then if sets A and $C \in \mathcal{P}(X)$ are such that $A \succsim B \succsim C$ for all sets $B \in \mathcal{P}(X)$, then $A \cap C = \varnothing$.*

The second result states that if $\succsim$ is a non-trivial ordering of $\mathcal{P}(X)$, then the restriction of $\succsim$ to $\mathcal{P}_*(X)$ is also a non-trivial ordering of that set.

Lemma 2 *Let $\succsim$ be a non-trivial ordering of $\mathcal{P}(X)$ satisfying Averaging. Then the restriction of $\succsim$ to $\mathcal{P}_*(X)$ is also a non-trivial ordering of $\mathcal{P}_*(X)$.*

The third result establishes a somewhat strong implication of the Richness condition when combined with the Averaging axiom, and applied to a non-trivial ordering. Indeed, in such a case, the Richness condition implies that, for any two non-maximal nor minimal decisions faced by the decision maker, it is possible to replace one of them by another that is indifferent to it and that leads to different consequences than those of the two initial decisions. The formal statement of this lemma is as follows.

Lemma 3 *Let $\succsim$ be a non-trivial ordering of $\mathcal{P}(X)$ satisfying Richness and Averaging. Then, for every A, $C \in \mathcal{P}_*(X)$, there exists $B \in \mathcal{P}(X)$ such that $B \sim A$ and $B \cap (A \cup C) = \varnothing$.*

An important implication of this lemma is that any ordering of $\mathcal{P}(X)$ satisfying Averaging and Richness must be trivial if the universe X is finite. Averaging and Richness, if they are to apply to a non-trivial ordering, force the sets X and $\mathcal{P}_*(X)$ to be both infinite. It also forces the set $\mathcal{P}_*(X)$ to be "unbounded" with respect to $\succsim$.in the sense that, for any decision $B \in \mathcal{P}_*(X)$, one can find decisions A and C in $\mathcal{P}_*(X)$ that are, respectively, strictly better and strictly worse than B. The formal statement of this fact is as follows.

Lemma 4 *If $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness and Averaging, then, for every set $B \in \mathcal{P}_*(X)$, there are decisions A and $C \in \mathcal{P}_*(X)$ such that $C \succ B \succ A$.*

Equipped with these four lemmas, we define, for any decision $E \in \mathcal{P}_*(X)$, the set $\mathcal{P}^E(X) = \{C \in \mathcal{P}(X) : C \sim E\}$ of all decisions indifferent to E. This set is not empty since it contains E itself by reflexivity. We then define the binary relation $\succsim_l^E$ on $\mathcal{P}^E(X)$ by: $A \succsim_l^E B$ iff there is a decision C disjoint from A and B such that $A \cup C \succsim B \cup C$ and $E \succ C$. Notice that, since we work on the set $\mathcal{P}_*(X)$, we do not define $\succsim_l^E$ on a maximal (or minimal) equivalence class. This binary relation $\succsim_l^E$ - which depends upon the reference set E - is naturally interpreted as meaning "is at least as likely as". Hence decision A is at least as likely as decision B if A and B provide the decision maker with the same utility as that of the benchmark E and if merging A to a strictly worse decision C is better than merging B with that same worse decision. By virtue of the Balancedness axiom, the choice of the set C used to define $\succsim_l^E$ is irrelevant. The following lemma establishes more precisely that the binary relation $\succsim_l^E$ is, in fact, an ordering of the set $\mathcal{P}^E(X)$.

Lemma 5 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Balancedness and Averaging. Then, for any decision $E \in \mathcal{P}_*(X)$, the relation $\succsim_l^E$ is an ordering of $\mathcal{P}^E(X)$.*

In the next lemma, we connect the definitions of the asymmetric factor $\succ_l^E$ and the symmetric factor $\sim_l^E$ of $\succsim_l^E$ to those of the primitive ordering $\succsim$.

Lemma 6 *Assume that $\succsim$ is an ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness and Averaging. Then, for any decision $E \in \mathcal{P}_*(X)$, any decisions A and B in $\mathcal{P}^E(X)$ and any decision $C \in \mathcal{P}(X)$ such that $E \succ C$ and $C \cap (A \cup B) = \varnothing$,*

1. *$A \succ_l^E B$ if and only if $A \cup C \succ B \cup C$.*
2. *$A \sim_l^E B$ if and only if $A \cup C \sim B \cup C$.*

The next lemma is quite important. It establishes the possibility of representing the likelihood ordering $\succsim_l^E$ (for any given E) by a disjoint-set-additive and strictly positive numerical function which behaves indeed like a probability measure. The proof of this lemma relies on an important theorem on additive numerical representation established in Krantz, Luce, Suppes, and Tversky (1971).

Lemma 7 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Then, for every decision $E \in \mathcal{P}_*(X)$, there exists a disjoint-set-additive mapping $p^E : \mathcal{P}^E(X) \to \mathbb{R}_{++}$ such that, for all $A, B \in \mathcal{P}^E(X)$, $A \succsim_l^E B$ iff $p^E(A) \geq p^E(B)$. Furthermore, p^E is unique up to a positive linear transformation.*

Given any decision $E \in \mathcal{P}_*(X)$, the ordering $\succsim_l^E$ enables the comparison of any two decisions that are indifferent to E as per the ordering $\succsim$. We now need to establish some connections between the various orderings $\succsim_l^E$ induced by all the sets E that are not indifferent to each other as per $\succsim$. A preliminary step for doing so consists in showing the possibility of constructing, starting from p^E, a disjoint-set-additive function which, for any decision E, indicates whether any other decision is weakly preferred to E, or weakly worse than E. We do this in the following lemma, that is very similar, in its statement and proof, to Lemma A10 in Ahn (2008).

Lemma 8 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. If $E \in \mathcal{P}_*(X)$, there exists a disjoint-set-additive mapping $\nu^E : \mathcal{P}(X) \to \mathbb{R}$ such that, for all $A \in \mathcal{P}(X)$, $\nu^E(A) \geq 0$ iff $A \succsim E$ and $\nu^E(A) \leq 0$ iff $E \succsim A$.*

The function ν^E constructed in the proof of Lemma 8 is a disjoint-set-additive extension of the function p^E of Lemma 7. An important thing to note about ν^E is that, while defined only with respect to a decision $E \in \mathcal{P}_*(X)$, it is in fact a function that maps every decision $A \in \mathcal{P}(X)$ to a real number. Hence, the domain of ν^E includes sets that belong to $m(X)$ or $M(X)$.

The function ν^E of Lemma 8 enables one to identify whether a decision is better or worse than the benchmark decision E. In order to obtain a numerical representation of the whole preference $\succsim$ over all sets, it is important to connect together the information conveyed by the functions ν^E for all benchmark decisions E. A first step in establishing this connection is the following lemma, which says that the function ν^E (for any E) numerically represents the likelihood ordering $\succsim_l^B$ defined on $\mathcal{P}^B(X)$ no matter what the reference set B is. The formal statement of this result is as follows.

Lemma 9 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Then, for any two sets $E, B \in \mathcal{P}_*(X)$ the function ν^E defined in Lemma 8 numerically represents the likelihood ordering $\succsim_l^B$ on $\mathcal{P}^B(X)$ in the sense that, for any two decisions S and $T \in \mathcal{P}^B(X)$, $S \succsim_l^B T \iff \nu^E(S) \geq \nu^E(T)$.*

We now establish, with the help of this result, that the set of all functions ν^E resulting from all reference decisions $E \in \mathcal{P}_*(X)$ is a two-dimensional space in the sense that any such function can be obtained as a linear combination of any two other linearly independent functions. An analogous result was proved as Lemma A12 in Ahn (2008).

Lemma 10 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Then the family $\{\nu^E : E \in \mathcal{P}_*(X)\}$ is spanned by any two of its members ν^A and ν^B provided that ν^A and ν^B are linearly independent. That is, for any two functions ν^A and ν^B for which there is no real number α such that $\frac{\nu^A(C)}{\nu^B(C)} = \alpha$ for all decisions $C \in \mathcal{P}(X)$, one can write any function ν^E as $\nu^E = \alpha^E \nu^A + \beta^E \nu^B$ for some real numbers α^E and β^E.*

The next lemma establishes a somewhat stronger result concerning the set of functions $\{\nu^E : E \in \mathcal{P}_*(X)\}$ defined in Lemma 8. Namely, that this set is a positive cone.

Lemma 11 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Let A, B and C be three sets in $\mathcal{P}_*(X)$. Then, there does not exist a strictly positive real number λ and a $\delta \in [0, 1]$ such that $-\lambda\nu^A(D) = \delta\nu^B(D) + (1 - \delta)\nu^C(D)$ holds for all sets $D \in \mathcal{P}(X)$.*

Endowed with these results on the (vector-like) structure of the set of functions $\{\nu^E : E \in \mathcal{P}_*(X)\}$ defined in Lemma 8, we now use these functions to construct a disjoint-set-additive function μ that will play a key role in the construction of the numerical representation of the form (1) that we are aiming at. Roughly speaking, the function μ will define the denominator of the numerical expression (1).

Lemma 12 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Then there exists a disjoint-set-additive function $\mu : \mathcal{P}(X) \to \mathbb{R}$ such that $\mu(C) > 0$ for all $C \in \mathcal{P}_*(X)$ and such that $\mu \in S$, where S is the span of any two linearly independent mappings ν^A and ν^B.*

The next lemma establishes that the set function $f^C : \mathcal{P}_*(X) \to \mathbb{R}$ defined, for any reference set C, by:

$$f^C(A) = \frac{\nu^C(A)}{\mu(A)} \tag{4}$$

provides a numerical representation of the ordering $\succsim$ on the set $\mathcal{P}_*(X)$.

Lemma 13 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Choose any $C \in \mathcal{P}_*(X)$. Then, for all sets A and $B \in \mathcal{P}_*(X)$, $\nu^C(A)/\mu(A) \geq \nu^C(B)/\mu(B)$ iff $A \succsim B$.*

In the next lemma, we show that each of the two disjoint-set-additive functions ν^C - for any set $C \in \mathcal{P}_*(X)$ - and μ serves as an index of the equivalence class associated to the intersection of the symmetric factors of the two orderings $\succsim$ and $\succsim_l^E$. That is, any two sets of consequences that are considered both equally desirable - from the view point of $\succsim$ - and equally "likely" - as per $\succsim_l^E$- will be assigned the same value by either the function ν^C or the function μ.

Lemma 14 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Then, for any $C \in \mathcal{P}_*(X)$, and any two decisions A and $B \in \mathcal{P}_*(X)$, $A \sim B$ and $A \sim_l^E B$ implies $\nu^C(A) = \nu^C(B)$ and $\mu(A) = \mu(B)$.*

We now establish the existence, in the universe X, of consequences that have arbitrarily small level of likelihood. More precisely, we show that the function μ that defines the denominator of the numerical expression (1) can take values arbitrarily close to zero if the set of consequences to which it applies is suitably chosen. Notice that this implies that the UEU criteria characterized in Gravel, Marchant, and Sen (2012) are not members of the family of CEU criteria that are represented as per (1) for some functions p and u (with p strictly positive) if the richness condition is assumed. Indeed, if a UEU criterion was a CEU criterion, the function p of expression (1) would be a constant (say $p(x) = a$ for all consequences x and some strictly positive number a). In this case, there would be no consequence in X with an arbitrarily small level of likelihood.

Lemma 15 *Assume that* $\succsim$ *is a non-trivial ordering on* $\mathcal{P}(X)$ *satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Then, for any set* $D \in \mathcal{P}_*(X)$ *and any strictly positive real number* ε*, one can find a decision* E *such that* $E \sim D$ *and* $\mu(E) < \varepsilon$*.*

The results collected so far have been dealing with decisions that are not maximal or minimal - for the ordering $\succsim$ - in the set $\mathcal{P}(X)$. We must now show that the functions μ and ν^C (for any given $C \in \mathcal{P}_*(X)$) defined for those non-minimal or maximal decisions can also be extended to minimal or maximal decisions (if any). We do this by first extending, in the next lemma, the function μ of Lemma 12 defined on $\mathcal{P}_*(X)$ to a closely related function μ_+ defined on the whole set $\mathcal{P}(X)$.

Lemma 16 *Assume that* $\succsim$ *is a non-trivial ordering on* $\mathcal{P}(X)$ *satisfying Richness, Balancedness, Averaging and the Archimedean axiom. There exists then a disjoint-set-additive mapping* $\mu_+ : \mathcal{P}(X) \to \mathbb{R}$ *such that* $\mu_+(S) > 0$ *for all* $S \in \mathcal{P}(X)$*. Moreover the function* μ_+ *belongs to the span of any two linearly independent mappings* ν^A *and* ν^B*.*

Endowed with this function, we now to prove in the next lemma an analogue of Lemma 13, but using μ_+ rather than μ.

Lemma 17 *Assume that* $\succsim$ *is a non-trivial ordering on* $\mathcal{P}(X)$ *satisfying Richness, Balancedness, Averaging and the Archimedean axiom, and let* C *be any decision in* $\mathcal{P}_*(X)$*. Then, for all sets* A *and* $B \in \mathcal{P}(X)$*,* $\nu^C(A)/\mu_+(A) \geq \nu^C(B)/\mu_+(B)$ *iff* $A \succsim B$*.*

The combination of these lemmas lead to the proof of Theorem 1, established in the Appendix.

4 Interpretation

4.1 Independence of the axioms

In the next three examples, we show that the axioms used in the characterization of the CEU family of orderings are independent when applied to an ordering satisfying richness. We first give an example of a non-CEU ordering of $\mathcal{P}(X)$ that violates the Archimedean axiom but that satisfies Averaging, Balancedness and Richness.

Example 1 *Let* $X = \mathbb{R}^2_{++} \times \mathbb{R}^2$*, with typical elements* (a_1, a_2, a_3, a_4)*,* (b_1, b_2, b_3, b_4)*, etc. For every* $A \in \mathcal{P}(X)$*, define*

$$U_1(A) = \frac{\sum_{a \in A} a_1 a_3}{\sum_{a \in A} a_1}$$

and

$$U_2(A) = \frac{\sum_{a \in A} a_2 a_3}{\sum_{a \in A} a_2}.$$

Define $\succsim$ on $\mathcal{P}(X)$ by

$$A \sim B \iff U_1(A) = U_1(B) \text{ and } U_2(A) = U_2(B);$$
$$A \succ B \iff \begin{cases} U_1(A) > U_1(B) \\ \text{or} \\ U_1(A) = U_1(B) \quad \text{and} \quad U_2(A) > U_2(B). \end{cases}$$

We show in the Appendix that this (lexicographic) ordering of $\mathbb{R}^2_{++} \times \mathbb{R}^2$ satisfies the richness condition and the Averaging and Balancedness axioms, but violates the Archimedean axiom.

The next example provides a non-CEU ordering that satisfies Balancedness, Richness and the Archimedean axiom but that violates Averaging.

Example 2 *Let $X = \mathbb{R}_{++} \times \mathbb{R}^2$, $p(x) = x_1$, $u(x) = x_2$,*

$$U(A) = \frac{\sum_{a \in A} p(a)u(a)}{\sum_{a \in A} p(a)^2}$$

and $A \succsim B$ iff $U(A) \geq U(B)$.
It is immediate to see that the ordering $\succsim$ satisfies Richness and the Archimedean axiom. This ordering violates Averaging because $A = \{(3/4, 2, 0)\} \sim B = \{(3/4, 2, 1)\} \succ A \cup B$. In the Appendix, we prove that $\succsim$ satisfies Balancedness.

Finally, the next example shows a non-CEU ordering that satisfies Richness, the Archimedean axiom and Averaging but that violates Balancedness.

Example 3 *Consider $\succsim$ defined on $\mathcal{P}(\mathbb{R}^2_+)$ by:*

$$A \succsim B \iff V(A) \geq V(B)$$

where, for any finite $D \subset \mathbb{R}^2_+$, $V(D)$ is defined by:

$$V(D) = \frac{\sum\limits_{(d_1,d_2)\in D} d_1\Big(d_2 + \frac{d_2^2}{\sum\limits_{(\delta_1,\delta_2)\in D} \delta_2}\Big)}{\sum\limits_{(d_1,d_2)\in D} \Big(d_2 + \frac{d_2^2}{\sum\limits_{(\delta_1,\delta_2)\in D} \delta_2}\Big)}.$$

It is easy to see that this ordering satisfies the Archimedean axiom. In the Appendix, we show that it satisfies Richness and Averaging, but violates Balancedness.

4.2 Some implications of the richness condition

We mentioned earlier that the richness condition restricts significantly the class of orderings that are *a priori* considered. We now more specifically show that the richness condition also restricts the functions p and u that define the family of CEU criteria and, as result, the family itself.

An example of the implication of our richness condition is provided in the following proposition, where we show that if $X = \mathbb{R}$ (for instance the consequences of a decision under ignorance are amounts of money), then it is impossible with our richness condition to have both the functions p and the function u to be monotonic if the function u is continuous.

Proposition 1 *Suppose that $X = I$ for some non-degenerate interval I of $\mathbb{R}$ and assume that $\succsim$ is a CEU ranking satisfying richness. Then, if the function u in expression (1) is continuous, it can not be weakly monotonic if p is weakly monotonic.*

In the next proposition, we establish that if X is a topological space (for instance a separable one of the kind considered in Gravel, Marchant, and Sen (2012)), then no Uniform Expected Utility criterion in which u is a continuous utility function satisfies the richness condition. This shows that the characterization of the CEU family of criteria provided in this paper does not contain all members of that family because it excludes, at least in topological environments, the UEU subclass of that family that is obtained by considering only constant functions p and continuous functions u.

Proposition 2 *Let X be a topological space, and let $\succsim$ be a non-trivial UEU ranking with u continuous. Then $\succsim$ violates the Richness condition.*

5 Conclusion

This paper characterizes the family of CEU rankings of decisions under ignorance or objective ambiguity with finitely many consequences, under the assumption that the rankings are defined in a "rich" environment. As we argued above, our focus to decisions with finitely many consequences (or probability distributions) makes our analysis complementary to the atomless environment considered by Ahn (2008) and the literature that derives from the Bolker-Jeffrey tradition (e.g. Bolker (1966), Bolker (1967), Jeffrey (1965) and Broome (1990)). We emphasize that the discrete framework makes the proof and the characterization quite different from what is obtained in this latter tradition. Moreover, we have shown that the three axioms that we use are independent.

Yet, the analysis conducted in this paper suffers from two limitations, of varying importance. First, as suggested in the preceding section, it relies on a richness assumption that is, probably, unduly strong. We use the qualifier "probably" because we do not have, at the moment, an alternative. We are therefore unable to assess the strength of the assumption. As shown in Proposition 2, the richness condition is sufficiently strong to exclude from the family of CEU rankings, all UEU rankings of the kind characterized in Gravel, Marchant, and Sen (2012) when the latter are defined on a topological space and are continuous on that space. Another limitation of the analysis is that it is conducted in an algebraic framework rather than a topological one (using Wakker (1988)'s terminology). Contrary to Gravel, Marchant, and Sen (2012), we do not provide a topological version of our theorem in which the richness condition and the Archimedean axiom are replaced by an appropriate - and necessary - continuity condition. We do not view this second limitation as being as important however. Indeed, as convincingly argued by Wakker (1988), the algebraic framework is more general than the topological one. Yet, it is fair to say that topological environments, and the continuity properties that they enable to define, are more familiar to decision theorists and economists than richness and Archimedean conditions. For this reason, it would be nice to have a topological version of our main theorem.

The analysis of this paper needs also to be extended in several directions. One of them is the comparative attitudes toward radical uncertainty or ambiguity. What does it mean for a CEU decision maker to exhibit more aversion to ambiguity/radical uncertainty than another ? Ahn (2008) discusses some of the issues in his atomless framework. Most of his discussion applies to the discrete framework of this paper. A difficulty in defining comparative attitudes toward radical uncertainty or ambiguity for CEU decision makers is that of disentangling the implication of differential attitudes toward ambiguity/uncertainty on the utility function u and on the likelihood function p. For example, one result in Ahn (2008) states, for two CEU decision makers who use the same likelihood function p, that the fact of having a more concave utility function is equivalent to exhibiting a larger aversion to ambiguity/uncertainty. The difficulty in applying such a definition is that of identifying empirically two decision makers with the same likelihood function. It would be desirable (but probably difficult) to have definitions of comparative attitudes toward ambiguity/uncertainty that are expressed directly in terms of two distinct CEU decision makers that are not *a priori* restricted with respect to their u or their p function. In Gravel, Marchant, and Sen (2012), we provided such definitions for UEU decision makers. There does not seem to be an easy way to do so for CEU counterparts.

Another required extension of the analysis is a proper understanding of the uniqueness properties of the functions u and p used in the characterization. As shown in Proposition 1, the two functions may not be totally independent from each other in specific environments. It would be nice to have a complete identification of the uniqueness properties of these functions.

Last, but not least, we believe that CEU models of decision making under radical uncertainty and/or ambiguity should be put to work, notably in public policies, to generate consistent rankings of radically uncertain decisions. While environmental policy discussed in the Introduction is an obvious field for such applications, there are many others that are awaiting empirical implementations.

6 Appendix: Proofs

6.1 Lemma 1

Let the sets A and $C \in \mathcal{P}(X)$ be such that $A \succsim B \succsim C$ for all sets $B \in \mathcal{P}(X)$ and assume by contradiction that there exists some consequence $x \in A \cap C$. Since $\succsim$ is not trivial, one must have $A \succ C$. Since $\succsim$ is an ordering, either $\{x\} \succsim A \succ C$ or $A \succ \{x\}$. In the first case, it follows from Averaging that $C \succ C\backslash\{x\}$, which contradicts the definition of the sets A and $C \in \mathcal{P}(X)$ to be such that $A \succsim B \succsim C$ for all sets $B \in \mathcal{P}(X)$. In the second case, it follows from Averaging again that $A\backslash\{x\} \succ A$ which is also a contradiction of the definition of the sets A and $C \in \mathcal{P}(X)$ to be such that $A \succsim B \succsim C$ for all sets $B \in \mathcal{P}(X)$.

6.2 Lemma 2

Since $\succsim$ is a non-trivial ordering of $\mathcal{P}(X)$, there are sets A and $B \in \mathcal{P}(X)$ such that $A \succ B$. If $m(X) = \{C \in \mathcal{P}(X) : E \succsim C \text{ for all } E \in \mathcal{P}(X)\} = \varnothing$, then $B \notin m(X)$ so that there exists some $B_1 \in \mathcal{P}(X)$ such that $B \succ B_1$ (because $\succsim$ is complete). Since $B_1 \notin m(X)$, one concludes again from the completeness of $\succsim$ in the existence of some $B_2 \in \mathcal{P}(X)$ such that $B \succ B_1 \succ B_2$. But this shows that B and $B_1 \in \mathcal{P}_*(X)$ and, as a result, that $\succsim$ is not trivial on $\mathcal{P}_*(X)$. A similar argument (but made "at the top") can be constructed if $M(X) = \{D \in \mathcal{P}(X) : D \succsim F \text{ for all } F \in \mathcal{P}(X)\}$ is empty. Suppose now that $m(X) \neq \varnothing$ and $M(X) \neq \varnothing$. Since $\#X \geq 3$, there are distinct x_1, x_2, x_3 such that $\{x_1\} \succsim \{x_2\} \succsim \{x_3\}$ (using again the completeness of $\succsim$). Without loss of generality, one can assume that $\{x_1\} \in M(X)$ and $\{x_3\} \in m(X)$. Indeed, by assumption, there are non-empty sets $A = \{a_1, ..., a_{\#A}\} \in M(X)$ and $B = \{b_1, ..., b_{\#B}\} \in m(X)$. By definition of these sets $A \succsim \{a_j\}$ for every $j = 1, ..., \#A$ and $\{b_i\} \succsim B$ for every $i = 1, ..., \#B$. If $A \succ \{a_j\}$ for some j (resp. $\{b_i\} \succ B$ for some i) then by Averaging $A\backslash\{a_j\} \succ A$ (resp. $B \succ B\backslash\{b_i\}$) and this is a contradiction of $A \in M(X)$ (resp. $B \in m(X)$). Hence $\{a_j\} \sim A$ for every $j = 1, ..., \#A$ and $B \sim \{b_i\}$ for every $i = 1, ..., \#B$ and this proves that x_1, x_2 and x_3 can be chosen so that $\{x_1\} \in M(X)$ and $\{x_3\} \in m(X)$. Three cases are compatible with $\{x_1\} \succsim \{x_2\} \succsim \{x_3\}$ and the non-triviality of the ordering $\succsim$ on $\mathcal{P}(X)$:

(i) $\{x_1\} \succ \{x_2\} \succ \{x_3\}$. In this case, it follows from averaging and transitivity that $\{x_1\} \succ \{x_1, x_2\} \succ \{x_2\} \succ \{x_3\}$ and, therefore, that both $\{x_1, x_2\}$ and $\{x_2\}$ belong to $\mathcal{P}_*(X)$. Hence, the restriction of $\succsim$ to $\mathcal{P}_*(X)$ is not trivial on this set.

(ii) $\{x_1\} \sim \{x_2\} \succ \{x_3\}$. In this case, it first follows from averaging and transitivity that $\{x_1\} \sim \{x_2\} \succ \{x_2, x_3\} \succ \{x_3\}$. Applying averaging and transitivity again, we conclude that $\{x_2\} \sim \{x_1\} \succ \{x_1, x_2, x_3\} \succ \{x_2, x_3\} \succ \{x_3\}$ and, therefore, that the sets $\{x_1, x_2, x_3\}$ and $\{x_2, x_3\}$ both belong to $\mathcal{P}_*(X)$ so that the result holds for that case as well.

(iii) $\{x_1\} \succ \{x_2\} \sim \{x_3\}$. This case can be treated just like the preceding one, and this completes the proof.

6.3 Lemma 3

Since $\succsim$ is non-trivial on $\mathcal{P}(X)$, it is non-trivial on $\mathcal{P}_*(X)$ by Lemma 2. Hence, there are sets A and D in $\mathcal{P}_*(X)$ such that either $A \succ D$ or $D \succ A$. We treat the case $D \succ A$ (the other case is handled symmetrically). We first prove that there are at least three equivalence classes in the set $\mathcal{P}_*(X)$ so that it will be possible to apply Richness. We consider two cases :
(1) $A \cap D = \varnothing$. Then Averaging yields $D \succ A \cup D \succ A$ (and we are done).
(2) $A \cap D \neq \varnothing$. We then consider three subcases:
(a) $A \cap D \sim A$. Then, by Averaging, $A \setminus D \sim A$ and $D \succ A \cup D \succ A \setminus D$.
(b) $A \succ A \cap D$. If this is the case, one can not have $D \subset A$. Indeed, if one had $D \subset A$, this would imply that $A \succ A \cap D = D$, which contradicts the initial assumption that $D \succ A$. Hence the set $D \setminus A \neq \varnothing$. Averaging then implies that $D \setminus A \succ D$.
(c) $A \cap D \succ A$. If $A \cap D \not\sim D$, then we are done. Otherwise, by Averaging, $D \setminus A \sim D$ and $D \setminus A \succ A \cup (D \setminus A) \succ A$.

We now apply richness to the three equivalence classes. A first application of Richness yields a set B_1 such that $B_1 \sim A$ and $B_1 \cap A = \varnothing$. If $B_1 \cap D = \varnothing$, then the proof is done. If $B_1 \cap D \neq \varnothing$, then use Richness again to find a set B_2 such that $B_2 \sim A \cup B_1$ and $B_2 \cap (A \cup B_1) = \varnothing$. By Averaging, $A \cup B_1 \sim A$ and, by transitivity, $B_2 \sim A$. We are now sure that B_2 does not contain any of the elements of $B_1 \cap D$. If $B_2 \cap D = \varnothing$, then the proof is done. If $B_2 \cap C \neq \varnothing$, then use Richness again to find a set B_3 such that $B_3 \sim A \cup B_1 \cup B_2$ and $B_3 \cap (A \cup B_1 \cup B_2) = \varnothing$. By Averaging, $A \cup B_1 \cup B_2 \sim A$ and, by transitivity, $B_3 \sim A$. Notice that $(B_1 \cup B_2) \cap D \supsetneq B_1 \cap D$. We are now sure that B_3 does not contain any of the elements of $(B_1 \cup B_2) \cap D$. If $B_3 \cap D = \varnothing$, then the proof is done. If $B_3 \cap D \neq \varnothing$, we iterate this construction and we find sets like $B_4, B_5, \ldots$ At each iteration, $(B_1 \cup \ldots \cup B_i) \cap D \supsetneq (B_1 \cup \ldots \cup B_{i-1}) \cap D$. Since D is finite, we are sure to reach some B_j satisfying the same conditions as B in the statement of the lemma.

6.4 Lemma 4

If $\succsim$ is not trivial on $\mathcal{P}(X)$, it is non-trivial on $\mathcal{P}_*(X)$ by Lemma 2. Hence there are decisions D and $E \in \mathcal{P}(X)$ such that $E \succ D$. By Lemma 3, there is a set $F \in \mathcal{P}(X)$ such that $F \sim D$ and $F \cap (D \cup E) = \varnothing$. By Averaging and Transitivity, $E \succ F \cup E \succ D$. Hence, the ordering $\succsim$ has at least three equivalence classes and, hence, $\mathcal{P}_*(X)$ is not empty. Let B be a decision in $\mathcal{P}_*(X)$ (we have just proved that it exists). We will prove that there is a set $A \in \mathcal{P}_*(X)$ such that $B \succ A$ (the proof that there is $C \in \mathcal{P}_*(X)$ such that $C \succ B$ is similar). If $m(X)$ is empty, then the proof is immediate. So, we consider that $m(X)$ is not empty. Let G be a decision in $m(X)$. By Lemma 3, there is a set $H \in P(X)$ such that $H \sim G$ and $H \cap (G \cup B) = \varnothing$. By Averaging and Transitivity, $B \succ H \cup B \succ H$.

6.5 Lemma 5

Let A, B, and C be three sets in $\mathcal{P}^E(X)$ such that $A \succsim_l^E B$ and $B \succsim_l^E C$. By definition of $\succsim_l^E$, this implies the existence of sets D and $D' \in \mathcal{P}_*(X)$ respectively disjoint from $A \cup B$ and $B \cup C$ such that $E \succ D, D'$, $A \cup D \succsim B \cup D$ and $B \cup D' \succsim C \cup D'$. Thanks to Lemma 3, we can find a set $D'' \in \mathcal{P}_*(X)$ disjoint from $A \cup B \cup C$, with $D \sim D''$. By Balancedness, $A \cup D'' \succsim B \cup D''$ and $B \cup D'' \succsim C \cup D''$. By transitivity, $A \cup D'' \succsim C \cup D''$ and, hence, $A \succsim_l^E C$. This proves the transitivity of $\succsim_l^E$. In order to establish its completeness, let A and B be two sets in $P^E(X)$ such that $A \not\succsim_l^E B$. By definition of $\succsim_l^E$, either:
(i) there is no set C disjoint from $A \cup B$ such that $E \succ C$ or,
(ii) there are such sets but for none of them it is true that $A \cup C \succsim B \cup C$.
Case (i) can be ruled out by Lemma 4. If case (ii) holds, then, since $\succsim$ is complete, we must have $B \cup C \succ A \cup C$ for all sets C disjoint from $A \cup B$ such that $E \succ C$. It follows that $B \succsim_l^E A$ and the relation $\succsim_l^E$ is therefore complete.

6.6 Lemma 6

For the "only if" part of the first part of the lemma, we know that, since $\succsim_l^E$ is complete, $A \succ_l^E B$ implies that $B \succsim_l^E A$ does not hold. Hence, either there is no D disjoint from $A \cup B$ for which $A \succ D$ or $A \cup D \succ B \cup D$ for all sets D such that $A \succ D$. The first of these two possibilities is ruled out by Lemma 4. The second one implies, as a particular case, that $A \cup C \succ B \cup C$. For the "if" part of the first part of the lemma, suppose $A \cup C \succ B \cup C$. This implies $A \succsim_l^E B$ (by definition of $\succsim_l^E$). Suppose by contradiction that $A \succ_l^E B$ does not hold. Since $\succsim_l^E$ is complete, $B \succsim_l^E A$ must hold so that, by definition of $\succsim_l^E$, there exists a set D such that $B \cup D \succsim A \cup D$, and $A \succ D$. But this contradicts Balancedness. Hence $A \succ_l^E B$ must hold.

For the "only if" part of the second part of the lemma, one knows that $A \sim_l^E B$ implies the existence of sets D and D' (both strictly dominated by A as per $\succsim$) such that $(D \cup D') \cap (A \cup B) = \varnothing$, $A \cup D \succsim B \cup D$ and $B \cup D' \succsim A \cup D'$. By Balancedness, $A \cup C \succsim B \cup C$ and $B \cup C \succsim A \cup C$ and, so, $A \cup C \sim B \cup C$. The proof of the "if" part of the second part of the lemma is obvious.

6.7 Lemma 7

Define the binary operation $\circ^E$ on $\mathcal{P}^E(X)$ as follows. If $A \cap B = \varnothing$, then $A \circ^E B = A \cup B$. Otherwise set $A \circ^E B = A' \cup B'$ for some $A', B' \in \mathcal{P}^E(X)$ such that $A' \cap B' = \varnothing$, $A' \cup C \sim A \cup C$ and $B' \cup D \sim B \cup D$ for some sets C and D such that $E \succ C$, D, $(A \cup A') \cap C = \varnothing$ and $(B \cup B') \cap D = \varnothing$. The existence of such sets A', B' does not pose any difficulty, thanks to Richness. Indeed, by Lemma 4 and Averaging, there exists a set $C \in \mathcal{P}(X)$ such that $A \sim B \succ C$. By Lemma 4, C can be chosen disjoint from A. By Averaging, $A \succ C \cup A \succ C$ and, using Richness, there exists a set A' such that $A' \cup C \sim A \cup C$, $A' \sim A$ and $A' \cap (C \cup A) = \varnothing$.

Using an analogous reasoning, one can establish the existence of a set B' such that $B' \cup C \sim B \cup C$, $B' \sim B$ and $B' \cap (C \cup A \cup A') = \varnothing$.

Hence $\circ^E$ is defined for every pair A, $B \in \mathcal{P}^E(X)$, and the choice of the sets A' and B' can be made by any rule whatsoever if there are several such sets for a given pair A and B. Finally we note that $\circ^E$ is closed in the set $\mathcal{P}^E(X)$ thanks to Averaging.

For any $E \in \mathcal{P}(X)$, we now show that the triple $\langle \mathcal{P}^E(X), \succsim_l^E, \circ^E \rangle$ is what Krantz, Luce, Suppes, and Tversky (1971) (p. 73, definition 1) call a "closed extensive measurement structure". Indeed, we establish that $\langle \mathcal{P}^E(X), \succsim_l^E, \circ^E \rangle$ satisfies the following.

1. $\succsim_l^E$ is a weak order: see Lemma 5.

2. $\circ^E$ is weakly associative so that $A \circ^E (B \circ^E C) \sim_l^E (A \circ^E B) \circ^E C$ for every A, B and $C \in \mathcal{P}^E(X)$. The proof of this is obvious if A, B, C are mutually disjoint. Consider now the case where $A \cap B \cap C \neq \varnothing$. Let $A', B', C' \in \mathcal{P}(X)$ be mutually disjoint sets such that $A' \cup M \sim A \cup M$, $B' \cup N \sim B \cup N$, $C' \cup O \sim C \cup O$ for some sets M, N and O satisfying $E \succ M$, N, O and $(A \cup A') \cap M = (B \cup B') \cap N = (C \cup C') \cap O = \varnothing$. They exist thanks to Richness (the argument is similar to that employed in the definition of the binary operation $\circ^E$). We have $B \circ^E C = B' \cup C'$ and $A \circ^E (B \circ^E C) = A' \cup B' \cup C'$. We also have $A \circ^E B = A' \cup B'$ and $(A \circ^E B) \circ^E C = A' \cup B' \cup C'$, so that $A \circ^E (B \circ^E C) = (A \circ^E B) \circ^E C$. The reasoning is similar when some but not all pairwise intersections between A, B, C are not empty.

3. $A \succsim_l^E B$ iff $A \circ^E C \succsim_l^E B \circ^E C$ iff $C \circ^E A \succsim_l^E C \circ^E B$. Since $\circ^E$ is obviously commutative, we just need to prove $A \succsim_l^E B$ iff $A \circ^E C \succsim_l^E B \circ^E C$. Choose A' and B' in $\mathcal{P}^E(X)$ such that $A' \cap C = \varnothing = B' \cap C$, $A' \cup D \sim A \cup D$ and $B' \cup D \sim B \cup D$ for some set D disjoint from C such that $E \succ D$. Thanks to Richness, this is always possible. Notice that $E \succ C \cup D$ by averaging. We have $A \succsim_l^E B$ iff $A \cup F \succsim B \cup F$ (by definition) iff $A' \cup D \succsim B' \cup D$ (by construction) iff $A' \cup C \cup D \succsim B' \cup C \cup D$ (by Balancedness and because $E \succ C \cup D$ thanks to Averaging) iff $A \circ^E C \succsim_l^E B \circ^E C$.

4. If $A \succ_l^E B$, then for any $C, D \in \mathcal{P}^E(X)$, there exists a positive integer n such that $n.A \circ^E C \succsim_l^E n.B \circ^E D$, where $n.A$ is defined inductively as: $1.A = A$, $(n+1).A = n.A \circ^E A$ and similarly for $n.B$. To prove this, assume by contradiction there are sets A, B, C and $D \in \mathcal{P}^E(X)$ such that $A \succ_l^E B$ and $n.B \circ^E D \succ^l n.A \circ^E C$ for every positive integer n. Notice that $2.A = A \circ^E A$ can be defined as $A \cup A_1$ for some $A_1 \in \mathcal{P}^E(X)$ such that $A_1 \cap A = \varnothing$, $A_1 \cup F \sim A \cup F$ for some set F such that $E \succ F$, $(A_1 \cup A) \cap F = \varnothing$. More generally, for any pair of distinct sets G, $H \in \mathcal{P}^E(X)$, such that $G \subset H$, one can define $G \circ^E H$ by $G' \cup H$ for some $G' \in \mathcal{P}^E(X)$ such that $G' \cap H = \varnothing$ and $G' \cap F \sim G \cap F$ for some $F \in \mathcal{P}(X)$ such that $E \succ F$ and $F \cap (G' \cup G) = \varnothing$. Indeed, by Lemma 4 and Averaging, there exists a set $S \in \mathcal{P}(X)$ such that $G \sim H \succ S$. By Lemma 4, S can be chosen disjoint from H. By Averaging, $H \succ H \cup S \succ S$ and, using Richness, there exists a

set G' such that $G' \cup S \sim H \cup S$, $G' \sim H$ and $G' \cap (S \cup H) = \varnothing$. Hence one can write, for any $n \geq 2$, $n.A = A \cup A_1 \cup ... \cup A_{n-1}$ for pairwise disjoint sets A_1,...,A_{n-1} also disjoint from A and F such that $A_j \cup F \sim_l^E A \cup F$ for all $j = 1, ..., n$ and $n.B = B \cup B_1 \cup ... \cup B_{n-1}$ for paiwise disjoint sets B_1,...,B_{n-1} disjoint from both B and F such that $B_j \cup F \sim_l^E B \cup F$ for all $j = 1, ..., n$. Hence assuming that $n.B \circ^E D \succ_l^E n.A \circ^E C$ for every positive integer n contradicts the Archimedean axiom.

By Theorem 1 of Krantz, Luce, Suppes, and Tversky (1971) (p.74), for any $E \in \mathcal{P}_*(X)$, there exists a mapping $p^E : \mathcal{P}^E(X) \to \mathbb{R}$ such that, for all $A, B \in \mathcal{P}^E(X)$, $A \succsim_l^E B$ iff $p^E(A) \geq p^E(B)$ and $p^E(A \circ^E B) = p^E(A) + p^E(B)$. Furthermore, p^E is unique up to a linear transformation.

We now show that $p^E(A) > 0$ for all $A \in \mathcal{P}^E(X)$. For any $A \in \mathcal{P}^E(X)$, we can find a set $B \in \mathcal{P}^E(X)$ such that $A \cap B = \varnothing$ (using Lemma 3). By definition of $\mathcal{P}_*(X)$, there is a set D' such that $E \succ D'$. By Lemma 3, there is a set $D \sim D'$ such that $D \cap (A \cup B) = \varnothing$. By Averaging, $B \sim A \succ B \cup D \succ D$. By Averaging again, $B \succ B \cup D \cup A \succ B \cup D$. By definition of $\succsim_l^E$, $A \cup B \succ_l^E B$. This implies $p^E(A \cup B) > p^E(B)$ and, since A and B are disjoint, $p^E(A) + p^E(B) > p^E(B)$ or, equivalently, $p^E(A) > 0$.

6.8 Lemma 8

For a fixed $E \in \mathcal{P}_*(X)$, let $\mathcal{L} = \{a \in X : E \succ \{a\}\}$ and $\mathcal{U} = \{a \in X : \{a\} \succ E\}$. These sets are not empty by Lemma 4. Define M to be an arbitrary set such that $M \succ E$.

We first define ν^E on $\mathcal{P}(\mathcal{L})$. Fix some $L \in \mathcal{P}(\mathcal{L})$. By Richness, there is $U \in \mathcal{P}(\mathcal{U})$ such that $U \sim M$ and $U \cup L \sim E$. Set $\nu^E(L) = -p^M(U)$. By construction, $\nu^E(L)$ does not depend on the choice of U. Indeed, suppose there are several such U, say U and U'. Notice that $U \sim M \sim U'$, $U \cup L \sim E$ and $U' \cup L \sim E$. So, $U \cup L \sim U' \cup L$. Hence $U \sim_l^E U'$ and $p^M(U) = p^M(U')$.

By repeated application of Lemma 4, there are sets $L_1, L_2 \in \mathcal{P}(\mathcal{L})$ such that $L_1 \cap L_2 = \varnothing$. By Averaging, $L_1 \cup L_2 \in \mathcal{P}(\mathcal{L})$. Using Richness as above, we find two disjoint sets $U_1, U_2 \in \mathcal{P}(\mathcal{U})$ such that $U_1 \sim U_2 \sim M$, $U_1 \cup L_1 \sim E$ and $U_2 \cup L_2 \sim E$. By Averaging, $U_1 \cup U_2 \cup L_1 \cup L_2 \sim E$ and $U_1 \cup U_2 \sim M$. Hence:

$$\begin{aligned} \nu^E(L_1 \cup L_2) &= -p^M(U_1 \cup U_2) \\ &= -p^M(U_1) - p^M(U_2) \\ &= \nu^E(L_1) + \nu^E(L_2). \end{aligned}$$

which proves that ν^E is disjoint-set-additive over $\mathcal{L}$.

We now define ν^E on $\mathcal{P}(\mathcal{U})$. Take any $U \in \mathcal{P}(\mathcal{U})$. By Richness used in a similar (but this time "downward") way as above, there is $L \in \mathcal{P}(\mathcal{L})$ such that $U \cup L \sim E$. Set $\nu^E(U) = -\nu^E(L)$. The mapping ν^E on $\mathcal{P}(\mathcal{U})$. does not depend on the choice of L. Indeed, suppose there are several such L, say L_1 and L_2 in $\mathcal{P}(\mathcal{L})$. We must prove that $\nu^E(L_1) = \nu^E(L_2)$. Suppose first $L_1 \cap L_2 = \emptyset$. Let $U_1, U_2 \in \mathcal{P}(\mathcal{U})$ be such that $U_1 \cap U = \emptyset = U_2 \cap U$, $U_1 \sim M \sim U_2$, $U_1 \cup L_1 \sim E \sim U_2 \cup L_2$.

By Richness, such sets exist. We also have $U \cup L_1 \sim E \sim U \cup L_2$. By Averaging, $U_1 \cup L_1 \cup U \cup L_2 \sim E \sim U_2 \cup L_2 \cup U \cup L_1$. Hence, $U_1 \sim_l^E U_2$, $p^M(U_1) = p^M(U_2)$ and $\nu^E(L_1) = \nu^E(L_2)$. Suppose now $L_1 \cap L_2 \neq \emptyset$. By Richness used in the same way as above, there is $L_3 \in \mathcal{P}(\mathcal{L})$ such that $L_3 \cap (L_1 \cup L_2) = \emptyset$ and $U \cup L_3 \sim E$. Define U_3 by $U_3 \sim M$ and $U_3 \cup L_3 \sim E$. By richness, U_3 can be chosen disjoint from both U_1 and U_2. Since $U_1 \cup L_1 \sim U \cup L_3 \sim E \sim U_3 \cup L_3 \sim U \cup L_1$ and U, U_1 and U_3 are disjoint as are L_1 and L_2, it follows from Averaging that $U_1 \cup L_1 \cup U \cup L_3 \sim E \sim U_3 \cup L_3 \cup U \cup L_1$. Hence, $U_1 \sim_l U_3$ and, therefore, $p^M(U_1) = p^M(U_3)$. A similar reasoning can be performed for U_2 and U_3. We therefore have $p^M(U_1) = p^M(U_2) = p^M(U_3)$ and, as a result, $\nu^E(L_1) = \nu^E(L_3) = \nu^E(L_2)$.

The mapping ν^E on $\mathcal{P}(\mathcal{U})$ is disjoint-set-additive. Indeed, consider two sets $U_1, U_2 \in \mathcal{P}(\mathcal{U})$, with $U_1 \cap U_2 = \emptyset$. Let us find two sets $L_1, L_2 \in \mathcal{P}(\mathcal{L})$ such that $U_1 \cup L_1 \sim E \sim U_2 \cup L_2$. Since the choice of L_1 and L_2 is not important, we can choose them disjoint (using Richness). By Averaging, $U_1 \cup U_2 \cup L_1 \cup L_2 \sim E$. So, $\nu^E(U_1 \cup U_2) = -\nu^E(L_1 \cup L_2) = -\nu^E(L_1) - \nu^E(L_2) = \nu^E(U_1) + \nu^E(U_2)$.

We define then ν^E on the whole set $P(X)$. Take any $S \in P(X)$. If $\{s\} \sim E$ for all $s \in S$, set $\nu^E(S) = 0$. Otherwise, we can express S as $S = L \cup U \cup R$ with $L = S \cap \mathcal{L}$, $U = S \cap \mathcal{U}$ and $R = S \setminus (\mathcal{L} \cup \mathcal{U})$. By Averaging, $S \succsim E$ iff $L \cup U \succsim E$. Set $\nu^E(S) = \nu^E(L) + \nu^E(U)$. Disjoint set additivity is inherited from ν^E on $\mathcal{P}(\mathcal{U})$ and ν^E on $\mathcal{P}(\mathcal{L})$.

We must finally check whether ν^E satisfies (ii). Suppose $S \succ E$. Then $(S \cap \mathcal{L}) \cup (S \cap \mathcal{U}) \succ E$. Using richness and averaging, one can find a superset L' of $S \cap \mathcal{L}$ belonging to $\mathcal{P}(\mathcal{L})$ such that $L' \cup (S \cap \mathcal{U}) \sim E$. As shown above, $-\nu^E(L') = \nu^E(S \cap \mathcal{U})$. Since $S \cap \mathcal{L} \subset L' \subseteq \mathcal{L}$, and , for every $L \in \mathcal{P}(\mathcal{L})$, $\nu^E(L) = -p^M(U) < 0$ for some set $U \in \mathcal{P}(\mathcal{U})$ we have that $0 > \nu^E(S \cap \mathcal{L}) > \nu^E(L')$ by disjoint set additivity. Now, by construction, $\nu^E(S) = \nu^E(S \cap \mathcal{L}) + \nu^E(S \cap \mathcal{U}) = \nu^E(S \cap \mathcal{L}) - \nu^E(L') > 0$.

Suppose now $E \succ S$. Then $E \succ (S \cap \mathcal{L}) \cup (S \cap \mathcal{U})$. Using Averaging and Richness again, there is a superset U' of $S \cap \mathcal{U}$ belonging to $\mathcal{P}(\mathcal{U})$ such that $U' \cup (S \cap \mathcal{L}) \sim E$. By definition of the mapping ν^E, one has that $\nu^E(U') = -\nu^E(S \cap \mathcal{L}) > 0$. Moreover, since $S \cap \mathcal{U} \subset U' \subseteq \mathcal{U}$ and $\nu^E(U) > 0$ for every $U \in \mathcal{P}(\mathcal{U})$, one has $\nu^E(U') > \nu^E(S \cap \mathcal{U}) > 0$ by disjoint set additivity. We therefore have, by construction, $\nu^E(S) = \nu^E(S \cap \mathcal{L}) + \nu^E(S \cap \mathcal{U}) = \nu^E(S \cap \mathcal{U}) - \nu^E(U') < 0$.

Suppose finally $S \sim E$. Then $(S \cap \mathcal{L}) \cup (S \cap \mathcal{U}) \sim E$ so that $\nu^E(S \cap \mathcal{L}) = -\nu^E(S \cap \mathcal{U})$. We have, by construction, $\nu^E(S) = \nu^E(S \cap \mathcal{L}) + \nu^E(S \cap \mathcal{U}) = \nu^E(S \cap \mathcal{U}) - \nu^E(S \cap \mathcal{U}) = 0$.

6.9 Lemma 9

Take any two decisions B and $E \in \mathcal{P}_*(X)$. The result is immediate if $B \sim E$. We provide the proof for the case where $B \succ E$ (the argument for the case where $E \succ B$ being symmetric). We must establish that, for any two sets S and $T \in \mathcal{P}^B(X)$ one has $S \succsim_l^B T \iff \nu^E(S) \geq \nu^E(T)$. By definition of the ordering $\succsim_l^B$ this amounts to showing that:

$$\nu^E(S) \geq \nu^E(T) \iff S \cup F \succsim T \cup F. \tag{5}$$

holds for every S and T such that $S \sim T \sim B$ and every F disjoint from S and T

such that $B \succ F$. Consider indeed such sets S and T with $B \sim S \sim T$. By Lemma 8, $\nu^B(S) = 0 = \nu^B(T)$. By construction, $\nu^E(S) > 0$. By richness and the fact that B and $E \in \mathcal{P}_*(X)$, one can find a set L_1 such that $L_1 \cap (S \cup T) = \varnothing$ and $S \cup L_1 \sim E$. By Averaging $E \succ L_1$. By Lemma 8, $\nu^E(S) + \nu^E(L_1) = \nu^E(S \cup L_1) = 0$. Suppose $\nu^E(T) \geq \nu^E(S)$. Then, $\nu^E(T \cup L_1) = \nu^E(T) + \nu^E(L_1) \geq 0$. By Lemma 8, $T \cup L_1 \succsim B \sim S \cup L_1$. By Balancedness, $T \cup F \succsim S \cup F$ for any F such that $B \succ F$ and $F \cap (S \cup T) = \varnothing$. A similar argument shows that $\nu^E(T) > \nu^E(S) \Rightarrow T \cup F \succ S \cup F$ for any F such that $B \succ F$ and $F \cap (S \cup T) = \varnothing$.

Conversely, suppose $S \cup F \succsim T \cup F$ for some F such that $B \succ F$ and $F \cap (S \cup T) = \varnothing$. By Richness, there is L_2 such that $L_2 \cap (S \cup T) = \varnothing$,and $T \cup L_2 \sim E$. By Averaging, $E \succ L_2$. By Balancedness, $S \cup L_2 \succsim T \cup L_2 \sim E$. By Lemma 8, $\nu^E(S) + \nu^E(L_2) \geq 0$. Since $\nu^E(T) + \nu^E(L_2) = 0$, we obtain $\nu^E(S) \geq \nu^E(T)$. The same argument holds if we suppose $T \cup F \succ S \cup F$, and this establishes (5) and, therefore, the proof of the lemma.

6.10 Lemma 10

Consider any two sets A and B such that $A \succ B$. We first show that the mappings ν^A and ν^B are linearly independent. By contradiction, suppose they are not, and that there is a real number α such that $\nu^A(C) = \alpha \nu^B(C)$ for all $C \in \mathcal{P}^*(X)$. If this was the case, one would have in particular

$$\nu^A(A) = 0 = \alpha \nu^B(A)$$

and

$$\nu^A(B) = \alpha \nu^B(B) < 0$$

which are two incompatible statements. Choose now some sets $S, T \in \mathcal{P}^A(X)$ and a set D in such a way that $D \cap (S \cup T) = \varnothing$ and $A \succ B, D$. By Richness, this choice is possible. Suppose without loss of generality that $T \cup D \succsim S \cup D$. By iterative application of Richness, there are sets $S_1, S_2, ...$ such that, for every $i \neq j \in N$, $S \cap (\bigcup_{i \in \mathbb{N}} S_i) = \varnothing = S_i \cap S_j = S_i \cap D$, $S_i \sim S$ and $\nu^B(S) = \nu^B(S_i)$. Similarly, there exist $T_1, T_2, ...$ such that, for every $i \neq j \in N$, $T \cap (\bigcup_{i \in \mathbb{N}} T_i) = \varnothing = T_i \cap T_j = T_i \cap D$, $T_i \sim T$ and $\nu^B(T) = \nu^B(T_i)$.

For every positive integer n, there is a largest integer $q(n)$ such that $\bigcup_{i=1}^{n} T_i \cup D \succsim \bigcup_{i=1}^{q(n)} S_i \cup D$ because $\nu^B(\bigcup_{i=1}^{p} S_i) = p\nu^B(S_i)$ (remember that ν^B is disjoint-set-additive) and is therefore unbounded when p increases. Notice that $q(n) \geq n$ because $\nu^B(T) \geq \nu^B(S)$. We thus have:

$$\bigcup_{i=1}^{q(n)+1} S_i \cup D \succ \bigcup_{i=1}^{n} T_i \cup D \succsim \bigcup_{i=1}^{q(n)} S_i \cup D$$

for every positive integer n. Since the sets $\bigcup_{i=1}^{q(n)} S_i$, $\bigcup_{i=1}^{n} T_i$ and $\bigcup_{i=1}^{q(n)+1} S_i$ are all equivalent to A (by Averaging) and thanks to Lemma 9, we have:

$$\nu^B(\bigcup_{i=1}^{q(n)+1} S_i) > \nu^B(\bigcup_{i=1}^{n} T_i) \geq \nu^B(\bigcup_{i=1}^{q(n)} S_i).$$

The function ν^B being disjoint-set-additive, we may write $(q(n)+1)\nu^B(S) > n\nu^B(T) \geq q(n)\nu^B(S)$ and:

$$\frac{q(n)+1}{n}\ \nu^B(S) > \nu^B(T) \geq \frac{q(n)}{n}\ \nu^B(S),\ \forall n \in \mathbb{N}_+$$

so that $\nu^B(T) = \lim_{n\to\infty} \frac{q(n)}{n}\ \nu^B(S)$. Following the same reasoning with any $C \in \mathcal{P}_*(X)$ such that $A \succ C$ instead of B yields $\nu^C(T) = \lim_{n\to\infty} \frac{q(n)}{n}\ \nu^C(S)$. So, $\nu^B(T)/\nu^B(S) = \nu^C(T)/\nu^C(S)$. Since this holds for any $S, T \sim A$, this proves that $\nu^B(S) = k\nu^C(S)$ for some positive constant k and for all S such that $\nu^A(S) = 0$.

Define the set $\nu_{ABC}(\mathcal{P}(X))$ by

$$\nu_{ABC}(\mathcal{P}(X)) = \{(\nu^A(S), \nu^B(S), \nu^C(S))\ for\ some\ S \in \mathcal{P}(X)\}.$$

Then $\{x \in \mathbb{R}^3 : x_1 = 0\} \cap \nu_{ABC}(\mathcal{P}(X))$ is contained in the ray $\{(0, kt, t) : t \geq 0\}$. Since $A \in \mathcal{P}_*(X)$, there is S such that either $S \succ A$ or $A \succ S$. Hence the set $\{x \in \nu_{ABC}(\mathcal{P}(X)) : x_1 \neq 0\}$ is not empty. We can therefore select vectors $x^0, x^1 \in \nu_{ABC}(\mathcal{P}(X))$ such that $x^0_1 = 0$ and $x^1_1 \neq 0$. Let the sets S^0 and S^1 be such that $\nu_{ABC}(S^0) = x^0$ and $\nu_{ABC}(S^1) = x^1$.

We show that x^0 and x^1 together, span $\nu_{ABC}(\mathcal{P}(X))$. Let $x \in \nu_{ABC}(\mathcal{P}(X))$ and the set S be such that $\nu_{ABC}(S) = x$. We proceed by cases, assuming $x^1_1 > 0$ (the case $x^1_1 < 0$ being symmetric).

1. Suppose $x_1 = 0$. Since $\{x \in \mathbb{R}^3 : x_1 = 0\} \cap \nu_{ABC}(\mathcal{P}(X))$ is contained in the ray $\{(0, kt, t) : t \geq 0\}$, we have $x = kx^0$.

2. Suppose $x_1 > 0$. By Richness, there is $T : T \cup S^1 \sim S^0$. Hence, $\nu^A(T) = -\nu^A(S^1)$. By Richness, there is a set R such that $R \cup T \sim S^0$ and $R \sim S$. Hence, $\nu^A(R) = \nu^A(S^1)$. Since $R \sim S \succ A, B, C$, we know that $\nu^C(R) = \alpha\nu^A(R)$ and $\nu^C(S) = \alpha\nu^A(S)$ for some $\alpha \in R$. For the same reason, $\nu^B(R) = \beta\nu^A(R)$ and $\nu^B(S) = \beta\nu^A(S)$ for some $\beta \in R$. So, $\nu^C(R)/\nu^C(S) = \nu^A(R)/\nu^A(S)$ and $\nu^B(R)/\nu^B(S) = \nu^A(R)/\nu^A(S)$. In other words, $\nu_{ABC}(R)$ and $\nu_{ABC}(S)$ are in the same ray and $\nu_{ABC}(S) = \gamma\nu_{ABC}(R)$ for some $\gamma \in R$.

   Since $T \cup S^1 \sim S^0$, we know that $\nu_{ABC}(T \cup S^1)$ is in the same ray as x^0. Hence, $\nu_{ABC}(T \cup S^1) = \nu_{ABC}(T) + x^1 = \lambda x^0$ for some $\lambda > 0$. Similarly, since $T \cup R \sim S^0$, we know that $\nu_{ABC}(T \cup R)$ is in the same ray as x^0. Therefore $\nu_{ABC}(T \cup R) = \nu_{ABC}(T) + \nu_{ABC}(R) = \lambda x^0 - x^1 + \nu_{ABC}(R) = \lambda' x^0$ for some $\lambda' > 0$. Whence $\nu_{ABC}(R) = \lambda' x^0 - \lambda x^0 + x^1$. We can therefore write $\nu_{ABC}(S) = \gamma[(\lambda' - \lambda)x^0 + x^1]$. This proves that x is spanned by x^0 and x^1.

3. Suppose $x_1 < 0$. By Richness, there is $T : T \cup S \sim S^0$ and, hence, $\nu_{ABC}(T \cup S)$ is in the same ray as x^0. So, $\nu_{ABC}(T \cup S) = \nu_{ABC}(T) + x = \lambda x^0$ for some $\lambda > 0$. So, $x = \lambda x^0 - \nu_{ABC}(T)$. In other words, x is spanned by x^0 and $\nu_{ABC}(T)$. We have seen in case 2 that $\nu_{ABC}(T)$ is spanned by x^0 and x^1. So, actually, x is spanned by x^0 and x^1.

Hence, there are two real numbers λ, γ such that, for any $S \in \mathcal{P}(X)$,

$$\nu_{ABC}(S) = \lambda \nu_{ABC}(S^0) + \gamma \nu_{ABC}(S^1). \tag{6}$$

In particular, $\nu^A(S) = \lambda \nu^A(S^0) + \gamma \nu^A(S^1) = \gamma \nu^A(S^1)$ because $\nu^A(S^0) = 0$. So, $\gamma = \nu^A(S)/\nu^A(S^1)$. From (6), we also derive $\nu^C(S) = \lambda \nu^C(S^0) + \gamma \nu^C(S^1)$ which yields $\lambda = (\nu^C(S) - \gamma \nu^C(S^1))/\nu^C(S^0)$. From (6), we finally derive $\nu^B(S) = \lambda \nu^B(S^0) + \gamma \nu^B(S^1)$. Substituting λ and γ in this equation yields

$$\nu^B(S) = \frac{\nu^C(S) - (\nu^A(S)/\nu^A(S^1))\nu^C(S^1)}{\nu^C(S^0)}\ \nu^B(S^0) + \nu^A(S)\nu^B(S^1)/\nu^A(S^1),$$

showing that ν^B is a linear combination of ν^A and ν^C.

Hence, for every $A, B, C \in \mathcal{P}_*(X)$, such that none of them are indifferent, there are two real numbers α, β such that $\nu^A = \alpha \nu^B + \beta \nu^C$. Consider now A, B and C such that $B \not\sim C$. Using richness, we select D and D' not indifferent to any of A, B or C and such that $D \not\sim D'$. We can express each of ν^A, ν^B, ν^C as a linear combination of ν^D and $\nu^{D'}$. For instance,

$$\nu^A = \alpha_A \nu^D + \beta_A \nu^{D'}, \tag{7}$$

$$\nu^B = \alpha_B \nu^D + \beta_B \nu^{D'} \tag{8}$$

$$\nu^C = \alpha_C \nu^D + \beta_C \nu^{D'}. \tag{9}$$

From (8) and (9), we derive

$$\nu^D = \frac{\nu^C \beta_B - \nu^B \beta_C}{\alpha_C \beta_B - \alpha_B \beta_C}$$

and

$$\nu_{D'} = \frac{\nu^C \alpha_B - \nu^B \alpha_C}{\beta_C \alpha_B - \beta_B \alpha_C}.$$

We substitute ν^D and $\nu^{D'}$ in (7) and we obtain that ν^A is a linear combination of ν^B and ν^C. This suffices to show the entire space $\{\nu^E : E \in \mathcal{P}_*(X)\}$ can be spanned by any two of its members ν^B, ν^C with $B \not\sim C$, since the selection of A, B, C in the proof was arbitrary.

6.11 Lemma 11

We consider two cases.

(1) $\nu^B(D) = k\nu^C(D)$ for some $k \in \mathbb{R}_{++}$ and all $D \in \mathcal{P}(X)$. In this case, assuming the existence of a strictly positive real number λ and a $\delta \in [0,1]$ for which $-\lambda \nu^A(D) = \delta \nu^B(D) + (1-\delta)\nu^C(D)$ holds for all sets $D \in \mathcal{P}(X)$ would amount to assume that

$$\nu^A(D) = -\frac{k\delta + (1-\delta)}{\lambda k}\nu^B(D) \tag{10}$$

for all D. But, since $\frac{k\delta+(1-\delta)}{\lambda k}$ is strictly positive (as is the ratio of convex combination of two strictly positive numbers and a strictly positive number), Equality (10) is not possible because, by Lemma 8, we know that, for any D such that $B \succ D$, and $A \succ D$, we have $\nu^B(D) < 0$ and $\nu^A(D) < 0$. The cases $\nu^A = k\nu^C$ and $\nu^A = k\nu^B$ are treated in the same way.

(2) $A \succ B \succ C$ (the 5 other orderings are treated in the same way). By Lemma 10, ν^A and ν^B span $\{\nu^E : E \in \mathcal{P}_*(X)\}$. For every $C \in \mathcal{P}_*(X)$, let $\alpha(C)$ and $\beta(C)$ be the solution of $\nu^C = \alpha(C)\nu^A + \beta(C)\nu^B$. Since $B \succ C$, we must have $\alpha(C) < 0$ or $\beta(C) < 0$ as assuming otherwise would imply, for any set S such that $B \succ S \succ C$, that it is impossible to have $\nu^C(S) > 0$. We must also have $\beta(C) > 0$ because $\nu^C(A)$ must be positive. Hence, we must have $\beta(C) > 0 > \alpha(C)$. Assume by contradiction that $-\lambda\nu^A = \delta\nu^B + (1-\delta)\nu^C$ for some $\lambda \in \mathbb{R}_{++}$ and $\delta \in]0,1[$. This implies that

$$\nu^C = \frac{\lambda}{\delta - 1}\nu^A + \frac{\delta}{\delta - 1}\nu^B$$

with $\delta/(\delta-1) < 0$, a contradiction of the fact that $0 < \beta(C)$.

6.12 Lemma 12

Consider two sets B and $C \in \mathcal{P}_*(X)$ such that $B \succ C$. As established in Lemma 10, ν^B and ν^C are linearly independent and span the set $\{\nu^E : E \in \mathcal{P}_*(X)\}$. For every $A \in \mathcal{P}_*(X)$, let $\alpha(A)$ and $\beta(A)$ be the solution of $\nu^A = \alpha(A)\nu^B + \beta(A)\nu^C$. If B, $C \succ A$, then we must have $\alpha(A) < 0$ or $\beta(A) < 0$. Indeed, assuming otherwise would imply, for any set S such that $B, C \succ S \succ A$, that it is impossible to have $\nu^A(S) > 0$. Simultaneously, we must also have $\beta(A) > 0$ because $\nu^A(B)$ must be positive. So, we must have $\beta(A) > 0 > \alpha(A)$. Define the function $\rho : \mathcal{P}(X) \longrightarrow \mathbb{R}$ by $\rho(A) = -\beta(A)/\alpha(A)$ for any set A. We have $\rho(A) > 0$ for all $A \in \mathcal{P}_*(X)$ such that $C \succ A$.

We now show that, for all $A' \in \mathrm{P}_*(X)$ such that $C \succ A \succ A'$, one has $\rho(A) > \rho(A')$ so that the function ρ numerically represent the ranking of decisions that are worse than A. Suppose to the contrary that $\rho(A') \geq \rho(A)$. Since $\nu^A(A) = \alpha(A)\nu^B(A) + \beta(A)\nu^C(A) = 0$, we have:

$$-\frac{\beta(A)}{\alpha(A)} = \rho(A) = \frac{\nu^B(A)}{\nu^C(A)} \leq -\frac{\beta(A')}{\alpha(A')} = \rho(A').$$

Hence $\nu^B(A)\alpha(A') \leq -\beta(A')\nu^C(A)$ and $\nu^{A'}(A) = \alpha(A')\nu^B(A) + \beta(A')\nu^C(A) \leq 0$, which implies $A' \succ A$, a contradiction. Notice that the converse is also true. Hence, for all A, $A' \in \mathcal{P}_*(X)$ satisfying $C \succ A$ and $C \succ A'$, $A \succsim A' \Longleftrightarrow \rho(A) \geq \rho(A')$.

Similarly, it is easy to prove that, for all sets A and $A' \in \mathcal{P}_*(X)$ such that $A \succ B$ and $A' \succ B$, it is the case that $\rho(A) > 0$ and $A \succsim A' \Longleftrightarrow \rho(A) \geq \rho(A')$.

Define now the set $Q = \{\rho(A) : A \in \mathcal{P}_*(X), C \succ A\}$. This set has a greatest lower bound $\rho^* \geq 0$ (because $\rho(A) > 0$ for all set $A \in \mathcal{P}(X)$ such that $C \succ A$). We can actually show that $\rho^* > 0$. Indeed, assume by contradiction that $\rho^* = 0$. Since $B \in \mathcal{P}_*(X)$, there exists a set $D \in \mathcal{P}_*(X)$ such that $D \succ B$. Because $\rho^* = 0$, there is also a set $F \in \mathcal{P}_*(X)$ with $\rho(F)$ sufficiently close to zero and such that ν^D, ν^B

and ν^F are just like the functions ν^A, ν^B and ν^C of Lemma 11, which is not possible. Hence we must conclude that $\rho^* > 0$.

Furthermore $\rho^* \notin Q$ because the set $\{A \in \mathcal{P}_*(X) : C \succ A\}$ has no minimal element. Let μ be any of the element in the ray $\{x(-\nu^B + \rho^* \nu^C) : x > 0\}$. To be specific, define μ by: $\mu = -\nu^B + \rho^* \nu^C$. By construction, μ belongs to the elements spanned by (ν^B, ν^C), as per Lemma 10.

We now prove that $\mu(A) > 0$ for all $A \in \mathcal{P}_*(X)$. Suppose on the contrary that $0 \geq \mu(A)$ for some $A \in \mathcal{P}_*(X)$. By definition of $\mathcal{P}_*(X)$, there are decisions B and C such that $B \succ C \succ A$. We know that $\mu(A) = -\nu^B(A) + \rho^* \nu^C(A) \leq 0$. Hence, it follows that $\rho^* \nu^C(A) \geq \nu^B(A)$ and, therefore, that $\rho^* \geq \nu^B(A)/\nu^C(A)$ because $\nu^C(A) < 0$. Since $\nu^A(A) = \alpha(A)\nu^B(A) + \beta(A)\nu^C(A) = 0$, we have $\rho^* \geq -\beta(A)/\alpha(A)$, which is impossible because $\rho^* \notin Q$. We must therefore conclude that $\mu(A) > 0$ for all $A \in \mathcal{P}_*(X)$.

Finally, we notice that the function μ is disjoint-set-additive because it is the linear combination of two disjoint-set-additive functions.

6.13 Lemma 13

Pick any set $C \in \mathrm{P}_*(X)$ and, for every set $A \in \mathrm{P}_*(X)$, let $\alpha(A)$ and $\beta(A)$ be the solution of $\nu^A = \alpha(A)\nu^C + \beta(A)\mu$. These $\alpha(A)$ and $\beta(A)$ exist because ν^C and μ are linearly independent and, by Lemma 10, can span the whole set $\{\nu^E : E \in \mathcal{P}_*(X)\}$. By construction, $\nu^A(A) = 0 = \alpha(A)\nu^C(A) + \beta(A)\mu(A)$ or, equivalently, $\frac{\nu^C(A)}{\mu(A)} = \frac{-\beta(A)}{\alpha(A)}$. Hence, in order to show that ν^C/μ is a numerical representation of $\succsim$ on $\mathcal{P}_*(X)$, it suffices to show that $-\beta/\alpha$ represents $\succsim$ on $\mathcal{P}_*(X)$. Notice first that $-\beta/\alpha$ is well-defined because $-\beta(A)/\alpha(A) = \nu^C(A)/\mu(A)$ and $\mu(A) > 0$ for all $A \in \mathcal{P}_*(X)$. Pick any two decisions A and $B \in \mathcal{P}_*(X)$ such that $A \succsim B$. By construction, $\nu^B(A) \geq 0$. Hence, we must have $\alpha(B)\nu^C(A) + \beta(B)\mu(A) \geq 0$ and $\nu^C(A) \geq -\beta(B)\mu(A)/\alpha(B)$. We also have $\alpha(A)\nu^C(A) + \beta(A)\mu(A) = 0$ or, equivalently, $\nu^C(A) = -\beta(A)\mu(A)/\alpha(A)$. Hence, $-\beta(A)\mu(A)/\alpha(A) \geq -\beta(B)\mu(A)/\alpha(B)$ or, after simplification, $-\beta(A)/\alpha(A) \geq -\beta(B)/\alpha(B)$. We have therefore proved that $A \succsim B$ implies $-\beta(A)/\alpha(A) \geq -\beta(B)/\alpha(B)$. Proving the converse is easily done by just reversing the argument.

6.14 Lemma 14

Consider any set $C \in \mathcal{P}_*(X)$. For any sets A and $B \in \mathcal{P}_*(X)$, $A \sim_l^C B$ implies, by Lemma 6, that $A \cup D \sim B \cup D$ for some set D such that $A \succ D$ and $D \cap (A \cup B) = \varnothing$. Since, thanks to Lemma 17, ν^C/μ numerically represents the ordering $\succsim$ on $\mathcal{P}_*(X)$, one has

$$\frac{\nu^C(A \cup D)}{\mu(A \cup D)} = \frac{\nu^C(B \cup D)}{\mu(B \cup D)}$$

or, using the disjoint-additivity of ν^C and μ,

$$\frac{\nu^C(A) + \nu^C(D)}{\mu(A) + \mu(D)} = \frac{\nu^C(B) + \nu^C(D)}{\mu(B) + \mu(D)}. \tag{11}$$

Moreover, since the statement $A \sim_l B$ is constructed from the statement that $A \sim B \sim E$ for some set E, it follows from the fact that ν^C/μ numerically represents the ordering $\succsim$ that

$$\frac{\nu^C(A)}{\mu(A)} = \frac{\nu^C(B)}{\mu(B)}$$

or, equivalently,

$$\nu^C(A) = \frac{\mu(A)\nu^C(B)}{\mu(B)}. \tag{12}$$

Substituting equality (12) into equality (11) yields:

$$\frac{\nu^C(B)\mu(A)/\mu(B) + \nu^C(D)}{\mu(A) + \mu(D)} = \frac{\nu^C(B) + \nu^C(D)}{\mu(B) + \mu(D)}$$

or, after some manipulations and simplifications,

$$[(\nu^C(B)\mu(D) - \mu(B)\nu^C(D)][\mu(A) - \mu(B)] = 0.$$

If $\mu(A) - \mu(B) \neq 0$, then $\nu^C(B)/\mu(B) = \nu^C(D)/\mu(D)$ and $B \sim D$, which is incompatible with the definition of D. We therefore conclude that $\mu(A) - \mu(B) = 0$ and, hence, $\mu(A) = \mu(B)$ and $\nu^C(A) = \nu^C(B)$.

6.15 Lemma 15

Take any reference set $C \in \mathcal{P}_*(X)$ and consider any set $D \in \mathcal{P}_*(X)$. By Richness, there are sets A and $B \in \mathcal{P}_*(X)$ such that $D \succ A \succ B$. By Lemma 3, there is a set A' such that $A' \sim A$ and $A' \cap B = \varnothing$. By Averaging, $A \succ A' \cup B \succ B$. By Richness, there are sets $A_1, A_2, \ldots$ such that, for $i \in \{1, 2, \ldots\}$, $A \sim A_i$, $A_i \cap (A \bigcup_{j=1}^{i-1} A_j) = \varnothing$ and $A_i \cup B \sim A' \cup B$. By Lemma 14, $\nu^C(A_i) = \nu^C(A)$ and $\mu(A_i) = \mu(A)$ for $i \in \{1, 2, \ldots\}$. Some of the sets $A_1, A_2, \ldots$ may intersect with D, but the number of such intersecting sets is necessarily finite (as these sets are pairwise disjoint). So, if we drop them, we can still end up with an infinite collection of sets $A_1, A_2, \ldots$ that are all disjoint from D. We therefore assume hereafter that $A_i \cap D = \varnothing$ for $i \in \{1, 2, \ldots\}$. By Averaging, $D \succ D \bigcup_{j=1}^{k} A_j \succ A$, for any $k \in \{1, 2, \ldots\}$. By Richness, for any $k \in \{1, 2, \ldots\}$, there is a set E such that $E \cap (A \cup D) = \varnothing$, $E \sim D$ and $E \cup A \sim D \bigcup_{j=1}^{k} A_j$. By Lemma 13, $\nu^C(D)/\mu(D) = \nu^C(E)/\mu(E)$ and, for all $i \in \{1, 2, \ldots\}$, one has $\nu^C(A)/\mu(A) = \nu^C(A_i)/\mu(A_i)$ and

$$\frac{\nu^C(E \cup A)}{\mu(E \cup A)} = \frac{\nu^C(D \bigcup_{j=1}^{i} A_j)}{\mu(D \bigcup_{j=1}^{i} A_j)}.$$

Using the disjoint-additivity of ν^C and μ, one can write, for any k:

$$\frac{\nu^C(E) + \nu^C(A)}{\mu(E) + \mu(A)} = \frac{\nu^C(D) + \sum_{j=1}^{k} \nu^C(A_j)}{\mu(D) + \sum_{j=1}^{k} \mu(A_j)} = \frac{\nu^C(D) + k\nu^C(A)}{\mu(D) + k\mu(A)}.$$

which can be equivalently written as:

$$(\nu^C(E) + \nu^C(A))\ (\mu(D) + k\mu(A)) = (\nu^C(D) + k\nu(A))\ (\mu(E) + \mu(A)).$$

If one substitutes $\nu^C(D)\mu(E)/\mu(D)$ for $\nu^C(E)$ in this expression and performs similar manipulation as in the proof of Lemma 14, one obtains:

$$[\nu^C(A)\nu^C(D) - \mu(A)\mu(D)][(\mu(D) - k\mu(E))] = 0.$$

One cannot have $\nu^C(A)\nu^C(D) - \mu(A)\mu(D) = 0$. Indeed, assuming such an equality would amount to assuming that $\nu^C(A)/\mu(A) = \nu^C(D)/\mu(D)$ and, since the function ν^C/μ numerically represents the ordering $\succsim$, that $A \sim D$, which is not case. We therefore conclude that $\mu(D) - k\mu(E) = 0$ and, hence, $\mu(E) = \mu(D)/k$. For any $\varepsilon > 0$, we can therefore guarantee that $\mu(E) < \varepsilon$ by choosing a suitably large k.

6.16 Lemma 16

If the function μ of Lemma 12 is such that $\mu(S) > 0$ for all set $S \in \mathcal{P}(X)$, we define $\mu_+ = \mu$ and the proof is done.

Otherwise, we first prove that $\mu(S) \geq 0$ for all $S \in m(X)$. Assume for contradiction that $\mu(S) < 0$ for some $S \in m(X)$ and choose (using richness) sets C and $T \in \mathcal{P}_*(X)$ satisfying $S \cap T = \varnothing$, $\nu^C(T) < 0$ and $\mu(T)$ sufficiently small (thanks to Lemma 15). Consider the set $D = T \cup S$ and its numerical representation by the function ν^C/μ:

$$\frac{\nu^C(T) + \nu^C(S)}{\mu(T) + \mu(S)}.$$

The numerator of this expression is negative because $S \in m(X)$ and $\nu^C(S) < 0$ if $\nu^C(T) < 0$ by definition of the function ν^C provided in Lemma 8. For a sufficiently small $\mu(T)$, one can also make the denominator of the expression negative. Hence $\nu^C(D)/\mu(D) > 0$ and, since the function ν^C/μ numerically represents the ordering $\succsim$ on $\mathcal{P}_*(X)$, one concludes that $D \succ T$. Yet this contradicts the Averaging axiom according to which $T \succ D = T \cup S$ (because $S \in m(X)$).

Using an analogous argument, we can prove that $\mu(S) \geq 0$ for all set $S \in M(X)$.

We now claim that it is impossible to have $\mu(S) = 0 = \mu(T)$ for some $S \in m(X)$ and some $T \in M(X)$. Assume indeed that $\mu(S) = 0 = \mu(T)$ for some $S \in m(X)$ and $T \in M(X)$. Remember from Lemma 1 that $S \cap T = \varnothing$. By Averaging $S \cup T \in \mathcal{P}_*(X)$ and, as a result, one has $\mu(S \cup T) > 0$ by Lemma 10. Yet, using the disjoint-additivity of μ, we find that $\mu(S \cup T) = 0$ although $S \cup T \in \mathcal{P}_*(X)$. This contradiction shows the impossibility of having $\mu(S) = 0 = \mu(T)$ for some $S \in m(X)$ and some $T \in M(X)$.

Suppose now that $\mu(S) = 0$ for some $S \in m(X)$. This implies $\mu(T) > 0$ for all $T \in \mathcal{P}_*(X) \cup M(X)$. We know from Lemma 12 that $\mu = -\nu^B + \rho^* \nu^C$ for some sets B and $C \in \mathcal{P}_*(X)$. If we choose a number $\rho_+ < \rho^*$ and we define $\mu_+ = -\nu^B + \rho_+ \nu^C$, we are sure that $\mu(S) > 0$. If, in addition, we choose the number ρ_+ to be as close as necessary to ρ^*, we can guarantee that $\mu(T) > 0$ for all $T \in \mathcal{P}_*(X)$. The mapping μ_+ is clearly disjoint-set-additive and can be spanned by two (linearly independent) elements of the family $\{\nu^E : E \in \mathcal{P}_*(X)\}$. We still have to prove that $\mu_+(T) > 0$

for all $T \in m(X)$. If $T \neq S$ and $\mu(T) > 0$, then the proof is obvious because we have chosen ρ_+ to be very close to ρ^*. If $T \neq S$ and $\mu(T) = 0$, one must remember that $\mu(T) = -\nu^B(T) + \rho^* \nu^C(T)$, where $\nu^B(T) < 0$ and $\nu^C(T) < 0$. Hence if we choose $\rho_+ < \rho^*$, then $\mu_+(T) = -\nu^B(T) + \rho_+ \nu^C(T)$ is necessarily larger that $\mu(T)$ and, hence, positive.

The case where $\mu(S) = 0$ for some $S \in M(X)$ can be handled in a similar fashion.

6.17 Lemma 17

For every set $E \in \mathcal{P}_*(X)$, let $\alpha(E)$ and $\beta(E)$ be the solution of the equation $\nu^E = \alpha(E)\nu^C + \beta(E)\mu_+$. As in the proof of Lemma 13, the existence of these real numbers $\alpha(E)$ and $\beta(E)$ is secured by the fact that ν^C and μ_+ are linearly independent and, thanks to Lemma 10, can span the whole set $\{\nu^E : E \in \mathcal{P}_*(X)\}$. By construction, $\nu^E(E) = 0 = \alpha(E)\nu^C(E) + \beta(E)\mu_+(E)$ or, equivalently, $\frac{\nu^C(E)}{\mu_+(E)} = \frac{-\beta(E)}{\alpha(E)}$. As in the proof of Lemma 13 again, the proof that ν/μ_+ is a numerical representation of $\succsim$ on $\mathcal{P}_*(X)$ amounts to showing that $-\beta/\alpha$ represents $\succsim$ on $\mathcal{P}_*(X)$. Notice first that $-\beta/\alpha$ is well-defined because $-\beta(E)/\alpha(E) = \nu^C(E)/\mu_+(E)$ and $\mu_+(E) > 0$ for all $E \in \mathcal{P}_*(X)$. Consider any two sets A and $B \in \mathcal{P}_*(X)$ with $A \succsim B$. From Lemma 8, $\nu^B(A) \geq 0$. Hence one has $\alpha(B)\nu^C(A) + \beta(B)\mu_+(A) \geq 0$ and $\nu^C(A) \geq -\beta(B)\mu_+(A)/\alpha(B)$. One also has $\alpha(A)\nu^C(A) + \beta(A)\mu_+(A) = 0$ or, equivalently, $\nu^C(A) = -\beta(A)\mu_+(A)/\alpha(A)$. Hence $-\beta(A)\mu_+(A)/\alpha(A) \geq -\beta(B)\mu_+(A)/\alpha(B)$ or, after simplification, $-\beta(A)/\alpha(A) \geq -\beta(B)/\alpha(B)$. We have therefore proved that $A \succsim B$ implies $-\beta(A)/\alpha(A) \geq -\beta(B)/\alpha(B)$. The converse implication is obtained by reversing the argument.

6.18 Theorem 1

If the ordering $\succsim$ is trivial, then the numerical representation provided by (1) trivially holds with u constant. We therefore assume in the rest of the proof that $\succsim$ is not trivial.

Take any reference set C in $\mathcal{P}_*(X)$. Just as in equation (4) preceding Lemma 13, define the function $f_+^C : \mathcal{P}(X) \to \mathbb{R}$ by $f_+^C(A) = \frac{\nu^C(A)}{\mu_+(A)}$ for every $A \in \mathcal{P}(X)$. Since ν^C and μ_+ are both disjoint-set-additive, one can write:

$$f_+^C(A) = \frac{\sum_{a \in A} \nu^C(\{a\})}{\sum_{a \in A} \mu_+(\{a\})} = \frac{\sum_{a \in A} f_+^C(\{a\})\mu_+(\{a\})}{\sum_{a \in A} \mu_+(\{a\})}.$$

Define the two functions $u : X \to \mathbb{R}$ and $p : X \to \mathbb{R}_{++}$ by $u(a) = f(\{a\})$ and $p(a) = \mu_+(\{a\})$. One has:

$$f_+^C(A) = \frac{\sum_{a \in A} u(a)p(a)}{\sum_{a \in A} p(a)}.$$

We already know from Lemma 17 that $A \succsim B \iff f_+^C(A) \geq f_+^C(B)$ for all decisions A and $B \in \mathcal{P}_*(X)$. We only need to prove that the equivalence must hold also for decisions A and $B \in \mathcal{P}(X)$ that can be maximal or minimal in that set. We consider several cases.

1. $A \in M(X)$ and $B \in \mathcal{P}_*(X)$. By Lemma 3, there is $B' \in \mathcal{P}(X)$ such that $B' \cap A = \varnothing$ and $B' \sim B$. By Lemma 17, $\nu^C(B)/\mu_+(B) = \nu^C(B')/\mu_+(B')$. By Averaging, $A \succ A \cup B' \succ B'$ and, hence, $A \cup B' \in \mathcal{P}_*(X)$. We therefore have:
$$\frac{\nu^C(A \cup B')}{\mu_+(A \cup B')} = \frac{\nu^C(A) + \nu^C(B')}{\mu_+(A) + \mu_+(B')} > \frac{\nu^C(B')}{\mu_+(B')} = \frac{\nu^C(B)}{\mu_+(B)}.$$
Since μ_+ is always strictly positive, this yields
$$\frac{\nu^C(A)}{\mu_+(A)} > \frac{\nu^C(B)}{\mu_+(B)},$$
a statement that is in line with the fact that $A \succ B$.

2. $A \in m(X)$ and $B \in \mathcal{P}_*(X)$. Similar to the previous case.

3. $A, B \in m(X)$. Choose a decision $D \in \mathcal{P}_*(X)$ in such a way that $D \cap (A \cup B) = \varnothing$. By Averaging, $B \cup D \succ B$ and, by transitivity, $B \cup D \succ A$. Using the result of the proof of case 2, $\nu^C(B \cup D)/\mu_+(B \cup D) > \nu^C(A)/\mu_+(A)$ and
$$\frac{\nu^C(B) + \nu^C(D)}{\mu_+(B) + \mu_+(D)} > \frac{\nu^C(A)}{\mu_+(A)}. \tag{13}$$
By Lemma 15, we can choose D in a given equivalence class of $\succsim$, with $\mu_+(D)$ as close to zero as required. Since all sets in a given equivalence class have the same ratio ν^C/μ_+, we can actually choose D in such a way that both $\mu_+(D)$ and $\nu^C(D)$ are arbitrarily close to zero. Assume now by contradiction that $f_+^C(A) > f_+^C(B) \iff \nu^C(A)/\mu_+(A) > \nu^C(B)/\mu_+(B)$.Then, if we choose the set D as described above, we clearly have
$$\frac{\nu^C(A)}{\mu_+(A)} > \frac{\nu^C(B) + \nu^C(D)}{\mu_+(B) + \mu_+(D)},$$
which contradicts (13).

4. $A, B \in M(X)$. Similar to the previous case.

5. $A \in M(X)$ and $B \in m(X)$. We know from Lemma 1 that $A \cap B = \varnothing$. Then $A \succ A \cup B \succ B$ by Averaging and, hence, $A \cup B \in \mathcal{P}_*(X)$. From $A \succ A \cup B$ and case 1, we derive $f_+^C(A) > f_+^C(A \cup B)$. From $A \cup B \succ B$ and case 2, we derive $f_+^C(A \cup B) > f_+^C(B)$ and the required conclusion $f_+^C(A) > f_+^C(B)$ follows from transitivity.

6.19 Example 1

We first show that the ordering of this example violates the Archimedean axiom. Let $A = \{(1,2,0,-1)\}$, $B = \{(1,1,0,-1)\}$, $A_i = \{(1,2,0,i)\}$, $B_i = \{(1,1,0,i)\}$, $C = \{(1,1,0,0)\}$ and $D = \{(2,1,0,0)\}$. Notice that $U_1(A) = U_1(A_i) = U_1(B) = U_1(B_i) = U_1(C) = U_1(D) = 0 = U_2(A) = U_2(A_i) = U_2(B) = U_2(B_i) = U_2(C) = U_2(D)$ so that $A \sim A_i \sim B \sim B_i \sim C \sim D$ for all $i \in \mathbb{N}$. Let $E = \{(1/10, 1/10, -1, 0)\}$. One has

$$U_1(E) = -1/10 = U_2(E)$$

so that $A \sim B \succ E$. Moreover

$$U_1(A \cup E) = U_1(A_i \cup E) = -1/11 = U_1(B \cup E) = U_2(B \cup E) = U_1(B_i \cup E) = U_2(B_i \cup E)$$

and

$$U_2(A \cup E) = U_2(A_i \cup E) = -1/21 > -1/11$$

so that $A \cup E \succ B \cup E$, $A_i \cup E \sim A \cup E$ and $B_i \cup E \sim B \cup E$ for all $i \in \mathbb{N}$. Yet,contrary to what the Archimedean axiom requires, $C \cup E \bigcup_{i=0}^{n} A_i \prec D \cup E \bigcup_{i=0}^{n} B_i$ for all $n \in \mathbb{N}$ because

$$\begin{aligned} U_1(C \cup E \bigcup_{i=0}^{n} A_i) &= \frac{-1/10}{1 + 1/10 + n} = \frac{-1}{11 + 10n} \\ &< U_1(D \cup E \bigcup_{i=0}^{n} B_i) = \frac{-1/10}{2 + 1/10 + n} = \frac{-1}{21 + 10n} \end{aligned}$$

for every n.

We next show that $\succsim$ satisfies Averaging. Suppose first that $A \succ B$. Using the definition of $\succsim$, this is either equivalent to

$$\begin{gathered} U_1(A) > U_1(B) \\ \iff \\ U_1(A) > U_1(A \cup B) > U_1(B) \\ \iff \\ A \succ A \cup B \succ B \end{gathered}$$

or to

$$\begin{aligned} &U_1(A) = U_1(B) \text{ and } U_2(A) > U_2(B) \\ &\iff \\ &U_1(A) = U_1(A \cup B) = U_1(B) \text{ and } U_2(A) > U_2(A \cup B) > U_2(B) \\ &\iff \\ &A \succ A \cup B \succ B. \end{aligned}$$

A similar reasoning holds when $A \sim B$.

In order to show that satisfies Richness, consider $A, B, C \in \mathcal{P}(X)$ such that $A \succ B \succ C$. We will show that there exists a set $D = \{d, e\}$ such that $D \cap (A \cup C) = \emptyset$, $D \sim A$ and $D \cup C \sim B$. So, we must have

$$\frac{d_1 d_3 + e_1 e_3}{d_1 + e_1} = U_1(A), \tag{14}$$

$$\frac{d_2 d_3 + e_2 e_3}{d_2 + e_2} = U_2(A), \tag{15}$$

$$\frac{d_1 d_3 + e_1 e_3 + \sum_{c \in C} c_1 c_3}{d_1 + e_1 + \sum_{c \in C} c_1} = U_1(B), \tag{16}$$

$$\frac{d_2 d_3 + e_2 e_3 + \sum_{c \in C} c_2 c_3}{d_2 + e_2 + \sum_{c \in C} c_2} = U_2(B). \tag{17}$$

Set $d_3 = \max(U_1(A), U_2(A)) + 1$ and $e_3 = \min(U_1(A), U_2(A)) - 1$. There clearly exist $d_1, e_1 \in \mathbb{R}_{++}$ such that (14) holds. Notice that d_1 and e_1 are not unique. They can be scaled by any positive constant and we can choose this constant so that (16) holds. Similarly, there clearly exist $d_2, e_2 \in \mathbb{R}_{++}$ such that (15) holds. They are unique up to a multiplication by a positive constant, that we can choose independently of the scaling constant for d_1, e_1. In particular therefore, we can choose the constants so that (17) holds. In order to guarantee that $D \cap (A \cup C) = \emptyset$, we can freely manipulate d_4 and e_4. Hence Richness holds.

Finally, to show that satisfies Balancedness, we consider any finite and non-empty subsets A, B, C, D of X such that $A \sim B \succ C, D$ and $(A \cup B) \cap (C \cup D) = \emptyset$. We have $A \cup C \quad B \cup C$ if and only if either:
$U_1(A \cup C) > U_1(B \cup C)$ iff $U_1(A \cup D) > U_1(B \cup D)$ iff $A \cup D \quad B \cup D$ or
$[U_1(A \cup C) = U_1(B \cup C)$ and $U_2(A \cup C) \geq U_2(B \cup C)]$ iff $[U_1(A \cup D) = U_1(B \cup D)$ and $U_2(A \cup D) \geq U_2(B \cup D)]$ iff $A \cup D \quad B \cup D$.

6.20 Example 2

Let us prove that satisfies Balancedness. We first observe that $A \sim B$ implies

$$\sum_{a \in A} p(a)u(a)[\sum_{b \in B} p(b)^2] = [\sum_{a \in A} p(a)^2] \sum_{b \in B} p(b)u(b) \tag{18}$$

while $A \cup C \quad B \cup C$ implies

$$[\sum_{a \in A} p(a)u(a) + \sum_{c \in C} p(c)u(c)]\ [\sum_{b \in B} p(b)^2 + \sum_{c \in C} p(c)^2]$$
$$\geq [\sum_{b \in B} p(b)u(b) + \sum_{c \in C} p(c)u(c)][\sum_{a \in A} p(a)^2 + \sum_{c \in C} p(c)^2]$$

or, after distributing

$$[\sum_{a \in A} p(a)u(a)][\sum_{b \in B} p(b)^2] + [\sum_{c \in C} p(c)u(c)][\sum_{b \in B} p(b)^2] + [\sum_{a \in A} p(a)u(a)][\sum_{c \in C} p(c)^2]$$
$$\geq [\sum_{b \in B} p(b)u(b)][\sum_{a \in A} p(a)^2] + [\sum_{c \in C} p(c)u(c)][\sum_{a \in A} p(a)^2] + [\sum_{b \in B} p(b)u(b)][\sum_{c \in C} p(c)^2].$$

Substituting (18) into this inequality yields (after elementary manipulations)

$$\sum_{c\in C} p(c)u(c)[\sum_{b\in B} p(b)^2 - \sum_{a\in A} p(a)^2] \geq [\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)][\sum_{c\in C} p(c)^2]. \quad (19)$$

It is clear that $\sum_{c\in C} p(c)^2 > 0$. Given this, there are three (mutually exclusive) cases to consider:

1. $\sum_{b\in B} p(b)^2 - \sum_{a\in A} p(a)^2 > 0$. In this case, Inequality (19) can equivalently be written as

$$\begin{aligned}\frac{\sum_{c\in C} p(c)u(c)}{\sum_{c\in C} p(c)^2} &\geq \frac{\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)}{\sum_{b\in B} p(b)^2 - \sum_{a\in A} p(a)^2} \\ &= \frac{\sum_{b\in B} p(b)u(b)}{\sum_{b\in B} p(b)^2} \text{ (using Equality (18))}\end{aligned}$$

2. $\sum_{b\in B} p(b)^2 - \sum_{a\in A} p(a)^2 < 0$. In this case, we obtain

$$\begin{aligned}\frac{\sum_{c\in C} p(c)u(c)}{\sum_{c\in C} p(c)^2} &\leq \frac{\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)}{\sum_{b\in B} p(b)^2 - \sum_{a\in A} p(a)^2} \\ &= \frac{\sum_{b\in B} p(b)u(b)}{\sum_{b\in B} p(b)^2}.\end{aligned}$$

3. $\sum_{b\in B} \big(p(b)\big)^2 - \sum_{a\in A} \big(p(a)\big)^2 = 0$. In this case, Equality (18) writes

$$\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a) = 0.$$

Case 1 is not possible because $B \succ C$. Suppose now that case 2 is true and consider any decision D such that $B \succ D$. Then, one has

$$\frac{\sum_{d\in D} p(d)u(d)}{\big(\sum_{d\in D} p(d)\big)^2} \leq \frac{\sum_{b\in B} p(b)u(b)}{\big(\sum_{b\in B} p(b)\big)^2} = \frac{\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)}{\big(\sum_{b\in B} p(b)\big)^2 - \big(\sum_{a\in A} p(a)\big)^2}$$

and, therefore, since $\sum_{b\in B} p(b)^2 - \sum_{a\in A} p(a)^2 < 0$,

$$\sum_{d\in D} p(d)u(d)[\sum_{b\in B} p(b)^2 - \sum_{a\in A} p(a)^2] \geq [\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)][\sum_{d\in D} p(d)^2]. \quad (20)$$

Notice that Inequality (20) holds also under Case 3. Hence it holds in all possible cases. Distributing the terms of Inequality (20) yields

$$\begin{aligned}&[\sum_{d\in D} p(d)u(d)][\sum_{b\in B} p(b)^2] + [\sum_{a\in A} p(a)u(a)][\sum_{d\in D} p(d)^2] \\ \geq\ &[\sum_{d\in D} p(d)u(d)][\sum_{a\in A} p(a)^2] + [\sum_{b\in B} p(b)u(b)][\sum_{d\in D} p(d)^2]\end{aligned}$$

or, using Equality (18),

$$
\begin{aligned}
& [\sum_{a\in A} p(a)u(a)][\sum_{b\in B} p(b)^2] + [\sum_{d\in D} p(d)u(d)][\sum_{b\in B} p(b)^2] + [\sum_{a\in A} p(a)u(a)][\sum_{d\in D} p(d)^2] \\
\geq \quad & \sum_{b\in B} p(b)u(b)[\sum_{a\in A} p(a)^2] + [\sum_{d\in D} p(d)u(d)][\sum_{a\in A} p(a)^2] + [\sum_{b\in B} p(b)u(b)][\sum_{d\in D} p(d)^2].
\end{aligned}
$$

Adding $[\sum_{d\in D} p(d)u(d)][\sum_{d\in D} p(d)^2]$ on both sides and factorizing yields

$$
\begin{aligned}
& [\sum_{a\in A} p(a)u(a) + \sum_{d\in D} p(d)u(d)][\sum_{b\in B} p(b)^2 + \sum_{d\in D} p(d)^2] \\
\geq \quad & [\sum_{b\in B} p(b)u(b) + \sum_{d\in D} p(d)u(d)][\sum_{a\in A} p(a)^2 + \sum_{d\in D} p(d)^2]
\end{aligned}
$$

$$\Longleftrightarrow$$

$$
\begin{aligned}
& \frac{\sum_{a\in A} p(a)u(a) + \sum_{d\in D} p(d)u(d)}{\sum_{a\in A} p(a)^2 + \sum_{d\in D} p(d)^2} \\
\geq \quad & \frac{\sum_{b\in B} p(b)u(b) + \sum_{d\in D} p(d)u(d)}{\sum_{b\in B} p(b)^2 + \sum_{d\in D} p(d)^2}.
\end{aligned}
$$

Hence, $A \cup D$ $B \cup D$ and this concludes the proof that satisfies Balancedness.

6.21 Example 3

We first show that the ordering defined in this example satisfies Richness. For this sake, consider five sets A, B, C, $\underline{D}$ and $\overline{D} \in \mathcal{P}(\mathbb{R}^2_+)$ such that $\overline{D} \sim C \succ B \succ A \sim \underline{D}$. Define $\overline{E} = \{(x,y)\}$, with $x = V(C)$. We have $\lim_{y\to 0} V(A \cup \overline{E}) = V(A)$ and $\lim_{y\to\infty} V(A \cup \overline{E}) = V(E)$. Since V is (Hausdorff) continuous and $V(A) < V(B) < V(\overline{E})$, there exists $y \in \mathbb{R}_+$ such that $V(A \cup \overline{E}) = V(B)$. If $(x,y) \notin A \cup C \cup \overline{D}$, then the set $\overline{E}$ has the characteristics required by the statement of the Richness condition. If $(x,y) \in A \cup C \cup \overline{D}$, then consider $\overline{E} = \{(x,y),(z,w)\}$, with $x = z = V(C)$. We have $\lim_{y,w\to 0} V(A \cup \overline{E}) = V(A)$ and $\lim_{y,w\to\infty} V(A \cup \overline{E}) = V(\overline{E})$. Since V is Hausdorff continuous and $V(A) < V(B) < V(\overline{E})$, there exist necessarily infinitely many pairs $(y,w) \in \mathbb{R}^2_+$ such that $V(A \cup \overline{E}) = V(B)$. Since $A \cup C \cup \overline{D}$ is finite, at least one of these pairs is such that $\overline{E} \cap (A \cup C \cup \overline{D}) = \emptyset$. A similar argument can be established for the existence of a set $\underline{E}$ that is as in the statement of the richness condition, which must therefore hold.

Let us now show that this ordering satisfies Averaging. Let A and B be two disjoint sets such that A B. One has therefore

$$V(A) \geq V(B)$$

$$\Longleftrightarrow$$

$$\frac{\sum_{(b_1,b_2)\in B} (b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B} \beta_2}) V(A)}{\sum_{(a_1,a_2)\in A} (a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A} \alpha_2}) + \sum_{(b_1,b_2)\in B} (b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B} \beta_2})}$$

$$
\begin{aligned}
&\geq \frac{\sum_{(b_1,b_2)\in B}(b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B}\beta_2})V(B)}{\sum_{(a_1,a_2)\in A}(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2}) + \sum_{(b_1,b_2)\in B}(b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B}\beta_2})} \\
&= \frac{\sum_{(b_1,b_2)\in B} b_1(b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B}\beta_2})}{\sum_{(a_1,a_2)\in A}(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2}) + \sum_{(b_1,b_2)\in B}(b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B}\beta_2})} \qquad (21)
\end{aligned}
$$

and (trivially)

$$
\begin{aligned}
V(A) \quad &\geq \quad V(A) \\
&\Longleftrightarrow \\
&\frac{\sum_{(a_1,a_2)\in A}(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2})V(A)}{\sum_{(a_1,a_2)\in A}(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2}) + \sum_{(b_1,b_2)\in B}(b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B}\beta_2})} \\
&\geq \frac{\sum_{(a_1,a_2)\in A}(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2})V(A)}{\sum_{(a_1,a_2)\in A}(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2}) + \sum_{(b_1,b_2)\in B}(b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B}\beta_2})} \\
&= \frac{\sum_{(a_1,a_2)\in A} a_1(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2})}{\sum_{(a_1,a_2)\in A}(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2}) + \sum_{(b_1,b_2)\in B}(b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B}\beta_2})}. \qquad (22)
\end{aligned}
$$

Summing inequalities (21) and (22) yields:

$$
\begin{aligned}
V(A) \quad &\geq \quad \frac{\sum_{(a_1,a_2)\in A} a_1(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2}) + \sum_{(b_1,b_2)\in B} b_1(b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B}\beta_2})}{\sum_{(a_1,a_2)\in A}(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2}) + \sum_{(b_1,b_2)\in B}(b_2 + \frac{b_2^2}{\sum_{(\beta_1,\beta_2)\in B}\beta_2})} \\
&= \quad V(A \cup B)
\end{aligned}
$$

as required by the first part of Averaging. The other part of the axiom can be obtained by an analogous reasoning.

Let us finally show that the ordering violates Balancedness. Consider for this sake the sets $A = \{(505, 16)\}$, $B = \{(10, 10), (1000, 10)\}$, $C = \{(504, 1)\}$ and $D = \{(1, 10)\}$. One has

$$
V(A) = \frac{\sum_{(a_1,a_2)\in A} a_1(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2})}{\sum_{(a_1,a_2)\in A}(a_2 + \frac{a_2^2}{\sum_{(\alpha_1,\alpha_2)\in A}\alpha_2})}
$$

$$
= 505 = \frac{10 \times (10 + \frac{100}{10+10}) + 1000 \times (10 + \frac{100}{10+10})}{(10 + \frac{100}{10+10}) + (10 + \frac{100}{10+10})}
$$

$$
\begin{aligned}
&= V(B) \\
&> V(C) = \frac{504 \times (1 + \frac{1}{1})}{1 + \frac{1}{1}} = 504 \\
&> V(D) = \frac{1 \times (10 + \frac{100}{10})}{10 + \frac{100}{10}} = 1
\end{aligned}
$$

One has also

$$V(A \cup C) = \frac{505 \times (16 + \frac{256}{17}) + 504 \times (1 + \frac{1}{17})}{(16 + \frac{256}{17}) + (1 + \frac{1}{17})} \simeq 504.967033$$

and

$$
\begin{aligned}
V(B \cup C) &= \frac{10 \times (10 + \frac{100}{10+10+1}) + 1000 \times (10 + \frac{100}{10+10+1}) + 504 \times (1 + \frac{1}{10+10+1})}{(10 + \frac{100}{10+10+1}) + (10 + \frac{100}{10+10+1}) + (1 + \frac{1}{10+10+1})} \\
&\simeq 504.965732.
\end{aligned}
$$

Hence, one has $A \cup C \quad B \cup C$. However, contrary to what Balancedness requires:

$$
\begin{aligned}
V(A \cup D) &= \frac{505 \times (16 + \frac{256}{26}) + 1 \times (1 + \frac{1}{26})}{(16 + \frac{256}{26}) + (1 + \frac{1}{26})} \\
&\simeq 485.53 \\
&< V(B \cup D) \\
&= \frac{10 \times (10 + \frac{100}{10+10+10}) + 1000 \times (10 + \frac{100}{10+10+10}) + 1 \times (1 + \frac{1}{10+10+10})}{(10 + \frac{100}{10+10+10}) + (10 + \frac{100}{10+10+10}) + (1 + \frac{1}{10+10+10})} \\
&\simeq 486.20.
\end{aligned}
$$

6.22 Proposition 1

Suppose that A, B and C are three finite and non-empty subsets of $\mathbb{R}$ such that $A \succ B \succ C$ or $C \succ B \succ A$. Richness implies the existence of a set D disjoint from A and C such that $A \sim D$ and $D \cup C \sim B$. For any set $E \in \mathcal{P}(X)$, define $U(E)$ by

$$U(E) = \frac{\sum_{e \in E} p(e)u(e)}{\sum_{e \in E} p(e)}.$$

Then

$$U(D) = \frac{\sum_{d \in D} p(d)u(d)}{\sum_{d \in D} p(d)} = U(A) \tag{23}$$

and

$$U(D \cup C) = \frac{\sum_{d \in D} p(d)u(d) + \sum_{c \in C} p(c)u(c)}{\sum_{d \in D} p(d) + \sum_{c \in C} p(c)} = U(B).$$

This last equation can be rewritten as

$$\sum_{d \in D} p(d)u(d) + \sum_{c \in C} p(c)u(c) = U(B) \left(\sum_{d \in D} p(d) + \sum_{c \in C} p(c) \right). \tag{24}$$

From (23), we obtain $\sum_{d\in D} p(d)u(d) = U(A)\sum_{d\in D} p(d)$. By definition of U, we also have $\sum_{c\in C} p(c)u(c) = U(C)\sum_{c\in C} p(c)$. If we replace in (24), we find

$$U(A)\sum_{d\in D} p(d) + U(C)\sum_{c\in C} p(c) = U(B)\left(\sum_{d\in D} p(d) + \sum_{c\in C} p(c)\right)$$

or

$$(U(A) - U(B))\left(\sum_{d\in D} p(d)\right) = (U(B) - U(C))\left(\sum_{c\in C} p(c)\right)$$

which amounts to

$$\frac{U(A) - U(B)}{U(B) - U(C)} = \frac{\sum_{c\in C} p(c)}{\sum_{d\in D} p(d)}. \tag{25}$$

Since this holds for any sets A, B and C, it holds in particular for $B = \{b\}$. Thanks to the continuity of u, we can choose b so that $U(B) = u(b)$ is between $U(C)$ and $U(A)$ and is as close as we want to $U(A)$ or $U(C)$. We can therefore make the ratio in the left-hand side of (25) as close to 0 or ∞ as we wish. Hence, for given A and C, Richness implies the existence of a set D with $\sum_{d\in D} p(d)$ arbitrary close to 0 or to ∞.

Suppose p is non-decreasing. If we want to make $\sum_{d\in D} p(d)$ arbitrarily close to 0, then $\max_{d\in D} p(d)$ must be arbitrarily close to 0. This implies that $\lim_{x\to\inf X} p(x) = 0$ and, hence, $\max_{d\in D} d$ must be arbitrarily close to $\inf X$ (with the convention that $\inf X = -\infty$ when X is unbounded from below).

- Suppose that u is non-decreasing. We choose A, B and C such that $A \succ C$. We then know that $U(A) > \inf_{x\in X} u(x)$. We have seen that we may choose D satisfying (23)-(24) with $\max_{d\in D} d$ arbitrarily close to $\inf_{x\in X} u(x)$. Since u is non-decreasing, we can thus manage to have $U(D)$ arbitrarily close to $\inf_{x\in X} u(x)$. We can therefore choose D in such a way that $U(A) > U(D)$, which contradicts (23).

- If u is non-increasing, then $U(D) > U(A)$ (if we have chosen $C \succ A$). This contradicts (23) and proves that u continuous and non-increasing is not compatible with p non-decreasing.

Suppose p is non-increasing. If we want to make $\sum_{d\in D} p(d)$ arbitrarily close to ∞, then $\min_{d\in D} p(d)$ must be arbitrarily large. This implies that $\lim_{x\to\sup X} p(x) = \infty$ and, therefore, $\min_{d\in D} d$ must be arbitrarily close to $\sup X$ (with the convention that $\sup X = +\infty$ when X is unbounded from above).

- If u is non-decreasing, then $U(D) > U(A)$ (if we have chosen $C \succ A$). This contradicts (23) and proves that u continuous and non-decreasing is not compatible with p non-increasing.

- If u is non-increasing, then $U(D) < U(A)$ (if we have chosen $A \succ C$). This contradicts (23) and proves that u continuous and non-increasing is not compatible with p non-increasing.

6.23 Proposition 2

Assume that is a continuous UEU ordering so that the p function of expression (1) is a constant function. Hence, for any two sets S and $S' \in \mathcal{P}(X)$, one has

$$S \quad S' \iff \sum_{s \in S} \frac{u(s)}{\#S} \geq \sum_{s' \in S'} \frac{u(s')}{\#S'}$$

for some continuous function u. For any set S, let $\text{UEU}(S) = \sum_{s \in S} \frac{u(s)}{\#S}$. Since is not trivial there are consequences a and $c \in X$ such that $u(a) > u(c)$. Let D be a set such that $D \sim \{a\}$. The set D can be a singleton ($D = \{d\}$ with $u(d) = u(a)$) or a set with several elements. If D is a singleton, then $\text{UEU}(D \cup C) = (u(a) + u(c))/2$. If D is not a singleton, then $\text{UEU}(D \cup C) > (u(a) + u(c))/2$. Hence, for all sets $D \sim \{a\}$, $\text{UEU}(D \cup C) \geq (u(a) + u(c))/2$. The continuity of u implies that, for any real number α between $u(a)$ and $u(c)$, there exists some $B = \{b\} \in \mathcal{P}(X)$ such that $u(b) = \alpha$. If α is chosen to be strictly smaller than $(u(a) + u(c))/2$, then $\text{UEU}(D \cup C) > \text{UEU}(B)$ and $\text{UEU}(D \cup C) \succ \text{UEU}(B)$, for any D with $D \sim \{a\}$. Hence, Richness does not hold.

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