Higher spin Dirac operators

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I. Introduction

In Clifford analysis, one of the most fundamental building blocks is the so called Dirac operator. This operator finds its origin in the Dirac equation [5], which describes the behaviour of the electron in the 4-dimensional spacetime. Our aim is to determine the generalized version of this operator for a general dimension m. We will call these operators 'higher spin Dirac operators'.

II. Method

To generalize the Dirac operator, we use some representation theory. First of all, it is well known that the Dirac operator $\underline{\partial}_x$ is a first order differential operator which maps $\mathbb{S}$-valued polynomials to the same space. Here, the spinor space $\mathbb{S}$ is an irreducible representation of the $Spin(m)$-group with highest weight $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$.
The main idea behind generalizing the Dirac operator, is to change the spinor space $\mathbb{S}$ with another irreducible representation of the $Spin(m)$-group. A first generalization has been made by Rarita and Schwinger [6]. Here, the irreducible representation with highest weight $\left(k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ was taken. It was determined that this representation was exactly the space of monogenic polynomials in 1 vector variable $\underline{u}$, homogeneous of degree k (see [4]). The explicit form of this operator then was (e.g. [3]) proved to be

$$\mathcal{R}_k = \left(1 + \frac{\underline{u}\,\underline{\partial}_u}{m + 2k - 2}\right)\underline{\partial}_x.$$

This principle can be generalized for general irreducible representations $\mathbb{V}_\lambda$ of the $Spin(m)$-group, where the highest weight $\lambda = \left(l_1 + \frac{1}{2}, \ldots, l_k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$. Part of the research is to find the explicit form of the first order differential operator

$$\mathcal{Q}_{l_1,\ldots,l_k} : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda).$$

The next step of the research will be to take a look at the null solutions of these operators and investigate the structure of their space.

III. Conclusion

In order to generalize classical Clifford analysis to higher spin Clifford analysis, representation theory can be used. The first step in the process is to find a Dirac-like operator. The null solutions of this operator are the solutions of the higher spin Dirac equation.

References

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