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ON SOLUTIONS OF THE HIGHER SPIN DIRAC OPERATORS OF ORDER TWO

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Abstract. *In this paper, we define twisted Rarita-Schwinger operators $\mathcal{R}_{l_1}^T$ and explain how these invariant differential operators can be used to determine polynomial null solutions of the higher spin Dirac operators $\mathcal{Q}_{l_1,l_2}$.*

1 INTRODUCTION

Classical Clifford analysis is usually defined as a function theory generalizing complex analysis to the case of arbitrary dimension $m \in \mathbb{N}$, where the role of the Cauchy–Riemann operator is played by an elliptic first order differential operator, see e.g. [4, 7, 9, 10]. One of these operators is the Dirac operator, being invariant with respect to the spin group. This operator is acting on spinor-valued functions, see e.g. [1, 6, 8].

Clifford analysis also offers an elegant framework to study function theoretical problems not only for the classical Dirac operator, but also for generalizations of it, acting on functions which take their values in arbitrary half-integer irreducible spin-representations, higher spin Dirac operators (HSD operators for short).

In a previous paper [11], we have already established their explicit definition. A special case is the operator $\mathcal{Q}_{l_1,l_2}$, studied in depth in e.g. [2, 3], which acts on polynomials taking values in irreducible Spin(m)-representations with highest weight $\left(l_1 + \frac{1}{2}, l_2 + \frac{1}{2}, \frac{1}{2} \cdots, \frac{1}{2}\right)$. We will introduce an alternative method to determine the polynomial null solutions for these operators, which can then be translated to the most general case.

The outline of this paper is as follows. In section 2, we will give some general Clifford analysis background in order to define the higher spin Dirac operators $\mathcal{Q}_{l_1,l_2}$ and the twisted Rarita-Schwinger operators $\mathcal{R}^T_{l_1}$. In section 3, we will then determine the structure of the polynomial null solutions of $\mathcal{Q}_{l_1,l_2}$.

2 CLIFFORD ANALYSIS BACKGROUND

Let $\mathbb{R}_m$ be the Clifford algebra generated by an orthonormal basis $(\underline{e}_1, \ldots, \underline{e}_m)$ for the m-dimensional vector space $\mathbb{R}^m$ and let $\mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C}$ be its complexification. The multiplication is governed by the relations

$$\underline{e}_i\underline{e}_j + \underline{e}_j\underline{e}_i = -2\delta_{ij}, \quad \text{for all } i, j = 1, \ldots, m$$

If $m = 2n + 1$ is odd, we denote the unique spinor space, which can be realized as a minimal left ideal inside $\mathbb{C}_m$, by $\mathbb{S}$. In case $m = 2n$, we have $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ with $\mathbb{S}^\pm$ the space of positive (resp. negative) spinors. Unless explicitly stated otherwise, we will disregard the parity of the spinors in even dimension, and we will speak, with a slight abuse of language, about 'the' spinor space.

The Dirac operator, acting on $\mathbb{S}$-valued polynomials $f(\underline{x})$, is defined as $\underline{\partial}_x = \sum_{j=1}^m \underline{e}_j \partial_{x_j}$.

For odd dimensions, the vector space $\mathbb{S}$ defines the basic half-integer representation for the spin group Spin(m), described by the highest weight $(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$. In even dimensions, the spinor space $\mathbb{S}$ is reducible, with the spaces $\mathbb{S}^\pm$ both irreducible representations of Spin(m) with highest weights $(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$. It is crucial to mention that in the language of Clifford analysis, other irreducible half-integer Spin(m)-representations can be characterized as spaces of polynomials, see e.g. [5]. This is done using several vector variables $\underline{u}_i \in \mathbb{R}^m$. In particular, these irreducible modules will be modelled in terms of the following spaces, where we will denote the Dirac operators $\underline{\partial}_{u_i}$ by $\underline{\partial}_i$.

These spaces will be characterized as particular classes of polynomials, which are introduced in the following definitions.

Definition 1. *A function $f : \mathbb{R}^{km} \to \mathbb{S} : (\underline{u}_1, \ldots, \underline{u}_k) \mapsto f(\underline{u}_1, \ldots, \underline{u}_k)$ is called simplicial monogenic if it satisfies the system*

$$\begin{aligned} \underline{\partial}_i f &= 0, \text{ for all } i = 1, \ldots, k \\ \langle \underline{u}_i, \underline{\partial}_j \rangle f &= 0, \text{ for all } 1 \leq i < j \leq k \end{aligned}$$

The vector space of $\mathbb{S}$-valued simplicial monogenic polynomials which are l_i-homogeneous in $\underline{u}_i$ will be denoted by $\mathcal{S}_{l_1,\ldots,l_k}$, where we assume that $l_1 \geq \cdots \geq l_k$ (dominant weight condition). The following definition involves weaker conditions on the $\mathbb{S}$-valued functions, but will nevertheless be crucial in what follows.

Definition 2. *A function $f : \mathbb{R}^{km} \to \mathbb{S} : (\underline{u}_1, \ldots, \underline{u}_k) \mapsto f(\underline{u}_1, \ldots, \underline{u}_k)$ is called monogenic if it satisfies $\underline{\partial}_i f = 0$, for all $1 \leq i \leq k$.*

The vector space of $\mathbb{S}$-valued monogenic polynomials which are l_i-homogeneous in $\underline{u}_i$ will be denoted by $\mathcal{M}_{l_1,\ldots,l_k}$, again with $l_1 \geq \cdots \geq l_k$. Each of these polynomial vector spaces can be seen as a module for the spin group, under the regular representation (or so-called L-representation) given by

$$L(s)P(\underline{x}_1, \ldots, \underline{x}_k) := sP(\overline{s}\underline{x}_1 s, \ldots, \overline{s}\underline{x}_k s), \quad s \in \text{Spin}(m)$$

In e.g. [5], it was proven that under this action, the Spin(m)-modules $\mathcal{S}_{l_1,\cdots,l_k}$ define a model for the irreducible highest weight representation characterized by means of

$$\mathcal{S}_{l_1,\cdots,l_k} \to \left(l_1 + \frac{1}{2}, \cdots, l_k + \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right) =: (l_1, \cdots, l_k)'.$$

Note that when $m = 2n$, one should also add a parity index to the spaces of simplicial monogenics, according to the one for spinors.

On functions taking values in the space $\mathcal{S}_{\underline{\lambda}}$, one can define operators playing the role of $\underline{\partial}_x$ for $\mathbb{S}$-valued functions. These operators are precisely the HSD operators. These are first order Spin(m)-invariant differential operators $\mathcal{Q}_{\underline{\lambda}}$, where $\underline{\lambda} = (l_1, \ldots, l_k)$, which are uniquely defined up to a multiplicative constant:

$$\mathcal{Q}_{\underline{\lambda}} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\underline{\lambda}}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\underline{\lambda}}) : f(\underline{x}; \underline{u}_1, \ldots, \underline{u}_k) \mapsto \mathcal{Q}_{\underline{\lambda}} f(\underline{x}; \underline{u}_1, \ldots, \underline{u}_k)$$

The existence and uniqueness of these invariant differential operators follows from Fegan's result [7]; their explicit form was determined in [11] as

$$\mathcal{Q}_{\underline{\lambda}} = \left(\prod_{i=1}^{k} \left(1 + \frac{\underline{u}_i \underline{\partial}_i}{m + 2l_i - 2i}\right)\right) \underline{\partial}_x$$

Let us now define the twisted Rarita-Schwinger operator as follows:

Definition 3.

$$\mathcal{R}^T_{l_1} = \mathbf{1} \otimes \mathcal{R}_{l_1} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_{l_1} \otimes \mathcal{H}_{l_2}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_{l_1} \otimes \mathcal{H}_{l_2}), \tag{1}$$

where $\mathcal{H}_{l_i}$ is the space of harmonic polynomials in $\underline{u}_i$, homogeneous of degree l_i.

Remark 1. *We call this operator twisted, as it is acting on spaces with the 'wrong' values.*

In [11] the following result has been proven.

Proposition 1. *For each $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$, it holds that*

$$\mathcal{R}_{l_1}^T f = \mathcal{Q}_{l_1,l_2} f + \pi_{l_1}[\underline{u}_2]\langle \underline{\partial}_2, \underline{\partial}_x \rangle f \tag{2}$$

where $\pi_{l_1}[\underline{u}_2]$ is an embedding operator, mapping $\mathcal{S}_{l_1,l_2-1}$-valued polynomials to $\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1}$-valued ones.

We will use this last result to determine the polynomial null solutions of the operator $\mathcal{Q}_{l_1,l_2}$ in the next section.

3 POLYNOMIAL SOLUTIONS IN THE CASE OF ORDER TWO

In view of the fact that $\mathcal{S}_{l_1,l_2} \subset \mathcal{M}_{l_1} \otimes \mathcal{H}_{l_2}$, the following holds:

$$\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}) \xrightarrow{\mathcal{R}_{l_1}^T} \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{l_2} \otimes \mathcal{M}_{l_1})$$

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}) \arrow[r, "\mathcal{Q}_{l_1,l_2}"] \arrow[dr, "\langle \underline{\partial}_2, \underline{\partial}_x \rangle"'] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}) \\
 & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2-1})
\end{tikzcd}$$

We may then define two types of solutions. The type I solutions will be the solutions both in the kernel of $\mathcal{R}_{l_1}^T$ and $\langle \underline{\partial}_2, \underline{\partial}_x \rangle$. The type II solutions are the remaining polynomial null solutions of $\mathcal{Q}_{l_1,l_2}$ which aren't of type I.

Lemma 1. *For any $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$ with $\mathcal{R}_{l_1}^T f = 0$, one has that $\langle \underline{\partial}_2, \underline{\partial}_x \rangle f = 0$.*

Proof. Suppose $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$ and $\mathcal{R}_{l_1}^T f = 0$. As $\langle \underline{u}_1, \underline{\partial}_2 \rangle f = 0$, one also has that $\underline{\partial}_2 \mathcal{R}_{l_1}^T f = 0$. In view of the fact that

$$\underline{\partial}_2 \mathcal{R}_{l_1}^T f = \left(-2 - \frac{4}{m + 2l_1 - 2}\right) \langle \underline{\partial}_2, \underline{\partial}_x \rangle f$$

where the constant is always different from 0, this proves the lemma. □

From this lemma, it follows that the type I solutions of $\mathcal{Q}_{l_1,l_2}$ may be recharacterized as follows: $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$ is of type I if $f \in \mathcal{R}_{l_1}^T$.

Next, it is our goal to reveal more of the structure of the set of these type I solutions. Therefore, we need the following two lemmata. From now on, we will denote the degree of homogeneity in the variable $\underline{x}$ by h. From now on, when acting on the function space $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$, we will use the notation $\cdot|_{\mathcal{S}_{l_1,l_2}}$.

Lemma 2. *For each f belonging to $\ker \mathcal{R}_{l_1}^T|_{\mathcal{S}_{l_1,l_2}}$, one has that $\langle \underline{\partial}_1, \underline{\partial}_x \rangle f$ belongs to $\ker \mathcal{R}_{l_1-1}^T|_{\mathcal{S}_{l_1-1,l_2}}$.*

Proof. Suppose that $f \in \ker \mathcal{R}^T_{l_1}|_{\mathcal{S}_{l_1,l_2}}$. We can verify that $\langle \underline{\partial}_1, \underline{\partial}_x\rangle f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1-1,l_2})$:

$$\underline{\partial}_1 \langle \underline{\partial}_1, \underline{\partial}_x\rangle f = \underline{\partial}_2 \langle \underline{\partial}_1, \underline{\partial}_x\rangle f = 0$$

since both sets of operators commute, and

$$\langle \underline{u}_1, \underline{\partial}_2\rangle \langle \underline{\partial}_1, \underline{\partial}_x\rangle f = -\langle \underline{\partial}_2, \underline{\partial}_x\rangle f = 0$$

due to Lemma 1. We then get that

$$\begin{aligned}
\mathcal{R}^T_{l_1-1} \langle \underline{\partial}_1, \underline{\partial}_x\rangle f &= \left(1 + \frac{\underline{u}_1 \underline{\partial}_1}{m+2l_1-4}\right) \underline{\partial}_x \langle \underline{\partial}_1, \underline{\partial}_x\rangle f \\
&= \langle \underline{\partial}_1, \underline{\partial}_x\rangle \frac{m+2l_1-2}{m+2l_1-4} \left(1 + \frac{\underline{u}_1 \underline{\partial}_1}{m+2l_1-4}\right) \underline{\partial}_x f = 0
\end{aligned}$$

which proves the lemma. □

Lemma 3. *For each f belonging to $\ker_h \mathcal{R}^T_{l_1}|_{\mathcal{S}_{l_1,l_2}}$, one has that*

$$\langle \underline{\partial}_1, \underline{\partial}_x\rangle^{l_1-l_2+1} f = 0$$

Proof. Suppose that $f \in \ker_h \mathcal{R}^T_{l_1}|_{\mathcal{S}_{l_1,l_2}}$. Lemma 2 tells us that $\langle \underline{\partial}_1, \underline{\partial}_x\rangle f \in \ker_{h-1} \mathcal{R}^T_{l_1-1}|_{\mathcal{S}_{l_1-1,l_2}}$. Recalling that $l_1 \geq l_2$, we thus get

$$\langle \underline{\partial}_1, \underline{\partial}_x\rangle^{l_1-l_2} f \in \ker \mathcal{R}^T_{l_2}|_{\mathcal{S}_{l_2,l_2}}.$$

Since $\langle \underline{\partial}_2, \underline{\partial}_x\rangle \langle \underline{\partial}_1, \underline{\partial}_x\rangle^{l_1-l_2} f = 0$ in view of Lemma 1, we also have that

$$\begin{aligned}
0 = \langle \underline{u}_2, \underline{\partial}_1\rangle \langle \underline{\partial}_2, \underline{\partial}_x\rangle \langle \underline{\partial}_1, \underline{\partial}_x\rangle^{l_1-l_2} f &= (-\langle \underline{\partial}_1, \underline{\partial}_x\rangle + \langle \underline{\partial}_2, \underline{\partial}_x\rangle \langle \underline{u}_2, \underline{\partial}_1\rangle) \langle \underline{\partial}_1, \underline{\partial}_x\rangle^{l_1-l_2} f \\
&= \langle \underline{\partial}_1, \underline{\partial}_x\rangle^{l_1-l_2+1} f
\end{aligned}$$

which proves the lemma. □

From Lemma 2, we get that

$$\ker_h \mathcal{R}^T_{l_1}|_{\mathcal{S}_{l_1,l_2}} \supseteq \left(\ker_h \mathcal{R}^T_{l_1} \cap \ker_h \langle \underline{\partial}_1, \underline{\partial}_x\rangle\right)|_{\mathcal{S}_{l_1,l_2}} \oplus \langle \underline{\partial}_1, \underline{\partial}_x\rangle^{-1} \ker_{h-1} \mathcal{R}^T_{l_1}|_{\mathcal{S}_{l_1-1,l_2}}$$

where we use the notation $\langle \underline{\partial}_1, \underline{\partial}_x\rangle^{-1}$ for the (not necessarily surjective) inverse operator of $\langle \underline{\partial}_1, \underline{\partial}_x\rangle$. Note that

$$\ker_h \mathcal{Q}^T_{l_1} \cap \ker \langle \underline{\partial}_1, \underline{\partial}_x\rangle|_{\mathcal{S}_{l_1,l_2}} = \{f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}) : \underline{\partial}_x f = 0\} = \mathcal{M}_{h,l_1,l_2} \cap \ker \langle \underline{u}_1, \underline{\partial}_2\rangle$$

We will denote the latter space by $\mathcal{M}^s_{h,l_1,l_2}$. Using Lemma 2 inductively, we get that

$$\ker_h \mathcal{R}^T_{l_1} \supseteq \bigoplus_{j=0}^{l_1-l_2} \left(\langle \underline{\partial}_1, \underline{\partial}_x\rangle^{-1}\right)^j \mathcal{M}^s_{h-j,l_1-j,l_2}$$

where the sum stops at $l_1 - l_2$, because of Lemma 3.

Finally, we have that

Lemma 4. *For each $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$, one has:*

$$f \in \ker \mathcal{Q}_{l_1,l_2} \Rightarrow \langle \underline{\partial}_2, \underline{\partial}_x \rangle f \in \ker \mathcal{Q}_{l_1,l_2-1}$$

Proof. Suppose $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$ and $f \in \ker \mathcal{Q}_{l_1,l_2}$. Then, direct calculations yield that

$$\mathcal{Q}_{l_1,l_2-1} \langle \underline{\partial}_2, \underline{\partial}_x \rangle f = \frac{m + 2l_2 - 4}{m + 2l_2 - 6} \langle \underline{\partial}_2, \underline{\partial}_x \rangle \mathcal{Q}_{l_1,l_2} f$$

which proves the lemma. □

Lemma 5. *Suppose $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$ and $f \in \ker \mathcal{Q}_{l_1,l_2}$. One then has that*

$$f \in ker \langle \underline{\partial}_2, \underline{\partial}_x \rangle^{j+1} \setminus \ker \langle \underline{\partial}_2, \underline{\partial}_x \rangle^{j} \Rightarrow \langle \underline{\partial}_1, \underline{\partial}_x \rangle^{l_1 - l_2 + 1} \langle \underline{\partial}_2, \underline{\partial}_x \rangle^{j} f = 0$$

Proof. Define the twistor operator $\mathcal{T}^{l_1-1,l_2}_{l_1,l_2}$ as follows:

$$\mathcal{T}^{l_1-1,l_2}_{l_1,l_2} := \langle \underline{\partial}_1, \underline{\partial}_x \rangle + \frac{\langle \underline{u}_2, \underline{\partial}_1 \rangle \langle \underline{\partial}_2, \underline{\partial}_x \rangle}{l_1 - l_2 + 2}$$

It has been shown in e.g. [3] that $\mathcal{T}^{l_1-1,l_2}_{l_1,l_2} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1-1,l_2})$. We thus get, for each $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}) \cap \ker \mathcal{Q}_{l_1,l_2}$, that

$$T_{l_1,l_2} f := \mathcal{T}^{l_2,l_2}_{l_2+1,l_2} \ldots \mathcal{T}^{l_1-2,l_2}_{l_1-1,l_2} \mathcal{T}^{l_1-1,l_2}_{l_1,l_2} f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_2,l_2})$$

Therefore, we have that

$$\langle \underline{u}_2, \underline{\partial}_1 \rangle T_{l_1,l_2} f = 0 \tag{3}$$

It can be shown by means of direct calculations that for all $g \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{a,b})$,

$$\langle \underline{\partial}_2, \underline{\partial}_x \rangle \mathcal{T}^{a-1,b}_{a,b} g = c \mathcal{T}^{a-1,b-1}_{a,b-1} \langle \underline{\partial}_2, \underline{\partial}_x \rangle g$$

where $c = \frac{l_1 - l_2 + 2}{l_1 - l_2 + 1}$. Since $\langle \underline{\partial}_2, \underline{\partial}_x \rangle^{j+1} f = 0$, we then have that

$$\langle \underline{\partial}_2, \underline{\partial}_x \rangle^j T_{l_1,l_2} f = C\, T_{l_1,l_2-j} \langle \underline{\partial}_2, \underline{\partial}_x \rangle^j f = C\, \langle \underline{\partial}_1, \underline{\partial}_x \rangle^{l_1 - l_2} \langle \underline{\partial}_2, \underline{\partial}_x \rangle^j f \tag{4}$$

where the constant C is not further explicited. On the other hand, we have that

$$\langle \underline{\partial}_2, \underline{\partial}_x \rangle^{j+1} T_{l_1,l_2} f = 0$$

whence also $\langle \underline{u}_2, \underline{\partial}_1 \rangle \langle \underline{\partial}_2, \underline{\partial}_x \rangle^{j+1} T_{l_1,l_2} f = 0$. Now,

$$\langle \underline{u}_2, \underline{\partial}_1 \rangle \langle \underline{\partial}_2, \underline{\partial}_x \rangle^{j+1} T_{l_1,l_2} f = (-(j+1) \langle \underline{\partial}_1, \underline{\partial}_x \rangle \langle \underline{\partial}_2, \underline{\partial}_x \rangle^j + \langle \underline{\partial}_2, \underline{\partial}_x \rangle^{j+1} \langle \underline{u}_2, \underline{\partial}_1 \rangle) T_{l_1,l_2} f$$

Due to (3) and (4), we thus have that

$$\langle \underline{\partial}_1, \underline{\partial}_x \rangle^{l_1 - l_2 + 1} \langle \underline{\partial}_2, \underline{\partial}_x \rangle^j f = 0$$

which completes the proof. □

We define the following grading on $\ker_h \mathcal{Q}_{l_1,l_2}$. If we set

$$\ker_h^a \mathcal{Q}_{l_1,l_2} := \ker_h(\mathcal{Q}_{l_1,l_2}, \langle \underline{\partial}_2, \underline{\partial}_x \rangle^a) \setminus \ker_h(\mathcal{Q}_{l_1,l_2}, \langle \underline{\partial}_2, \underline{\partial}_x \rangle^{a-1})$$

we get that

$$\ker_h \mathcal{Q}_{l_1,l_2} = \bigoplus_{j=0}^{l_2} \ker_h^j \mathcal{Q}_{l_1,l_2}$$

On account of Lemma 5, we then obtain, omitting the embedding factors,

$$\ker_h \mathcal{Q}_{l_1,l_2} \supseteq \bigoplus_{i=0}^{l_1-l_2} \bigoplus_{j=0}^{l_2} \mathcal{M}^s_{h-i-j,l_1-i,l_2-j}$$

Dimensional analysis done in [3] then reveals that the above inclusion in fact is an equality.

REFERENCES

[1] Brackx, F., Delanghe, R., Sommen, F., *Clifford Analysis*, Research Notes in Mathematics **76**, Pitman, London, 1982.

[2] Brackx, F., Eelbode, D., Van de Voorde, L., *Invariant operators between spaces of simplicial monogenics*, submitted.

[3] Brackx, F., Eelbode, D., Van de Voorde, L., *Polynomial solutions for higher spin Dirac operators in two vector variables*, submitted.

[4] Branson, T., *Stein-Weiss operators and ellipticity*, J. Funct. Anal. **151** No. 2 (1997), pp. 334-383.

[5] Constales, D., Sommen. F., Van Lancker, P., *Models for irreducible representations of Spin*(m), Adv. Appl. Clifford Algebras **11** No. S1 (2001), pp. 271-289.

[6] Delanghe, R., Sommen, F., Souček, V., *Clifford analysis and spinor valued functions*, Kluwer Academic Publishers, Dordrecht, 1992.

[7] Fegan, H. D., *Conformally invariant first order differential operators*, Quart. J. Math. **27** (1976), pp. 513-538.

[8] Gilbert, J., Murray, M.A.M., *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge University Press, Cambridge, 1991.

[9] Slovak, J., *Natural operators on conformal manifolds*, Masaryk University Dissertation (Brno, 1993)

[10] Stein, E.W. , Weiss, G., *Generalization of the Cauchy-Riemann equations and representations of the rotation group*, Amer. J. Math. **90** (1968), pp. 163-196.

[11] De Schepper, H., Eelbode, D., Raeymaekers, T., *On a special type of solutions for arbitrary higher spin Dirac Operators*, J. Phys. A: Math. Theor. **43** (2010) 325208 (13pp).