Department of Telecommunications and Information Processing (TELIN),
Group for Artificial Intelligence and Sparse Modeling (GAIM),
Faculty of Engineering and Architecture,
Ghent University, Belgium

LEARNING IN REPRODUCING KERNEL HILBERT SPACES BY USING FRAME THEORY TECHNIQUES

Supervisors: Aleksandra Pižurica and Ivana Slamić
Guidance: Srđan Lazendić
Student: Tomislav Bukić

Index terms: Frames, Reproducing Kernels, Hilbert Spaces, Learning, Sparse Coding

Problem formulation. Although the reproducing kernels were introduced at the beginning of the 19th century, a huge breakthrough has been done by Aronszajn who systematically organized the theory of reproducing kernels in his paper [1] in the early 1950s. Since then, the theory of reproducing kernels evolved, finding applications in many disciplines. More recently, theory of reproducing kernels gained huge popularity in Machine Learning in the context of *Support Vector Machines* [2,11]. This powerful tool allows one to treat a wide class of different nonlinearities in a unifying way. In particular, the idea is in obtaining the feature map that allows us to solve an equivalent linear task in a space different than the one where the original measurements lie (Figure 1).

In linear algebra, a frame theory exists already for a very long time, but was brought to a broader audience by Daubechies, Grossmann and Meyer in their seminal paper [4] in 1986. Frames represent generalization of orthonormal basis. The linear independence property for a basis, which allows every vector to be uniquely represented as a linear combination, is very restrictive for practical problems. A frame on the other hand allows each element in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required. This means that there exists more than one representation of a given signal and this fact becomes important in signal and image processing, coding theory, sampling theory etc. In the last decade the frame theory found a huge application in image processing, although under the name of *dictionary learning*.

On one hand we have reproducing kernel theory and on the other hand frame theory, each spectacular in its own way. As shown in [3, 7, 8] the two can be combined. A recent trend in signal processing and machine learning attempts to build an improved version of reproducing kernels by introducing the frame theory techniques into kernel construction. More precisely, it has been shown that there exist methods for building a reproducing kernel Hilbert space (RKHS) from a Hilbert space with frame elements having special properties [8]. The potentials of the frame-based kernels are yet to be explored, starting from the first encouraging results on toy and real-world problems.

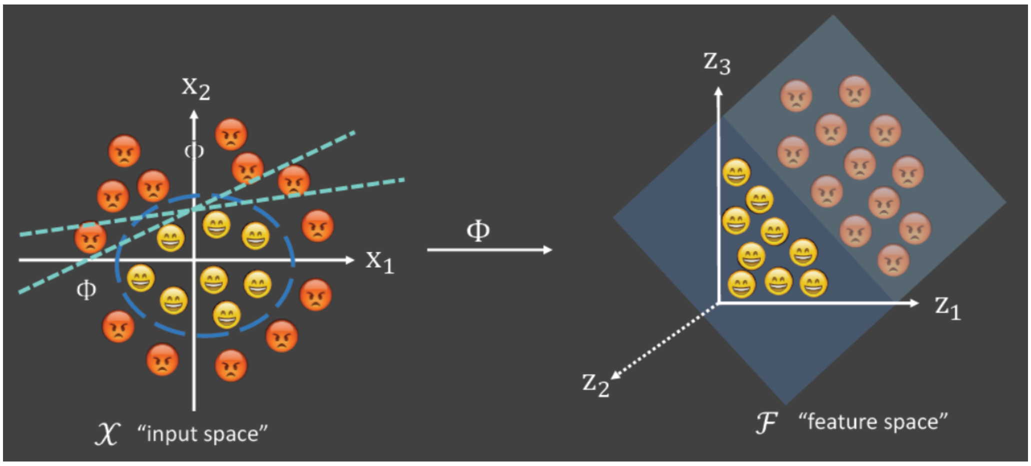

FIGURE 1. Mapping a nonlinear to a linear task

Goal of the thesis. The first goal of this thesis is to study and understand well the construction of both the reproducing kernel Hilbert spaces and the frame/dictionary learning techniques [5, 6, 9, 11]. A recent application of frame based reproducing kernels showed compelling results in efficiency over the classical kernel based theory [8, 10]. Starting from the already existing literature and developed algorithms [7–10], the main goal of the thesis is to further investigate the potential of frame-based kernel techniques. This thesis should combine most popular and emerging machine learning algorithms with the solid mathematical theory of frames and functional analysis in order to build a sound theoretical framework with concrete practical applications. Practical applications should be demonstrated for the concrete image processing problems instead of just artificially and toy made examples. Those will be chosen in the agreement with the student.

REFERENCES

[1] ARONSZAJN, N. Theory of reproducing kernels. *Transactions of the American mathematical society 68*, 3 (1950), 337–404.

[2] BOSER, B. E., GUYON, I. M., AND VAPNIK, V. N. A training algorithm for optimal margin classifiers. In *Proceedings of the fifth annual workshop on Computational learning theory* (1992), ACM, pp. 144–152.

[3] DAS, S. Frames and reproducing kernels in a hilbert space. Master's thesis, Concordia University, 2000.

[4] DAUBECHIES, I., GROSSMANN, A., AND MEYER, Y. Painless nonorthogonal expansions. *Journal of Mathematical Physics 27*, 5 (1986), 1271–1283.

[5] GAVRUTA, L. Atomic decompositions for operators in reproducing kernel Hilbert spaces. *arXiv preprint arXiv:1401.8150* (2014).

[6] GOLTS, A., AND ELAD, M. Linearized kernel dictionary learning. *IEEE Journal of Selected Topics in Signal Processing 10*, 4 (2016), 726–739.

[7] JORGENSEN, P., AND SONG, M.-S. Reproducing kernel Hilbert space vs. frame estimates. *Mathematics 3*, 3 (2015), 615–625.

[8] RAKOTOMAMONJY, A., AND CANU, S. Frames, reproducing kernels, regularization and learning. *Journal of Machine Learning Research 6*, Sep (2005), 1485–1515.

[9] SKRETTING, K. *Sparse signal representation using overlapping frames.* PhD thesis, Norwegian University of Science and Technology (NTNU), Trondheim, Norway, 2002.

[10] SLAVAKIS, K., BOUBOULIS, P., AND THEODORIDIS, S. Online learning in reproducing kernel Hilbert spaces. In *Academic Press Library in Signal Processing*, vol. 1. Elsevier, 2014, pp. 883–987.

[11] VINCENT, P., AND BENGIO, Y. Kernel matching pursuit. *Machine learning 48*, 1-3 (2002), 165–187.