GHENT UNIVERSITY

FACULTY OF ENGINEERING AND ARCHITECTURE

Hypercomplex Algebras for Dictionary Learning

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Image Processing and Interpretation Group

AGACSE 2018
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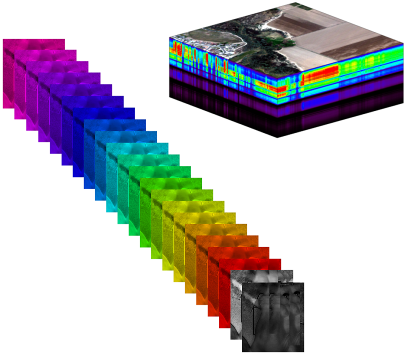
Campinas, Brazil

Outline

Outline

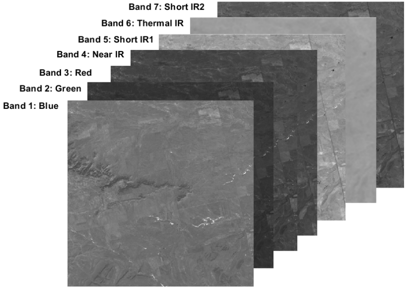
Highdimensional data



Remote-sensing data
(hyperspectral, multispectral, visible. . .)

Multimodal images
(infrared, X-ray, visible . . .)

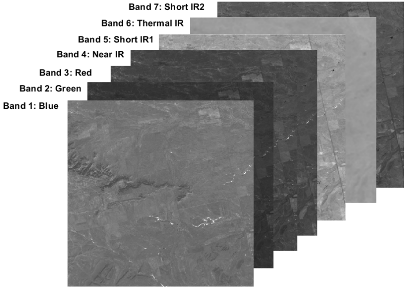
Motivation



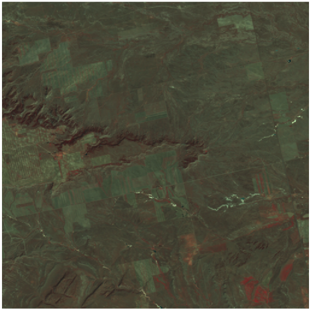
LANDSAT 7 multispectral image

Contrast adjusted RGB image

Motivation



LANDSAT 7 multispectral image



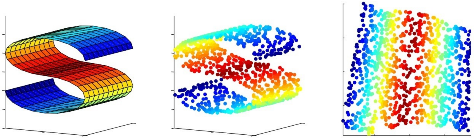
Near infrared image

Outline

Outline

Sparse representation

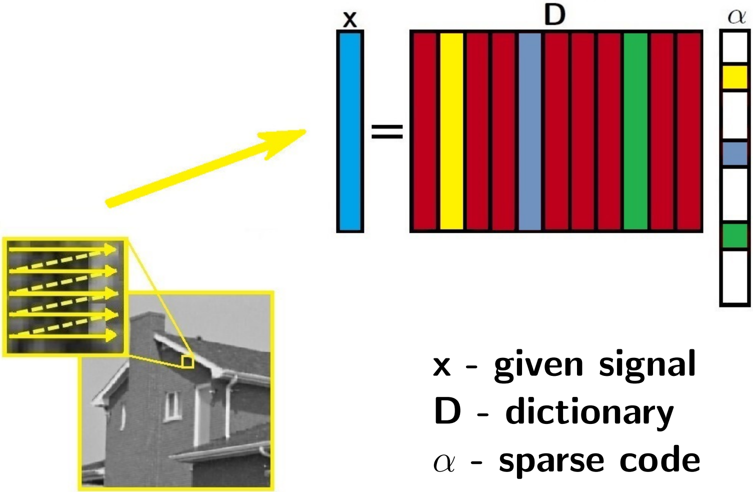
- High-dimensional data are often not truly high-dimensional
  - Points of high-dimensional data usually reside on a much low-dimensional manifold (manifold learning)



L.K. Saul, S.T. Roweis,
Think Globally, Fit Locally: Unsupervised Learning of Low Dimensional Manifolds,
Journal of Machine Learning Research, v4, pp. 119-155, 2003.

- High-dimensional data inherently has a sparse representation with respect to some basis (usually called dictionary)

Sparse representation



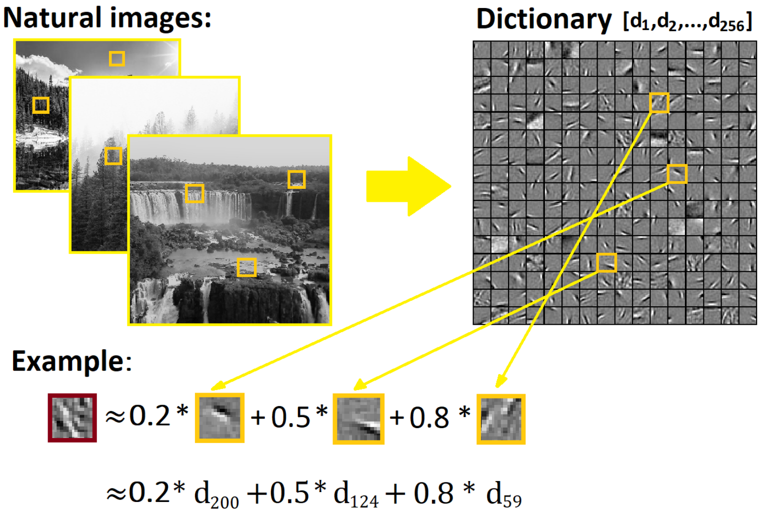
x - given signal

D - dictionary

α - sparse code

A. Pižurica, *Sparse Coding and Multimodal Dictionary Learning in Computer Vision*, Plenary Lecture. Mathematics for Big Data Workshop, ECMI, Novi Sad, Serbia, 2017. https://telin.ugent.be/~sanja/Presentation/MBD2017_final.pdf

Sparse representation

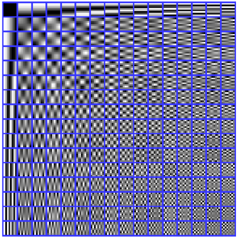


Y. LeCun, Y. Bengio, and G. Hinton, *Deep Learning*,
Nature, 521(7553), p. 436, 2015.

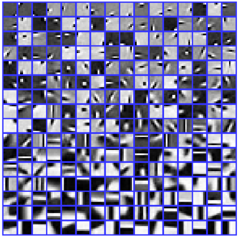
The quest for a dictionary

- **Predetermined dictionaries:** wavelets, curvelets, shearlets . . .
  - lead to sparse representation of signals and images
  - simple and fast algorithms for sparse representation

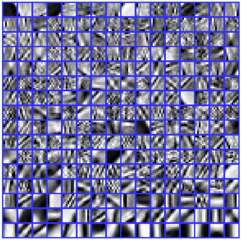
- **Learned dictionaries:**
  - trained on a set of representative examples
  - outperform the use of predetermined dictionaries
  - goal: optimally sparse representation for a given class of signals



Discrete cosine transform dictionary

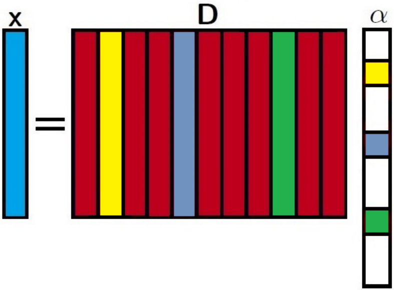


Dictionary based on natural images



Dictionary trained on a noisy image

Sparse coding: ℓ_0-minimization



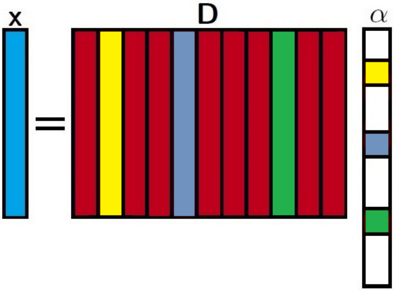
$$\hat{\alpha} = \arg\min_{\alpha} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2 \qquad \text{s.t.} \qquad \|\alpha\|_0 \leq K$$

$$\hat{\alpha} = \arg\min_{\alpha} \|\alpha\|_0 \qquad \text{s.t.} \qquad \|\mathbf{x} - \mathbf{D}\alpha\|_2 \leq \varepsilon$$

Greedy algorithms:

- Matching Pursuit (MP) [Mallat and Zhang, '93]
- Orthogonal Matching Pursuit (OMP) [Tropp, '04]

Convex relaxation: ℓ_1-minimization



$$\hat{\alpha} = \arg\min_{\alpha} \|\alpha\|_1 \qquad \text{s.t.} \qquad \|\mathbf{x} - \mathbf{D}\alpha\|_2 \leq \varepsilon$$

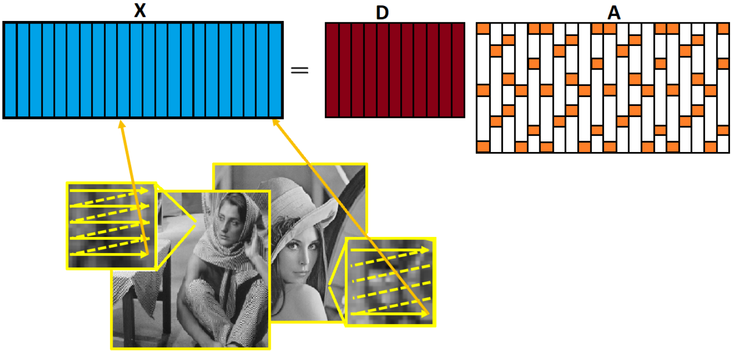
$$\hat{\alpha} = \arg\min_{\alpha} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2 + \lambda \|\alpha\|_1$$

Convex relaxation techniques:

- LASSO [Tibshirani, '96]
- Basis Pursuit Denoising (BPDN) [Chen et al, '01]

Outline

(Unsupervised) dictionary learning

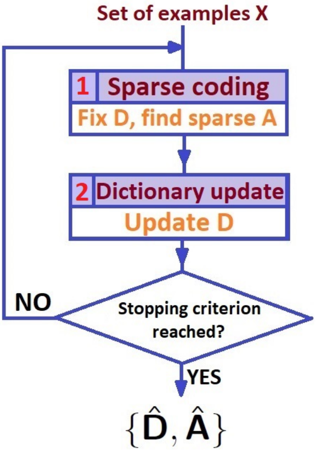


$$\{\hat{\mathbf{D}}, \hat{\mathbf{A}}\} = \underset{\mathbf{D},\mathbf{A}}{\arg\min}\{\|\mathbf{X} - \mathbf{DA}\|_F^2\} \qquad \text{s.t.} \qquad \forall i: \quad \|\alpha_i\|_0 \leq K$$

$$\{\hat{\mathbf{D}}, \hat{\mathbf{A}}\} = \underset{\mathbf{D},\mathbf{A}}{\arg\min} \sum_i \|\alpha_i\|_0 \qquad \text{s.t.} \qquad \|\mathbf{X} - \mathbf{DA}\|_F \leq \varepsilon$$

A. Pižurica, *Sparse Coding and Multimodal Dictionary Learning in Computer Vision*, Plenary Lecture. Mathematics for Big Data Workshop, ECMI, Novi Sad, Serbia, 2017.
https://telin.ugent.be/~sanja/Presentation/MBD2017_final.pdf

Sparse coding and dictionary update



Sparse coding step:

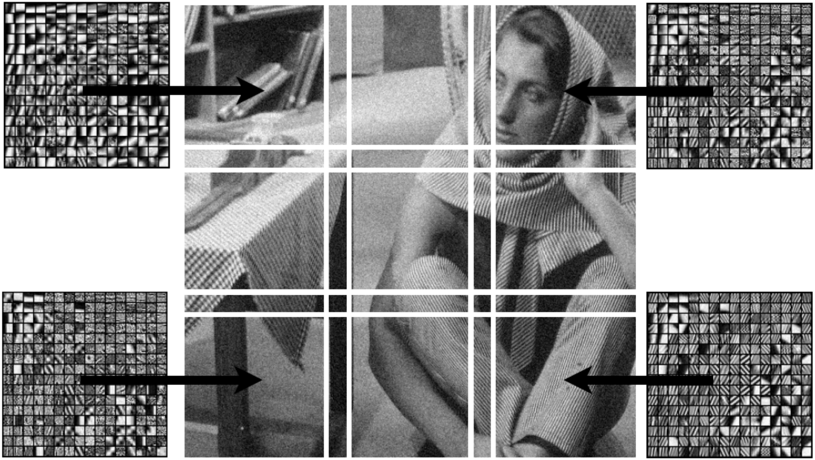
- Orthogonal Matching Pursuit (OMP) [Tropp, '04]
- Basis Pursuit Denoising (BPDN) [Chen et al, '01]

Dictionary update step:

- MOD [Engan, Aase and Hakon-Husoy, '99]
- K-SVD [Aharon, Elad and Bruckstein, '06]

A. Pižurica, *Sparse Coding and Multimodal Dictionary Learning in Computer Vision*, Plenary Lecture. Mathematics for Big Data Workshop, ECMI, Novi Sad, Serbia, 2017. https://telin.ugent.be/~sanja/Presentation/MBD2017_final.pdf

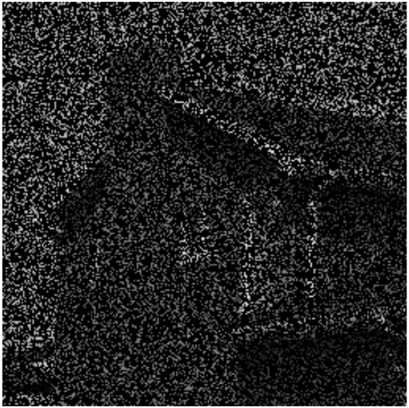
Learned dictionaries



Dictionaries learned on different parts of the Barbara image

J. Mairal, G. Sapiro, and M. Elad,
Learning multiscale sparse representations for image and video restoration,
Multiscale Modeling & Simulation, 7(1), 214-241, 2008.

Image restoration



Damaged image (75% missing)

Restored image

J. Mairal, G. Sapiro, and M. Elad,
Learning multiscale sparse representations for image and video restoration,
Multiscale Modeling & Simulation, 7(1), 214-241, 2008.

Outline

Unsupervised VS. supervised dictionary learning

- **Unsupervised** dictionary learning

$$\{\hat{\mathbf{D}}, \hat{\mathbf{A}}\} = \arg\min_{\mathbf{D},\mathbf{A}} \{\|\mathbf{X} - \mathbf{DA}\|_F^2\} \quad \text{s.t.} \quad \forall i : \quad \|\alpha_i\|_0 \leq K$$

  - minimizes the reconstruction error
  - inverse problems (restoration, inpainting. . .)

- **Supervised** dictionary learning (task-driven)

$$\{\hat{\mathbf{D}}, \hat{\mathbf{C}}, \hat{\mathbf{A}}\} = \arg\min_{\mathbf{D},\mathbf{C},\mathbf{A}} \{\|\mathbf{X} - \mathbf{DA}\|_F^2 + \eta\|\mathbf{H} - \mathbf{CA}\|_F^2 + \mu\|\mathbf{C}\|_F^2\}$$
$$\text{s.t.} \quad \forall i : \quad \|\alpha_i\|_0 \leq K$$

  - classification problems (**H - label information**, **C - classifier parameters**)

A. Pižurica, *Sparse Coding and Multimodal Dictionary Learning in Computer Vision*,
Plenary Lecture. Mathematics for Big Data Workshop, ECMI, Novi Sad, Serbia, 2017.
https://telin.ugent.be/~sanja/Presentation/MBD2017_final.pdf

Ghent Altarpiece - Adoration of the Mystic Lamb, Hubert and Jan Van Eyck - 1432.

A. Pižurica, Lj. Platiša, T. Ružić, B. Cornelis, A. Dooms, M. Martens, H. Dubois, B. Devolder, M. De Mey, and I. Daubechies,
Digital image processing of the Ghent altarpiece: supporting the painting's study and conservation treatment,
IEEE Signal Processing Magazine 32 (4): 112122, 2015.

Central panel enlarged

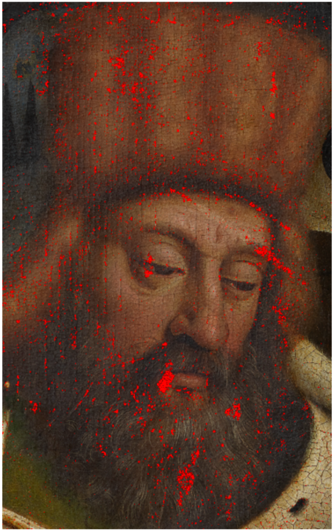
Image copyright: Ghent, Kathedrale Kerkfabriek, Lukasweb

https://www.eldiario.es/cultura/arte/Destapan-verdadera-cordero-hermanos-Eyck_0_784621776.html

Image copyright: Ghent, Kathedrale Kerkfabriek, Lukasweb

Image copyright: Ghent, Kathedrale Kerkfabriek, Lukasweb

Automatic paint loss detection



S. Huang, W. Liao, H. Zhang, and A. Pižurica,
Paint Loss Detection in Old Paintings by Sparse Representation Classification,
iTWIST 2016, pp. 62-64.

Virtual restoration results

Image copyright: Ghent, Kathedrale Kerkfabriek, Lukasweb

T. Ružić and A. Pižurica,
Context-Aware Patch-Based Image Inpainting Using Markov Random Field Modeling,
IEEE Transactions on Image Processing, 2015.

L. Meeus, S. Huang, B. Devolder, M. Martens, and A. Pižurica,
Deep learning for paint loss detection: A case study on the Ghent Altarpiece
IP4AI 2018.

Outline

TWO REPRESENTATIVES

- **Cayley-Dickson algebras**: obtained by doubling a smaller algebra and adding an additional imaginary unit

- Examples:
  - $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\imath$
  - $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\jmath$
  - $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$

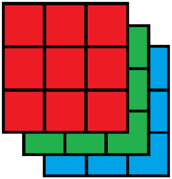
- **Real Clifford algebras:** a real associative algebra with identity 1, with generators $(e_1, \ldots, e_n)$ satisfying

$$\begin{aligned} e_i^2 &= -1, \\ e_i e_j &= -e_j e_i, \quad i \neq j. \end{aligned}$$

- Examples:
  - $\mathbb{R}_1 \cong \mathbb{C}$
  - $\mathbb{R}_2 \cong \mathbb{H}$
  - $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$

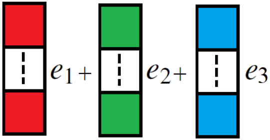
Outline

Color image models



Three color channels

Concatenated model



Quaternion based model

Outline

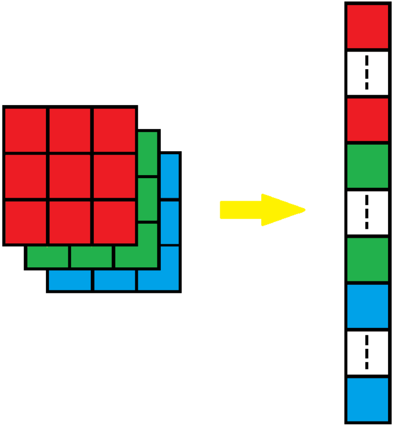


Image patch:

$$\mathbf{x} = [\mathbf{x}_r, \mathbf{x}_g, \mathbf{x}_b] \in \mathbb{R}^{3n}$$

Dictionary:

$$\mathbf{D} = [\mathbf{D}_r, \mathbf{D}_g, \mathbf{D}_b] \in \mathbb{R}^{3n \times N}$$

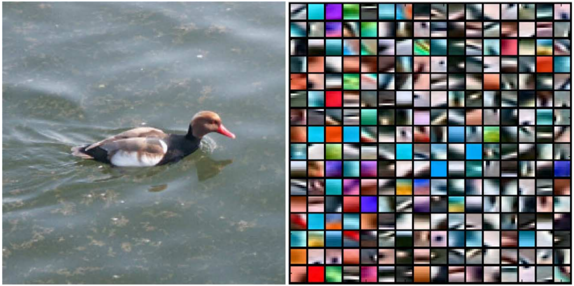
Sparse code:

$$\alpha \in \mathbb{R}^N$$

Representation model:

$$\mathbf{x} = \mathbf{D}\alpha = \sum_{i=1}^{N} \mathbf{d}_i \alpha_i$$

Vectorized model



Dictionary learned on the image on the left

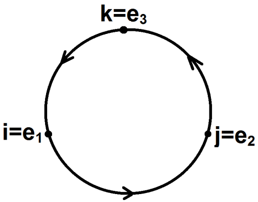
- **Shortcomings:**
  gray atoms introduce artifacts (lack of color saturation, washing effects . . .)

J. Mairal, M. Elad and G. Sapiro,
Sparse representation for color image restoration,
IEEE Trans. on Image Processing, vol. 17, pp. 53-69, 2008.

Outline

Quaternion

- The quaternion algebra $\mathbb{H} := \mathbb{C} \oplus \mathbb{C}\jmath$



Quaternion multiplication

For two quaternions

$$a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$$

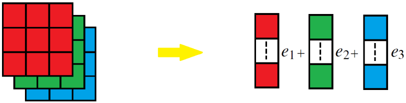
and

$$b = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3$$

then

$$\begin{aligned} \overrightarrow{ab} &= \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= L_q(a)\overrightarrow{b} \end{aligned}$$

Quaternion representation of an image patch



Color channels as quaternion imaginary units

- The color patch is defined as a quaternionic vector

$$\mathbf{x} = 0 + \mathbf{x_r} e_1 + \mathbf{x_g} e_2 + \mathbf{x_b} e_3$$

and the quaternionic dictionary as a quaternionic matrix

$$\mathbf{D} = \mathbf{D}_s + \mathbf{D}_r e_1 + \mathbf{D}_g e_2 + \mathbf{D}_b e_3$$

- Quaternion representation model learns the representation vector

$$\alpha = \alpha_s + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

for $\mathbf{x}$ such that

$$\mathbf{x} = \mathbf{D}\alpha$$

Yi Xu, Licheng Yu, Hongteng Xu, Hao Zhang, Truong Nguyen,
Vector Sparse Representation of Color Image Using Quaternion Matrix Analysis,
IEEE Trans. Image Process., vol. 24, no. 4, pp. 13151329, Apr. 2015.

Quaternion representation of an image patch

- The expression can be written as

$$\mathbf{x} = \begin{bmatrix} 0 \\ \mathbf{x}_r \\ \mathbf{x}_g \\ \mathbf{x}_b \end{bmatrix} = \begin{bmatrix} \mathbf{D_s}\alpha_0 - \mathbf{D_r}\alpha_1 - \mathbf{D_g}\alpha_2 - \mathbf{D_b}\alpha_3 \\ \mathbf{D_r}\alpha_0 + \mathbf{D_s}\alpha_1 - \mathbf{D_b}\alpha_2 + \mathbf{D_g}\alpha_3 \\ \mathbf{D_g}\alpha_0 + \mathbf{D_b}\alpha_1 + \mathbf{D_s}\alpha_2 - \mathbf{D_r}\alpha_3 \\ \mathbf{D_b}\alpha_0 - \mathbf{D_g}\alpha_1 + \mathbf{D_r}\alpha_2 + \mathbf{D_s}\alpha_3 \end{bmatrix} = \mathbf{D}\alpha$$

- Equivalently

$$[0 \quad \mathbf{x_r} \quad \mathbf{x_g} \quad \mathbf{x_b}] = [\mathbf{D_s} \quad \mathbf{D_r} \quad \mathbf{D_g} \quad \mathbf{D_b}]\ \mathbf{C_Q}$$

- The coefficient matrix $\mathbf{C_Q}$ is obtained as

$$\mathbf{C_Q} = \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}$$

Yi Xu, Licheng Yu, Hongteng Xu, Hao Zhang, Truong Nguyen,
Vector Sparse Representation of Color Image Using Quaternion Matrix Analysis,
IEEE Trans. Image Process., vol. 24, no. 4, pp. 13151329, Apr. 2015.

- **Quaternion-based learning algorithm K-QSVD**
  - Generalization of the classical K-SVD method to the quaternion setting
  - Key role: SVD decomposition of quaternion matrices

- **Advantages:**
  - The coefficient matrix preserves the correlation among the channels
  - The orthogonality property of the coefficient matrix is obtained
  - Each color channel is linearly correlated with other channels

C. Zou, K. I. Kou and Y. Wang,
Quaternion collaborative and sparse representation with application to color face recognition,
IEEE Trans. Image Process., vol. 25, no. 7, pp. 32873302, Jul. 2016.

Yi Xu, Licheng Yu, Hongteng Xu, Hao Zhang, Truong Nguyen,
Vector Sparse Representation of Color Image Using Quaternion Matrix Analysis,
IEEE Trans. Image Process., vol. 24, no. 4, pp. 13151329, Apr. 2015.

Comparison of the previous methods for image denoising

(a) Original image

(b) Noisy image $\sigma = 25$

(c) K-SVD denoising result

(d) K-QSVD denoising result

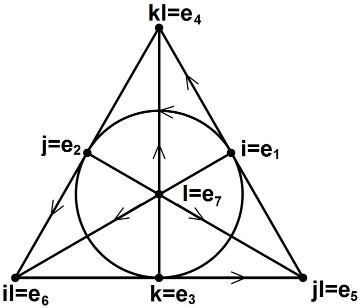
Yi Xu, Licheng Yu, Hongteng Xu, Hao Zhang, Truong Nguyen,
Vector Sparse Representation of Color Image Using Quaternion Matrix Analysis,
IEEE Trans. Image Process., vol. 24, no. 4, pp. 13151329, Apr. 2015.

Outline

Octonions

- The octonion algebra $\mathbb{O} := \mathbb{H} \oplus \mathbb{H}\ell$
- Every element $a \in \mathbb{O}$ can be written as

$$a = a_0 + a_1 e_1 + a_2 e_2 + \cdots + a_7 e_7 = \sum_{i=0}^{7} a_7 e_7$$



Octonionic multiplication

Octonionic multiplication

For two octonions

$$a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7$$

and

$$b = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5 + b_6 e_6 + b_7 e_7$$

we obtain

$$\begin{aligned}\overrightarrow{ab} &= \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} \\ &= L_o(a)\,\overrightarrow{b}\end{aligned}$$

Octonion problem formulation

Octonion sparse representation

Let $\mathbf{x} \in \mathbb{O}^{m\times 1}$ be an $\sqrt{m} \times \sqrt{m}$ image patch, then it can be represented as

$$\mathbf{x} = 0 + \mathbf{x}_1 e_1 + \cdots + \mathbf{x}_7 e_7, \quad \mathbf{x}_i \in \mathbb{R}^{m\times 1}.$$

The goal: find a dictionary $\mathbf{D} \in \mathbb{O}^{m\times n}$ and a sparse code $\alpha \in \mathbb{O}^{n\times 1}$ such that

$$\mathbf{x} \approx \mathbf{D}\alpha.$$

As before, **the coefficient matrix** is obtained as

$$\mathbf{C}_\mathbb{O} = \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 & -\alpha_5 & \alpha_4 & \alpha_7 & -\alpha_6 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 & -\alpha_6 & -\alpha_7 & \alpha_4 & \alpha_5 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 & -\alpha_7 & \alpha_6 & -\alpha_5 & \alpha_4 \\ \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_5 & -\alpha_4 & \alpha_7 & -\alpha_6 & \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_6 & -\alpha_7 & -\alpha_4 & \alpha_5 & \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_6 & -\alpha_5 & -\alpha_4 & \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{bmatrix}$$

S. Lazendić, A. Pižurica and H. De Bie,
Hypercomplex algebras for dictionary learning,
Early Proceedings of AGACSE 2018, Campinas, Brazil, 2018.

Octonion problem formulation

Octonion sparse coding problem

For a given dictionary $\mathbf{D} = \{\mathbf{d}_k\}_{k=1}^{m} \in \mathbb{O}^{m\times n}$ and signal $\mathbf{x} \in \mathbb{O}^{m\times 1}$ solve

$$\hat{\alpha} = \arg\min_{\alpha} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2 \quad \text{s.t.} \quad \|\alpha\|_0 \leq L.$$

Idea: OMP over $\mathbb{O}$

Octonion problem formulation

Octonion sparse coding problem

For a given dictionary $\mathbf{D} = \{\mathbf{d}_k\}_{k=1}^{m} \in \mathbb{O}^{m \times n}$ and signal $\mathbf{x} \in \mathbb{O}^{m \times 1}$ solve

$$\hat{\alpha} = \arg\min_{\alpha} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2 \quad \text{s.t.} \quad \|\alpha\|_0 \leq L.$$

Idea: OMP over $\mathbb{O}$

- initialize: $k = 1, \mathbf{r}^0 = \mathbf{y}, \mathbf{D}^0 = \emptyset$
- for $i = 1, \ldots, n$ compute inner products: $\mathbf{I}_i^k = \langle \mathbf{r}^{k-1}, \mathbf{d}_i \rangle$
- select them atom $\mathbf{d}_{i^k}$ s.t. $i^k = \arg\max_i |I_i^k|$
- update the active dictionary $\mathbf{D}^k = [\mathbf{D}^{k-1}, \mathbf{d}_{i^k}]$
- compute the coefficients α^k s.t. $\alpha^k = \arg\min_{\alpha} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2$
- update the residual $\mathbf{r}^k = \mathbf{x} - \mathbf{D}^k \alpha^k$
- set $k = k + 1$
- repeat until the stopping criterion is reached

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Octonion problem formulation

Octonion sparse coding problem

For a given dictionary $\mathbf{D} = \{\mathbf{d}_k\}_{k=1}^{m} \in \mathbb{O}^{m \times n}$ and signal $\mathbf{x} \in \mathbb{O}^{m \times 1}$ solve

$$\hat{\alpha} = \arg\min_{\alpha} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2 \quad \text{s.t.} \quad \|\alpha\|_0 \leq L.$$

Idea: OMP over $\mathbb{O}$

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Idea: OMP over $\mathbb{O}$

Difficulty: $\arg\min_{\alpha} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2$ **over** $\mathbb{O}$

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Difficulty: $\arg\min_{\alpha} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2$ **over** $\mathbb{O}$

- $\nu : \mathbb{O}^{n \times 1} \to \mathbb{R}^{8n \times 1}$
- $\chi : \mathbb{O}^{m \times n} \to \mathbb{R}^{8m \times 8n}$

s.t.

- $\|\alpha\|_2^2 = \|\nu(\alpha)\|_2^2$
- $\nu(\mathbf{D}\alpha) = \chi(\mathbf{D})\nu(\alpha)$

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Difficulty: $\arg\min_{\alpha} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2$ **over $\mathbb{O}$**

- $\nu : \mathbb{O}^{n\times 1} \to \mathbb{R}^{8n\times 1}$
- $\chi : \mathbb{O}^{m\times n} \to \mathbb{R}^{8m\times 8n}$

$$\begin{aligned} \|\mathbf{x} - \mathbf{D}\alpha\|_2^2 &= \|\nu(\mathbf{x} - \mathbf{D}\alpha)\|_2^2 \\ &= \|\nu(\mathbf{x}) - \chi(\mathbf{D})\nu(\alpha)\|_2^2 \end{aligned}$$

s.t.

- $\|\alpha\|_2^2 = \|\nu(\alpha)\|_2^2$
- $\nu(\mathbf{D}\alpha) = \chi(\mathbf{D})\nu(\alpha)$

$$\Longrightarrow \quad \alpha = \nu^{-1}\left(\chi(\mathbf{D})^{\dagger}\nu(\mathbf{x})\right)$$

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Octonion dictionary learning problem

For a given training set $\mathbf{X} \in \mathbb{O}^{m \times p}$ find a dictionary $\mathbf{D} \in \mathbf{O}^{m \times n}$ which best adapts to the training set, and a sparse code $\mathbf{A} \in \mathbb{O}^{n \times p}$ such that $\mathbf{X} \approx \mathbf{DA}$. Formally, it can be expressed as the following minimization problem:

$$\{\hat{\mathbf{D}}, \hat{\mathbf{A}}\} = \arg\min_{\mathbf{D},\mathbf{A}} \{\|\mathbf{X} - \mathbf{DA}\|_F^2\} \qquad \text{s.t.} \qquad \forall i : \quad \|\alpha_i\|_0 \leq K.$$

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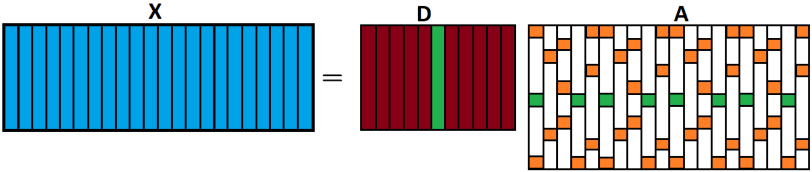
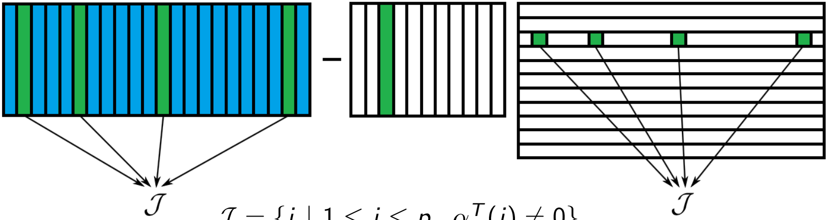


Update one dictionary atom at a time

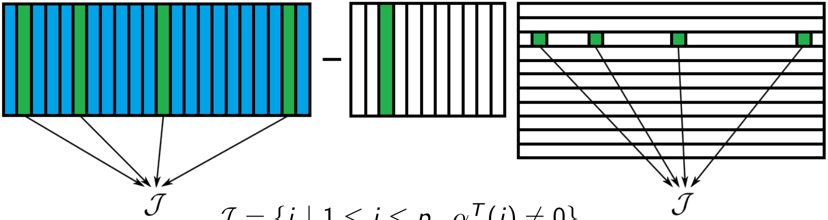
$$\|\mathbf{X} - \mathbf{DA}\|_F^2 = \left\|\mathbf{X} - \sum_{\mathbf{j}} \mathbf{d_j}\alpha_{\mathbf{j}}^{\mathbf{T}}\right\|_F^2 = \Bigg\| \underbrace{\left(\mathbf{X} - \sum_{\mathbf{j}\neq\mathbf{k}} \mathbf{d_j}\alpha_{\mathbf{j}}^{\mathbf{T}}\right)}_{\mathbf{E}_k \,=\, \text{error due to omitting } \mathbf{d}_k} - \mathbf{d_k}\alpha_{\mathbf{k}}^{\mathbf{T}} \Bigg\|_F^2$$

Octonion dictionary learning problem (continued)



$$\mathcal{J} = \{j \mid 1 \leq j \leq p,\ \alpha_k^T(j) \neq 0\}$$

Octonion dictionary learning problem (continued)



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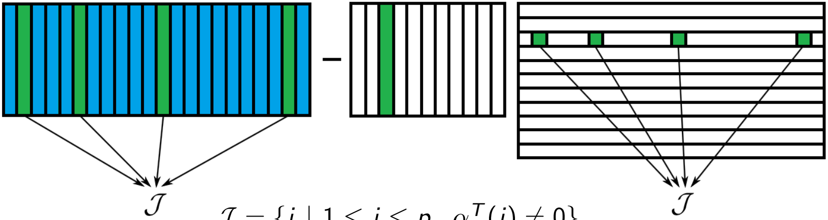
Idea: Approximate K-SVD over $\mathbb{O}$

For a residual matrix $\mathbf{E}_{\mathcal{J}}$, find the rank-one matrix approximation, i.e., $\mathbf{d}, \alpha$ s.t.

$$f(\mathbf{d}, \alpha) = \|\mathbf{E}_{\mathcal{J}} - \mathbf{d}\alpha^{\mathbf{T}}\|_F^2$$

is minimal.

Octonion dictionary learning problem (continued)



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For $\mathbf{E}_{\mathcal{J}} = [\mathbf{E_1}| \dots |\mathbf{E_n}]$ there holds that

$$f(\mathbf{d}, \alpha) = \|\mathbf{E}_{\mathcal{J}} - \mathbf{d}\alpha^{\mathbf{T}}\|_F^2 = \|[\mathbf{E_1}| \dots |\mathbf{E_n}] - \mathbf{d}\alpha^T\|_F^2 = \sum_{i=1}^{n} \|\mathbf{E_i} - \mathbf{d}\alpha_i\|_2^2$$

is a separable function.

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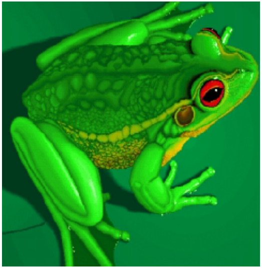
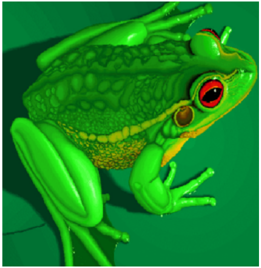
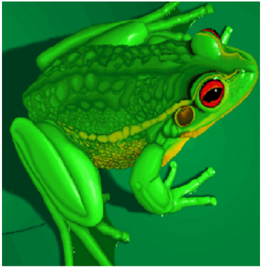
Outline



(a) K-SVD 30.99 dB



(b) K-QSVD 33.61 dB



(c) ODL 36.47 dB

	Average values of PSNR/SSIM		
	K-SVD	K-QSVD	ODL
64×128	33.15dB/0.990	33.06dB/0.994	**35.25dB/0.996**
64×256	33.92dB/0.990	33.64dB/0.995	**36.13dB/0.996**
64×512	34.94dB/0.992	34.23dB/0.995	**36.69dB/0.997**

Reconstruction of color images

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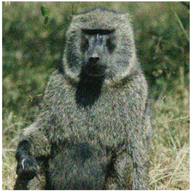
Color image denoising



(d) $\sigma = 25$

(e) K-SVD 29.28 dB

(f) K-QSVD 30.43 dB

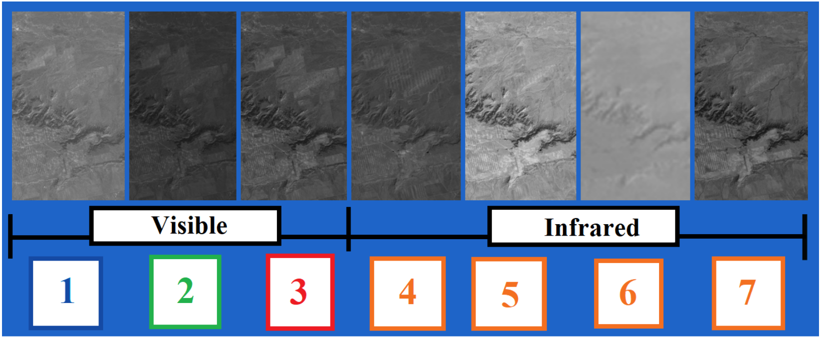
(g) ODL 30.45 dB

	Average values of PSNR/SSIM		
	K-SVD	K-QSVD	ODL
$\sigma = 10$	34.40dB/0.966	34.53dB/0.902	**35.32dB/0.971**
$\sigma = 25$	29.00dB/0.880	**30.65**dB/0.819	29.94dB/**0.889**

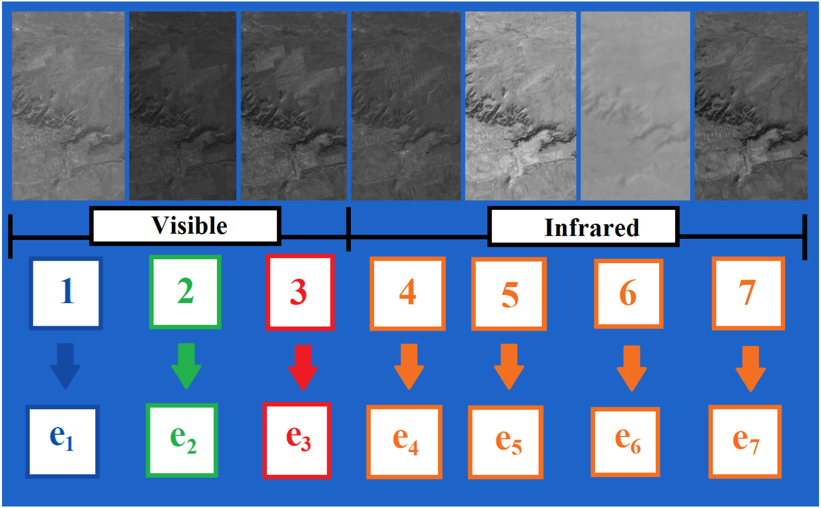
Denoising of color images

Outline

Landsat 7



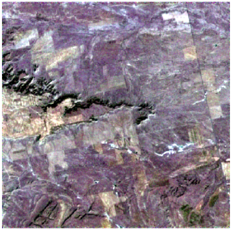
Seven bands from Landsat 7 dataset



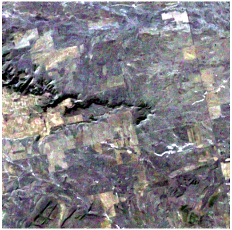
Seven bands as seven imaginary units

Image denoising

(a) Noisy Montana image



(b) K-SVD



(c) ODL

	Average values for PSNR/SSIM			
	Montana image		Mississippi image	
	K-SVD	ODL	K-SVD	ODL
$\sigma = 10$	34.79/**0.991**	**36.79**/0.900	36.21/**0.989**	**38.71**/0.918
$\sigma = 25$	31.63/**0.981**	**32.30**/0.765	32.54/**0.969**	**33.09**/0.772
Average	33.21/**0.986**	**34.54**/0.832	34.37/**0.979**	**35.90**/0.845

Denoising of Landsat 7 data

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Outline

TWO POSSIBILITIES

Cayley-Dickson algebras

$\mathbb{S} = \mathbb{O} \oplus \mathbb{O}k$ (order: 16)

Non-associativity of the octonions does not guarantee the orthogonality of the coefficient matrix in the sedenion algebra $\mathbb{S}$

Clifford algebras

$\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$ (order: 8)

It contains zero divisors and moreover $a\bar{a}$ is not necessarily a real number, so the orthogonality property is not guaranteed

S. Lazendić, A. Pižurica and H. De Bie ,
Hypercomplex algebras for dictionary learning,
Early Proceedings of the 7th Conference on Applied Geometric Algebras in Computer Science and Engineering (AGACSE 2018), Campinas, Brazil, 2018.

Conclusion

- Many applications in various inverse problems and detection/classification tasks
- Generalization of the real and the quaternion sparse model
- Octonion sparse representation model and the octonion OMP
- Octonion dictionary model (ODL)
- Processing of multichannel images in a holistic manner and preservation of interchannel dependencies

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- **Outlook:**
  - Validation of the ODL model for image inpainting
  - More thorough investigation of the ODL model
  - Validation of the model for multimodal image processing

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