Classification results on purely electric or magnetic perfect fluids

Lode Wylleman and Norbert Van den Bergh
Faculty of Applied Sciences TW16, Gent University, Galglaan 2, 9000 Gent, Belgium
E-mail: lode.wylleman@ugent.be, norbert.vandenbergh@ugent.be

Abstract. Non-conformally flat, non-vacuum perfect fluid models for which the magnetic, resp. electric, part of the Weyl tensor w.r.t. the fluid congruence vanishes, are called purely electric (PE), resp. purely magnetic (PM), perfect fluids. Their Petrov type is necessarily I or D. They are of particular interest, since the electric part is the relativistic generalization of the tidal tensor in Newtonian theory, whereas the magnetic part has no Newtonian analogue. A survey of recent classification results is presented, with focus on Petrov type D and on the Petrov type I non-accelerating subclass. Important known universe models are naturally embedded in the analysis, while new and physically interesting PE and PM families emerge.

1. Subject and motivation
Within the theory of general relativity (GR) in the full non-linear regime, one may be interested in determining which space-time metrics $g_{ab}$ obey certain algebraic properties for the Riemann curvature $R_{abcd}$, which can be decomposed as [1]

$$R_{cd}^{ab} = C_{cd}^{ab} - \frac{1}{3} R g^{a[c} g^{b]d} + 2 g^{[a} [c R^{b]} d]$$

where $R_{ab}$ is the Ricci tensor, $R$ the Ricci scalar and $C_{abcd}$ the (conformal) Weyl tensor, which is completely trace-free and represents the locally free gravitational field.

In a cosmological context, models with a perfect fluid source term in the Einstein field equation (EFE) are studied. From a classification perspective, the assumption of a perfect fluid translates in the Ricci tensor being of Segre type $[(111), 1]$, i.e., it has the particular algebraic structure

$$R_{ab} = (\mu + p) u_a u_b + \frac{1}{2} (\mu - p) g_{ab},$$

where the vector field $u^a$ is unit timelike ($u^a u_a = -1$) and plays the role of the average 4-velocity field of the fluid. It is uniquely determined, except in the $\Lambda$-type case $\mu + p = 0$, which we exclude. A possible cosmological constant $\Lambda$ has been absorbed in the fluid’s energy density $\mu'$ and pressure $p'$ by means of defining $\mu = \mu' + \Lambda$ and $p = p' - \Lambda$. Together with $u^a$ these scalar functions represent the complete Ricci freedom, such that a perfect fluid may be symbolized by $(u^a, \mu, p)$. Denote $h_{ab} = g_{ab} + u_a u_b$ for the spatial projector onto the comoving rest space of $u^a$, $\epsilon_{abc} \equiv \eta_{abcd} u^d$ for the spatial projection of the space-time permutation tensor $\eta_{abcd}$, and $S_{(ab)} = h_a^c h_b^d S_{(cd)} - \frac{1}{3} S_{cd} h^c d h_{ab}$ for the spatially projected, symmetric and trace-free part of
a 2-tensor $S_{ab}$. The kinematic quantities $\theta$ (expansion rate), $\sigma_{ab}$ (shear), $\omega_a$ (vorticity) and $\dot{u}_a$ (acceleration) of the fluid are then defined by

$$u_{ab} = \frac{1}{3} \theta h_{ab} + \sigma_{ab} + \epsilon_{abc} \omega^c - \dot{u}_a u_b$$

where $\sigma_{ab} = u_{(ab)}$.

The Weyl tensor of a perfect fluid $(u^a, \mu, p)$ can be represented by its so called electric part $E_{ab}$ and magnetic part $H_{ab}$ w.r.t. $u^a$, according to the formula [2]

$$C_{abcd} = 4 \left( u^a u^c + h^a_{\ [c} \right) E_{b]d} + 2 \epsilon_{abc} u^c H_{d]e} + 2 \epsilon_{cde} u^c H_{b]e}.$$  

These parts are defined by

$$E_{ab} \equiv C_{acbd} u^c u^d = E_{(ab)}, \quad H_{ab} \equiv \frac{1}{2} \epsilon_{acd} C_{cdbe} u^e = H_{(ab)}.$$  

In terms of $E_{ab}$ and $H_{ab}$ the Bianchi identities take a form analogous to Maxwell’s equations for the electromagnetic field, which explains the electric/magnetic terminology (see [3] for further discussion).

In the present study we focus on the classification of purely electric and purely magnetic perfect fluids (further abbreviated to PEpf’s and PMpf’s), defined as non-conformally flat perfect fluids satisfying $H_{ab} = 0$, resp. $E_{ab} = 0$. As for both situations the tensor $Q_{ab} := E_{ab} + i H_{ab}$ used in the eigenvector based Petrov classification (see §4.2 of [1]) is obviously diagonalizable, the Petrov type is either $D$ or $I$.

Whereas the electric part $E_{ab}$ is the GR generalization of the tidal tensor in Newtonian theory [4], the magnetic part $H_{ab}$ has no Newtonian analogue. Therefore PEpf’s may be regarded as ‘Newtonian-like’ and PMpf’s as ‘pure GR’ or ‘anti-Newtonian’ ([5]). Strictly speaking, the statement about $E_{ab}$ is only valid if $u^a$ is non-accelerating. In this case, denote $\xi^a$ for the deviation vector of the flow lines. The geodesic deviation equation for $u^a$ reads

$$\ddot{\xi}^a = R_{acbd} u^c u^d \xi^b = \left( E_{ab} + \frac{\mu + 3p}{6} h_{ab} \right) \xi^b,$$

and indeed reduces to Poisson’s equation in the limit of Newtonian gravity under the appropriate correspondences (see e.g. [6]). In a non-conformally flat vacuum, geodesic time-like congruences w.r.t. which $E_{ab}$ vanishes were formally excluded in [7].

This serves as a particular motivation for the study of PEpf’s and PMpf’s. Petrov type D has been investigated in general, and the most important results are summarized in section 3.1. Section 3.2 deals with the algebraically general, non-accelerating case. One might expect a rich class of PEpf solutions, representing general relativistic generalizations of classical Newtonian models. However, the non-linearity of GR may give rise to chains of severe integrability conditions for any imposed constraint (e.g. $E_{ab} = 0$); this will be briefly clarified in the next section.

2. Framework

In the fully covariant approach to the curvature of space-time metrics $g_{ab}$, the equations

$$g_{ab;c} = 0, \quad f_{[cd]} = 0, \quad 2Y_{a;}_{[cd]} \equiv R^b_{acd} Y_b,$$

$f$ being an arbitrary scalar function and $Y_a$ an arbitrary one-form, are fundamental. The first and second equations express the metric condition and the torsion-free character of the connection,
respectively, while the third equation is called the Ricci identity (for \( Y_0 \)) and provides a definition of the Riemann curvature tensor. Their respective integrability conditions are

\[
R_{(ab)cd} = 0, \quad R_{a[bc]d} = 0, \quad 0 = R_{ab[cd;e]},
\]

where the latter two are called the first and second Bianchi identity, and where indices are lowered by contraction with \( g_{ab} \). In four dimensions the second Bianchi identity is equivalent to its single contraction

\[
C_{abcd}^d = R_{[a|bc]} - \frac{1}{6} g_{[a} R_{b|c]}.
\]

In a tetrad description, one makes use of a basis of vector fields \( \mathcal{B} = (e_A^a) \) in which the metric is fixed, i.e. \( dg_{AB} = 0 \), where \( A, B, \ldots \) denote tetrad indices (values \( 1, 2, 3, 4 \)). One introduces the connection coefficients

\[
\Gamma_{ABC} \equiv g_{ef} e_A^e e_B^f : e_C^c = - \Gamma_{BAC}
\]

w.r.t. \( \mathcal{B} \). When written out in tetrad components the Ricci identity and the second Bianchi identity form a closed, first order differential system with the \( \Gamma_{ABC} \) and \( R_{ABCD} \) as scalar variables and the \( \partial_A \equiv e_A \) as formal derivation operators.

Suppose now we want to decide which space-time metrics represent e.g. non-accelerating PMPf’s \((u^a, \mu, p)\). The resulting analysis goes in two steps. Taking \( u^a \) for \( e_4^a \), we must impose \( \Gamma_{a4} = 0 \) and \( E_{\alpha \beta} = 0 \), \( \alpha, \beta \) and \( \gamma \) running from 1 to 3. This means that we put an algebraic constraint on the above system and hence we should, in a first step, perform a consistency analysis. In practice this means that we aim to arrive at the full set of consequences of this constraint, closing the system again, by taking consecutive directional derivatives, and by interacting with the basic equations and with the commutator relations \( f_{[cd]} = 0 \), i.e.,

\[
[\partial_A, \partial_B] (f) = -2 \Gamma_{C[AB]} \partial_C f,
\]

applied to the connection and curvature variables \( f \). On our way we may encounter equations of the form \( P_1 P_2 = 0 \), where \( P_1 \) and \( P_2 \) are typically polynomials in the scalar variables and their derivatives; this points to subdivision of the solution class into different branches or families. We may also run into inconsistency (e.g. PM irrotational dust [8]) or at least find compelling evidence for it (e.g. \( \S \) 3.2.1). In a second step we may then endeavour to find suitable coordinate forms of the metrics, by integration of (11); the line elements normally contain free functions or non-rescaling constants, but sometimes uniqueness theorems are obtained (e.g. \( \S \) 3.2.2).

It is advantageous to further try to fix the tetrad geometrically, in order to render the governing equations as compact as possible. For PEPf’s and PMPf’s of Petrov type I the orthonormal tetrad of \( Q_{ab} \) eigenvectors (principal Weyl tetrad) is a natural choice. For PEPf’s and PMPf’s of Petrov type D, the fluid 4-velocity \( u^a = e_4^a \) lies in the the plane of principal null directions \( \Sigma \), at each point (in accordance with the literature (e.g. [9]) the terminology \( \text{aligned} \) and PMpf’s of Petrov type D, the fluid 4-velocity \( u^a \) respectively, while the third equation is called the Ricci identity (for \( Y_0 \)) and provides a definition of the Riemann curvature tensor. Their respective integrability conditions are

\[
R_{(ab)cd} = 0, \quad R_{a[bc]d} = 0, \quad 0 = R_{ab[cd;e]},
\]

where the latter two are called the first and second Bianchi identity, and where indices are lowered by contraction with \( g_{ab} \). In four dimensions the second Bianchi identity is equivalent to its single contraction

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applied to the connection and curvature variables \( f \). On our way we may encounter equations of the form \( P_1 P_2 = 0 \), where \( P_1 \) and \( P_2 \) are typically polynomials in the scalar variables and their derivatives; this points to subdivision of the solution class into different branches or families. We may also run into inconsistency (e.g. PM irrotational dust [8]) or at least find compelling evidence for it (e.g. \( \S \) 3.2.1). In a second step we may then endeavour to find suitable coordinate forms of the metrics, by integration of (11); the line elements normally contain free functions or non-rescaling constants, but sometimes uniqueness theorems are obtained (e.g. \( \S \) 3.2.2).

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using a formalism which stands to ONP as GHP stands to NP. The relations with the connection coefficients of an orthonormal tetrad, where \( \mathbf{m} = (e_1 - i e_2)/\sqrt{2} \), are given in the Appendix.

For full details we refer to [10]. See also [11, 12] for illuminating discussions about and subtleties involved in tetrad formalisms.

3. Classification of PEpf’s and PMpf’s

3.1. Petrov type D

All locally rotationally symmetric (LRS) perfect fluids are of Petrov type D, and both the fluid 4-velocity and the spatial axis of rotation lie in the plane of principal null directions \( \Sigma \) \([13, 14]\).

Using ONP and working w.r.t. a canonical tetrad \( \mathcal{B}_{can} \), corollary 1 of theorem 2.1 in \([14]\) then tells us that an aligned Petrov type D perfect fluid \( (u^a, \mu, p) \) is LRS if and only if

\[
U = V = W = X = Y = Z = A = 0. \tag{12}
\]

It is worth noting that, referring to the Stewart-Ellis classification \([15]\), LRS classes I and III always exhibit an equation of state while this is not the case for class II, and that all members of class II and no members of class III are PEpf’s.

We provide a survey of the obtained classification results for PMpf’s and PEpf’s of Petrov type D and with regard to the above, special attention is given to the question how far such space-times deviate from (12) and hence from being LRS.

3.1.1. PMpf’s of Petrov type D. We recently showed that any PMpf is LRS I or III \([16]\). \( U = V = 0 \) immediately follows from the Bianchi identities, one then proofs \( A = X = Y = 0 \) by contradiction (also using the Ricci equations), and finally two commutator relations are needed for \( Z = W = 0 \) \([10]\). General coordinate expressions for the metrics of LRS PMpf’s had been determined previously in \([17]\) (up to one third-order ordinary differential equation in the LRS I case, and in closed form in the LRS III case), such that Petrov type D PMpf’s are fully classified. Some important examples in the literature previous to these two results are the axistationary rigidly rotating PMpf’s with circular motion \([18]\) in the LRS I case, and the \( p = \mu / 5 \) \([19]\) and Taub-NUT-like \([20]\) PMpf’s in the LRS III case.

3.1.2. PEpf’s of Petrov type D. The class of Petrov type D PEpf’s with an equation of state \( p = p(\mu) \), \( |dp/d\mu| \leq 1 \), was investigated in \([21]\). The main result was that either \( dp/d\mu = 0 \) (dust with cosmological constant) or \( dp/d\mu = 1 \) (stiff fluid-like space-time). At some places the analysis heavily depended on the assumption \( p = p(\mu) \) such that, certainly from a classification perspective, the question remained which differences arise and which parts may be recovered when this assumption is dropped.

Working in ONP, the remaining curvature variables are \( \mu, S \equiv \mu + p \) and \( \Psi_r \equiv -E_{33}/2 \). The Bianchi equations readily yield \( U = 0 \) and \( m \) real. Together with four Ricci equations they further lead to a natural division into three subclasses, which we denote by \( \mathcal{K}, \mathcal{E}_- \) and \( \mathcal{E}_+ \). Here \( \mathcal{K} \) is characterized by \( Y = V = X = A = 0 \) and \( r \) real, that is, its members are vorticity free while \( \sigma_{ab} \) and \( E_{ab} \) commute, both tensors being degenerate in \( \Sigma^\perp \); \( \mathcal{E}_- \) and \( \mathcal{E}_+ \) are characterized by \( 6\Psi_r - S = 0 \) and \( 6\Psi_r + S = 0 \), respectively. Note that the intersection of \( \mathcal{E}_- \) and \( \mathcal{E}_+ \) is empty, whereas this is not the case for \( \mathcal{E}_- \cap \mathcal{K} \) and \( \mathcal{E}_+ \cap \mathcal{K} \). In \([21]\) on the contrary, the analysis was subdivided into the cases \( 6\Psi_r - S = 0, 6\Psi_r + S = 0 \) and \( 36\Psi_r^2 - S^2 \neq 0 \), but our choice of defining \( \mathcal{K} \) turns out to be very natural. Indeed, we proved

**Theorem 1.** \( \mathcal{K} \) consists of the non-conformally flat members of three fully known families: the shearfree non-rotating Barnes family \([22]\) \( (r = \theta_b) \), the non-accelerating, non-rotating Szafron family \([23]\) \( (Z = u_3 = 0, \text{including the Szekeres dust inhomogeneous space-times}[24, 25]) \).
and the LRS II class \((Z = W = 0)\).

Non-rotating PEPf’s in general were investigated in [26]. It follows that their case 4.1.3 (corresponding to possible members of \(\mathcal{K}\) satisfying \(u_3 = 0, r \neq \theta_3, Z \neq 0\), for which a metric form metric was derived, is actually inconsistent. This stresses that at all times consistency should be checked, and that both steps mentioned in §2 should be preferably separated.

For \(\mathcal{E}_-\) one has \(U = V = Y = Z = A = 0\) and \(\partial_0 p = \delta p = 0\). We were able to perform a full integration of its rotating dust \((\partial_3 p = 0, X \neq 0)\) subclass, leaving 8 invariantly defined free functions of one coordinate \(z\) [27]. The vorticity of each of these dust-filled space-times lies in \(\Sigma^\perp\) (\(r\) is real).

Finally for \(\mathcal{E}_+\) we were able to prove:

**Theorem 2.** Any member of \(\mathcal{E}_+\) satisfies \(U = W = X = Y = A = 0\) (in particular \(\sigma_{ab}\) and \(E_{ab}\) commute). It is either LRS II or it has a stiff fluid-like equation of state \(p = \mu + c\), where \(c\) is space-time constant. In the latter case and when moreover \(Z = 0\) (acceleration lies in \(\Sigma\)), we get (possibly rotating) generalizations of the Allnutt metrics [28]. When moreover \(V = 0\) (\(\sigma_{ab}\) degenerate in \(\Sigma^\perp\)) then the space-time belongs either to \(\mathcal{K}\) or to the shear-free, expansion-free, rotating families found by Collins [29].

See [30] for further details of the analysis.

### 3.2. Petrov type I and zero acceleration

The 1+3 covariant approach for perfect fluids \((u^a, \mu, p)\) [31, 13, 3] provides direct insight into the origin of the results. The Ricci identity for \(u_a\) include the covariant derivatives along \(u^a\) (called ‘(time) evolutions’ in the sequel) of the kinematic quantities, whereas the second Bianchi identity consists of the evolutions of \(E_{ab}, H_{ab}\) and \(\mu\), the ‘\(\text{div}E\)’ and ‘\(\text{div}H\)’ constraint equations, and the pressure conservation equation.

For perfect fluids with zero acceleration, this latter equation and the Frobenius theorem imply that either the pressure is constant (dust with cosmological constant) or the fluid congruence is non-rotating (\(\omega_a = 0\)). If the pressure is a function of the scalar polynomial invariants built from \(\mu, \theta, E_{ab}, H_{ab}, \sigma_{ab}\) and \(\omega_a\) it follows for PEPf’s and PMpf’s that the evolution equations of these tensorial quantities form autonomous systems. In this case the space-times may therefore be termed ‘silent’, a terminology introduced in [32] for non-rotating purely electric dust.

#### 3.2.1. PM dust \((p = -\Lambda)\)

The basic key here is the \(\text{div}E\) equation, reading

\[
e_{abc}\sigma^d_{\ n}H^{cd} - 3H_{ab}\omega^b + \frac{1}{3}D_a\mu = 0,
\]

where \(D\) is the covariant spatial derivative, acting on arbitrary tensors \(A_{b_1...b_s}\) as

\[
D_aA_{b_1...b_s} = h_{a}^{c}h_{b_1}^{d_1}...h_{b_s}^{d_s}A_{d_1...d_s,c}.
\]

It turns out that the projections of the repeated time evolutions of (13) w.r.t. any orthonormal triad \((e_a^\nu)\) form an infinite chain of linear and homogeneous equations in the components of \(\partial_\mu, \partial_\theta, \partial_u U_2, \partial_u U_3\) and \(H_{03}\), parametrized by \(\Lambda, \mu, \theta, \omega_a\) and \(\sigma_{a\beta}\), where \(U_2 = \sigma_{ab}\sigma^{ab} - 2\omega^a\omega_a\) and \(U_3 = \sigma^{a\beta}\sigma^b_{\beta}\sigma_c^a + 3\sigma_{ab}\omega^a\omega^b\). These integrability conditions turn out to be very restrictive and lead to the following

**Conjecture 1.** Purely magnetic dust-filled space-times do not exist.
This generalizes the conjecture stated in [5] for the subcase of zero vorticity, which was proved in [8] for general Petrov type and cosmological constant. Examples of further subcases which support conjecture 1 are those where \( D_a\mu = 0 \) [33] and where the shear tensor is degenerate [10].

### 3.2.2. PMpf’s, zero vorticity.

Whereas \( \omega_a = 0 \) or \( D_a\mu \) are inconsistent for purely magnetic dust, we proved [33] Theorem 3. Up to a constant rescaling, the line element

\[
ds^2 = \exp(-2e^{-t})(-dt^2 + e^t dx^2) + e^t(e^{-2}dy^2 + e^2 dz^2).
\]

represents the unique algebraically general PMpf which satisfies any two of the three conditions \( \dot{u}_a = 0 \), \( \omega_a = 0 \), \( D_a\mu = 0 \). This space-time is orthogonally spatially homogeneous (OSH) of Bianchi type \( V I_0 \). The Petrov type is \( I(M^\infty) \) in the extended Arianrhod-McIntosh classification [34], while \( \sigma_{ab} \) commutes with \( H_{ab} \) and is degenerate in the plane orthogonal to the 0-eigendirection of \( H_{ab} \). The equation of state reads

\[
\mu + p = \frac{1}{2}(\mu - p) \ln \left| \frac{2}{3}(\mu - p) \right|.
\]

The space-time starts off with a stiff matter-like big-bang singularity at a finite proper time in the past and expands indefinitely towards an Einstein space.

### 3.2.3. PEpf’s, zero vorticity.

In this case it was proved in [26] that the Weyl principal tetrad is Fermi-propagated along the matter flow, and that either the space-time is shear-free and static or the Weyl principal vectors are hypersurface orthogonal. Later, it was observed independently in [35] and [36] that for non-rotating dust, repeated evolution of \( E_{ab} = 0 \) leads to an infinite chain of integrability conditions which, when projected on e.g. the principal Weyl tetrad, result in systems of massive polynomials in a fixed, finite number of variables (integrability conditions arise more generally and in exactly the same way for non-accelerating and non-rotating PEpf’s, but here the \( \partial_0^p(p) \) become also involved as variables). These conditions are identically satisfied for Petrov type D and for the OSH solutions of Bianchi type I, but not for Petrov type I. It was conjectured in [35] and [36] that the OSH solutions are the only algebraically general PE irrotational dust space-times. This was confirmed for vacuum in [37] and for space-times which admit a G3 isometry group in [38]. In all these works the cosmological constant \( \Lambda \) was equate to zero, as it seemed of little relevance to the consistency problem. However, in [39] a family \( \mathcal{F} \) of non-OSH PE irrotational dust solutions with \( \Lambda > 0 \) was constructed, which was further characterized by one space-like principal Weyl vector being geodesic. Further investigations point towards a uniqueness result for \( \mathcal{F} \). Most recently, an attempt to prove this uniqueness has been published [40]. However, the proof is incomplete (or incorrect) and a note on this will be published in a forthcoming contribution. Finally, it is fairly easy to show that non-rotating, non-accelerating PEpf’s which admit a non-dust equation of state are OSH Bianchi type I [10]. We conclude with

**Conjecture 2.** Non-rotating, non-accelerating PEpf’s which admit an equation of state either belong to \( \mathcal{F} \) or are OSH Bianchi type I.

### 3.2.4. PE rotating dust (\( p = -\Lambda \)).

In the non-rotating case the condition \( \text{div} \ H_a = 0 \) is equivalent to \( \epsilon_{abc} \sigma^b_d E^{cd} = 0 \), i.e., \( \sigma_{ab} \) and \( E_{ab} \) commute, and this is consistent under time evolution. In the rotating case on the contrary, two extra terms are involved; repeated time evolution leads to a chain of integrability conditions which projects into a system of polynomials in \( \Lambda, \mu, \theta, H_{11}, H_{22} \) and the components of \( \sigma_{ab} \) and \( \omega_a \). Careful examination of this system has recently led to the
surprising result [41]:

**Theorem 4.** Every algebraically general PE rotating dust space-time \((\omega_a \neq 0)\) necessarily possesses a space-like principal Weyl vector field which is geodesic, and the vorticity field is parallel to it. The cosmological constant is necessarily different from zero.

Taking \(e^a_i\) for the normalized geodesic space-like vector field, the latter fact is due to the relation

\[\left(\sigma_{11} - \sigma_{22}\right)\left(\sigma_{11} - \sigma_{33}\right) - 4\left(\sigma_{23} - \omega_1\right)\left(\sigma_{23} + \omega_1\right)\Lambda + 4\left(E_{11} - E_{22}\right)\left(E_{11} - E_{33}\right) = 0,\]

such that in the case \(\Lambda = 0\) the Petrov type is D. When setting the vorticity equal to zero in the defining set of invariant relations of this class, one recovers the above family \(F\). Some manipulation shows that the variables are constrained by a relation of the form

\[P\omega^2_1 + (\mu - 2\Lambda)^2 - 4\Lambda(\sigma_{22} - \sigma_{33})^2 = 0,\]

which makes clear that \(\Lambda < 0\) is allowed in general, but not for members of \(F\). Finally, some well chosen ansatze result in rotating subclasses which may be easily integrated.

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**Appendix**
Relations between the connection coefficients used in the ONP formalism (left hand sides, tetrad \((m, \bar{m}, e_3, u)\)) and the connection coefficients of an orthonormal (ON) tetrad \((e_1, e_2, e_3, u)\) with \(m = (e_1 - ie_2)\sqrt{2}\) (right hand sides, including ON components of the kinematic quantities (3) of \(u^a\)):

\[U = \frac{1}{2}(\Gamma_{232} - \Gamma_{131} + i(\Gamma_{231} - \Gamma_{312})),\quad V = \frac{1}{2}(\sigma_{11} - \sigma_{22} - 2i\sigma_{12}),\]
\[A = \frac{\sqrt{2}}{2}(\sigma_{13} + \omega_2 - i(\sigma_{23} - \omega_1)),\quad X = \frac{\sqrt{2}}{2}(s_{13} - \omega_2 - i(\sigma_{23} + \omega_1)),\]
\[W = \frac{\sqrt{2}}{2}(\Gamma_{313} - i\Gamma_{323}),\quad Y = -\frac{\sqrt{2}}{2}(\Gamma_{314} - i\Gamma_{324}),\quad Z = \frac{\sqrt{2}}{2}(\dot{u}_1 - i\dot{u}_2),\]
\[m = -\frac{1}{2}(\Gamma_{131} + \Gamma_{232} - i(\Gamma_{231} + \Gamma_{312})),\quad r = -\frac{s_{33}}{2} + \frac{\theta}{3} + i\omega_3,\]
\[\theta_3 = \sigma_{33} + \frac{1}{3}\theta,\quad \dot{u}_3 = \dot{u}_3,\]
\[B = -\frac{\sqrt{2}}{2}(\Gamma_{212} - i\Gamma_{121}),\quad N_3 = i\Gamma_{123},\quad \Omega_3 = i\Gamma_{124}.\]

**References**
[8] Wylleman L 2006 Class. Quantum Grav. 23, 2727
[9] Lozanovski C 2002 Class. Quantum Grav. 19, 6377
[16] Van den Bergh N and Wylleman L 2006 Class. Quantum Grav. 23, 3353
[17] Lozanovski C and Carminati J 2003 Class. Quantum Grav. 20, 215
[20] Lozanovski C and Aarons M 1999 Class. Quantum Grav. 16, 4075
[22] Barnes A 1973 Gen. Rel Grav. 4, 105
[26] Barnes A and Rowlingson R 1989 Class. Quantum Grav. 6, 949
[27] Van den Bergh N and Wylleman L, Full integration of purely electric rotating dust, to be submitted to Class. Quantum Grav.
[34] Arianrhod R and McIntosh C B G 1992 Class. Quantum Grav. 9, 1969
[37] Mars M 1999 Class. Quantum Grav. 16, 3245
[38] Wylleman L and Van den Bergh N 2006 Class. Quantum Grav. 23, 329
[39] Van den Bergh N and Wylleman L 2004 Class. Quantum Grav. 21, 2291
[40] Apostolopoulos P S and Carot J, gr-qc/0605130
[41] Wylleman L, Classification of algebraically general purely electric rotating dust, to be submitted to Phys. Rev. D.