1. Introduction

Using a 3+1 formalism based on a timelike congruence of observers, with the covariant derivative of the velocity field \( u \) being decomposed in the usual way,

\[
\nabla_b u_a = -\dot{u}_a u_b + D_b u_a
\]

(1)

with

\[
D_b u_a = \sigma_{ab} + 1/3 \theta h_{ab} + \epsilon_{abc} \omega^c
\]

(2)

\[
h_{ab} = g_{ab} + u_a u_b
\]

(3)

\[
\epsilon_{abc} = \epsilon_{abcd} u^d,
\]

(4)

reveals a close analogy between electromagnetism and gravity [Maartens et al. 1997,1998]:

the source free Maxwell equations become

\[
D.E = 2i\omega.E
\]

(5)

\[
\dot{E} + iD \times E = -2/3 \theta E + \sigma E - (\omega + i\dot{u}) \times E
\]

(6)

with

\[
E_a = F_{ab} u^b
\]

(7)

\[
H_a = *F_{ab} u^b
\]

(8)

\[
E = E + iH
\]

(9)

while the vacuum Einstein equations are given by

(Bianchi identities)

\[
D.E = 3i\omega.E - i\sigma \times E
\]

(10)

\[
\dot{E} + iD \times E = -\theta E - (\omega + 2i\dot{u}) \times E + 3\text{tr}(\sigma \otimes E)
\]

(11)
with
\begin{align}
E_{ab} &= C_{abcd}u^c u^d 
\quad (12) \\
H_{ab} &= {}^*C_{abcd}u^c u^d 
\quad (13) \\
\mathcal{E} &= E + iH 
\quad (14)
\end{align}

and

(Ricci identities)

\begin{align}
H &= D \times \sigma + D\omega + 2\dot{u} \otimes \omega 
\quad (15) \\
E &= -\dot{\sigma} - \frac{2}{3} \theta \sigma + D \otimes \dot{u} - \text{tr}(\sigma \otimes \sigma) - \omega \otimes \omega + \dot{u} \otimes \dot{u} 
\quad (16)
\end{align}

\begin{align}
\dot{\theta} + \frac{1}{3} \theta^2 - 2|\omega|^2 + |\sigma|^2 - |\dot{u}|^2 - D.\dot{u} - \Lambda = 0 
\quad (17) \\
\dot{\omega} + \frac{2}{3} \theta \omega + \frac{1}{2} D \times \dot{u} - \omega \cdot \sigma = 0 
\quad (18)
\end{align}

\begin{align}
\frac{2}{3} D\theta - D.\sigma + D \times \omega = 2\omega \times \dot{u} 
\quad (19) \\
D.\omega = \dot{u} \cdot \omega 
\quad (20)
\end{align}

Based on this analogy the (spatial and trace-free) tensors $E$ and $H$ are called resp. the electric and magnetic part of the Weyl curvature. While the non-vanishing of both is essential for the existence of gravitational waves, only $E$ has a Newtonian analogue. It is the tidal tensor, making its appearance in the equation of geodesic deviation:

\begin{align}
\ddot{\xi} + R(\xi, u)u &= 0 
\quad (21) \\
\Leftrightarrow \ddot{\xi} + C_{abcd}u^c \xi^b u^d &= 0 \quad \text{(in vacuum)} 
\quad (22) \\
\Leftrightarrow \ddot{\xi} + E.\xi &= 0 
\quad (23)
\end{align}

Whereas large classes of purely electric vacuum or $\Lambda$ type solutions ("gravito-electric monopoles") exist (for example all the D vacua with a real $\Psi_2$ in the canonical frame, the static Weyl solutions, the Gödel universe), not a single "gravito-magnetic monopole" is known.

Some partial results concerning the latter are

. Trümper (1965): no solutions exist having $\omega = \sigma = 0$

. McIntosh, Arianrhod, Wade and Hoenselaers (1994): the Petrov type is necessarily I; more particularly a canonical frame exists in which $\Psi_1 = \Psi_3 = 0$, while $\Psi_0 = \Psi_4$ and $\Psi_2$ are imaginary.

. Haddow (1995): no solutions exist with $\sigma = 0$
Purely magnetic vacuum solutions

. VdB (2002): no solutions exist with $\omega = 0$; this was generalised to space-times with vanishing Cotton-tensor by Ferrando and Saez (2003): using eq. (11) one obtains $\sigma \times H = 0$, acting on which with the $D$ operator results in

$$(D \times \sigma)H - \sigma(D \times H) = 0$$

(24)

and hence, using the Ricci identities $|H|^2 = 0$.

Maartens (1998) tries to construct stationary purely magnetic solutions (with $u$ aligned with the timelike Killing vector), which however is impossible in view of the Haddow result. He notices that, although only the Bianchi identities are invariant under the duality rotation $E \rightarrow e^{ia}E$, the full system of linearized equations allows non-rotating gravito-electric monopoles (e.g. linearized Schwarzschild) to be converted in a non-accelerating gravito-magnetic monopole (linearized Taub-NUT). The question whether the full non-linear equations allow the existence of such a solution was left open. Existence of such vacuum models would provide possibly interesting examples of spacetimes in which the genericity condition was violated.

2. Non-accelerating gravito-magnetic monopoles: do they exist?

The relevant dynamical equations for a purely gravito-magnetic vacuum space-time in which the timelike congruence $u$ is geodesic are given below, using the notations and conventions of the orthonormal tetrad formalism [10]. The coefficients $u_{\alpha \alpha}$ being redefined as follows:

$$n_{11} = (n_2 + n_3)/2, \ n_{22} = (n_3 + n_1)/2, \ n_{33} = (n_1 + n_2)/2$$

(25)

and with the tetrad being specified as an eigenframe of $H$ (each equation represents a triplet of equations, obtained by cyclic permutation of the spatial indices). The vanishing of the gravito-electric part of the Weyl-tensor can be expressed by the 9 equations

$$-\partial_0 \theta_1 - \theta_1^2 - \sigma_{12}^2 - \sigma_{13}^2 + \omega_2^2 + \omega_3^2 + 2\sigma_{12}\Omega_3 - 2\sigma_{13}\Omega_2 + \frac{1}{3}\Lambda = 0$$

(26)

$$-\partial_0(\sigma_{12} + \omega_3) - (\theta_1 + \theta_2)(\sigma_{12} + \omega_3) - (\sigma_{13} - \omega_2)(\sigma_{23} - \omega_1) + \Omega_1(\sigma_{13} - \omega_2) - \Omega_2(\sigma_{23} - \omega_1) + \Omega_3(\theta_2 - \theta_1) = 0$$

(27)

$$-\partial_0(\sigma_{12} - \omega_3) - (\theta_1 + \theta_2)(\sigma_{12} - \omega_3) - (\sigma_{23} + \omega_1)(\sigma_{13} + \omega_2) + \Omega_1(\sigma_{13} + \omega_2) - \Omega_2(\sigma_{23} + \omega_1) + \Omega_3(\theta_2 - \theta_1) = 0$$

(28)

while the vanishing of the off-diagonal components of $H$ leads to

$$-\partial_0(n_{12} + a_3) - \partial_1 \Omega_2 - \theta_1(n_{12} + a_3) - (n_{23} - a_1)(\sigma_{13} + \omega_2) + \frac{1}{2}n_2(\sigma_{12} - \omega_3) + \Omega_1(n_{13} - a_2) - \Omega_2(n_{23} - a_1) + \frac{1}{2}\Omega_3(n_1 - n_2) = 0$$

(29)

$$-\partial_0(n_{12} - a_3) - \partial_2 \Omega_1 - \theta_2(n_{12} - a_3) - (n_{13} + a_2)(\sigma_{23} - \omega_1) + \frac{1}{2}n_1(\sigma_{12} + \omega_3) + \Omega_1(n_{13} + a_3) - \Omega_2(n_{23} + a_1) + \frac{1}{2}\Omega_3(n_1 - n_2) = 0$$

(30)
Together with the Ricci-identities, we obtain from these equations the evolution for \( \theta_a \), \( \sigma_{ab} \), \( \omega_a \), \( a_a \), \( n_{ab} \) and \( n_a \), while the curl of the shear can be expressed in terms of the diagonal components of \( \mathbf{H} \):

\[
H_{11} = \partial_2 \sigma_{13} - \partial_3 \sigma_{12} + \partial_1 \omega_1 - \frac{1}{2} (\theta_1 (n_2 + n_3) - \theta_2 n_3 - \theta_3 n_2) + 2 n_{23} \sigma_{23}
\]

\[-n_{12} (\sigma_{12} + \omega_3) - a_2 (\sigma_{13} + \omega_2) - n_{13} (\sigma_{13} - \omega_2) + a_3 (\sigma_{12} - \omega_3)
\]

The remaining Ricci identities contain spatial gradients of the kinematical scalars only and can be used to simplify the integrability conditions which result by considering the following commutators:

\[ [\partial_0, \partial_1] \theta_2 - [\partial_0, \partial_2] (\sigma_{12} - \omega_3) \quad \text{and} \quad [\partial_0, \partial_1] \theta_3 - [\partial_0, \partial_3] (\sigma_{13} + \omega_2) \]

One obtains then three pairs of equations,

\[
(\sigma_{33} - \omega_1 + \Omega_1) H_{11} - (\sigma_{23} - \omega_1 - 2 \Omega_1) H_{22} = 0
\]

\[
(2 \sigma_{23} + 2 \omega_1 + \Omega_1) H_{11} + (\sigma_{23} + \omega_1 + 2 \Omega_1) H_{22} = 0
\]

in which one recognizes the previously obtained relation [2, 5, 9] between \( \sigma \), \( \omega \) and \( \mathbf{H} \),

\[
\sigma \times \mathbf{H} = 3 \omega \cdot \mathbf{H},
\]

together with a relation between \( \sigma \), \( \omega \) and the rotation rate \( \Omega \) of the \( \mathbf{H} \)-eigenframe with respect to a Fermi-propagated triad:

\[
\omega_1^2 + 2 \omega_1 \Omega_1 = \sigma_{23}^2
\]

(+ cyclic permutations). This leads to the following expressions for \( \sigma \) and \( \Omega \):

\[
\sigma_{12} = 3 \omega_3 \frac{H_{11} + H_{22}}{H_{11} - H_{22}}
\]

\[
\Omega_1 = 2 \omega_1 \frac{(H_{11} - H_{22})(H_{11} - H_{33})}{(H_{22} - H_{33})^2}
\]

(note that \( \mathbf{H} \) is not allowed to have equal eigenvalues [1]). Substituting the latter in the propagation of the shear and making use of the time evolution of the curvature (given by the Bianchi identities), there result algebraic relations between \( \omega \) and \( \mathbf{H} \):

\[
2 \omega_3 (H_{22} - H_{33}) (H_{11} - H_{33}) \left[ \theta_1 (H_{22} - H_{33}) + \theta_2 (H_{11} - H_{33}) + 3 \theta_3 (H_{11} - H_{22}) \right]
\]

\[
+ 3 \omega_1 \omega_2 (H_{11} - H_{22}) (5 H_{11}^2 + 8 H_{11} H_{22} + 5 H_{22}^2) = 0
\]

Elimination of \( \theta_a \) from this equation and its cyclic permutations eventually results in

\[
\left( 2 H_{11}^2 + 2 H_{11} H_{22} + 5 H_{22}^2 \right) (H_{11} - H_{33})^4 \omega_1^2 \omega_3^2
\]

\[
+ \left( 2 H_{22}^2 + 2 H_{22} H_{33} + 5 H_{33}^2 \right) (H_{22} - H_{11})^4 \omega_2^2 \omega_1^2
\]

\[
+ \left( 2 H_{33}^2 + 2 H_{33} H_{11} + 5 H_{11}^2 \right) (H_{33} - H_{22})^4 \omega_3^2 \omega_2^2 = 0
\]

showing that a purely magnetic vacuum is inconsistent with the assumption of the congruence being geodesic.

(The analysis becomes slightly more complicated when some of the components of the vorticity vanish: higher order time derivatives are then needed to obtain an inconsistency between the signs of the involved vorticity and curvature terms.)
References