EQUIVALENT FORMULATIONS OF THE MÜNTZ-SZÁSZ COMPLETENESS CONDITION FOR SYSTEMS OF COMPLEX EXPONENTIALS

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Abstract New results pertaining to the completeness of a system of complex exponentials in L_p spaces are presented. It is shown that the well-known Müntz-Szász condition can be interpreted in terms of an equivalence relation, thereby proving that the standard Müntz-Szász formulation is equivalent with certain alternative formulations.

Key words Exponentials, Completeness, L_p spaces, Kullback information.

1 INTRODUCTION

Recently an upsurge in research relating to the orthonormal Laguerre basis for use in system identification [1] and reduced-order modeling [2] has been noticed. The obvious generalization of the Laguerre basis to a full exponential basis, also called Kautz [3] basis, requires knowledge of the conditions under which an infinite set of exponentials is complete in $L_2[R_+]$, and more generally in $L_p[R_+]$. We show that there is a fundamental link with Müntz-Szász theory and the Müntz-Szász condition [4]-[8]. Based on Kullback information [9], we obtain a new completeness result in $L_p[R_+]$ for $1 \leq p < 2$. Furthermore we show that the Müntz-Szász condition can be interpreted in terms of an equivalence relation, thereby proving that the standard Müntz-Szász formulation is equivalent with different alternative formulations found in the literature [5], [10].

2 THE MÜNTZ-SZÁSZ CONDITION

The importance of the Müntz-Szász condition

$$\sum_{n=0}^{\infty} \frac{\Re(\lambda_n)}{1+|\lambda_n|^2} = \infty \tag{1}$$

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where $\Re(z)$ stands for the real part of z, in connection with the completeness of a system of exponentials $\{e^{-\lambda_n t}\}$ in $L_p[R_+]$ spaces with norm

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p \, dt\right)^{1/p} \qquad p \geq 1 \tag{2}$$

cannot be overestimated. The most important result is due to Crum [4]:

Theorem 1: Let $\{\lambda_n\}$ be a sequence of distinct complex numbers with positive real parts, such that $\{\lambda_n\}$ has at most a finite number of limit points on the imaginary axis. Then the system of exponentials $\{e^{-\lambda_n t}\}$ is complete in $L_p[R_+]$ if and only if the Müntz-Szász condition (1) holds.

Corollary 1: Let $\{\lambda_n\}$ be a sequence of distinct positive real numbers. Then the system of exponentials $\{e^{-\lambda_n t}\}$ is complete in $L_p[R_+]$ if and only if

$$\sum_{n=0}^{\infty} \frac{\lambda_n}{1+\lambda_n^2} = \infty \tag{3}$$

Proof: Straightforward.

Corollary 2: Full Müntz theorem in $L_p[0,1]$. Let $\{\lambda_n\}$ be a sequence of distinct real numbers greater than $-1/p$. Then the system of powers $\{x^{\lambda_n}\}$ is complete in $L_p[0,1]$ if and only if

$$\sum_{n=0}^{\infty} \frac{\lambda_n + 1/p}{1+(\lambda_n + 1/p)^2} = \infty \tag{4}$$

Proof: From Corollary 1 we know that condition (3) implies that for any $f(t)$ in $L_p[R_+]$ and for any $\epsilon > 0$ we can find coefficients $\{a_0, a_1, \ldots, a_n\}$ such that

$$\left\| f(t) - \sum_{k=0}^{n} a_k e^{-\lambda_k t} \right\|_p \leq \epsilon \tag{5}$$

Now the coordinate stretching $x = e^{-t}$ transforms $L_p[R_+]$ into $L_p[0,1]$, in the sense that $f(t) \in L_p[R_+]$ is equivalent with $f(-\ln x)x^{-1/p} \in L_p[0,1]$. Hence

$$\left\| f(t) - \sum_{k=0}^{n} a_k e^{-\lambda_k t} \right\|_p = \left(\int_0^1 \left| f(-\ln x)x^{-1/p} - \sum_{k=0}^{n} a_k x^{\lambda_k - 1/p} \right|^p dx\right)^{1/p} \leq \epsilon \tag{6}$$

and the proof follows, since the argumentation remains valid when taken in the reverse order. Note that Corollary 2 was presented as a conjecture in [5] and proved in [6] by operator-theoretic means.

Remark 1: Note that $L_p[R_+]$ for $p = \infty$ corresponds with $C[R_+]$, the space of continuous functions with the uniform norm

$$\|f\|_\infty = \sup_{t \geq 0} |f(t)| \tag{7}$$

Since the constant function belongs to $C[R_+]$, the system of exponentials $\{e^{-\lambda_n t}\}$ must include it, and hence for completeness in $C[R_+]$ we need to take $\lambda_0 = 0$.

Remark 2: Some or all of the exponents λ_n may coalesce, and in that case the system $\{e^{-\lambda_n t}\}$ must be replaced with the system $\{e^{-\lambda_n t} t^m\}$ where $m = 0, 1, 2, \ldots$ See e.g. [7].

Other important results in [4] are:

Theorem 2: Let $1 \leq p \leq 2$ and $\{\lambda_n\}$ a sequence of distinct complex numbers with positive real parts such that the system of exponentials $\{e^{-\lambda_n t}\}$ is complete in $L_p[R_+]$. Then the Müntz-Szász condition (1) holds.

Theorem 3: Let $p \geq 2$ and $\{\lambda_n\}$ a sequence of distinct complex numbers with positive real parts, such that the Müntz-Szász condition (1) holds. Then the system of exponentials $\{e^{-\lambda_n t}\}$ is complete in $L_p[R_+]$.

Theorem 3 is important since it states that the Müntz-Szász condition implies completeness in $L_p[R_+]$ for $p \geq 2$. Theorems 2 and 3 taken together imply that the system of exponentials $\{e^{-\lambda_n t}\}$ is complete in $L_2[R_+]$ if and only if the Müntz-Szász condition (1) holds.

Another important result is due to Sedletskii [8]:

Theorem 4: Let $\xi(\lambda) = \Re(\lambda)/(1 + |\lambda|^2)$ where λ is a complex number with positive real part and let $\phi(t)$ be a positive decreasing function on $(0, \infty)$ such that $\int_0^\infty \phi(t)dt < \infty$. If

$$\sum_{n=0}^{\infty} \xi(\lambda_n)\phi(-\ln(\xi(\lambda_n))) = \infty \tag{8}$$

then the system $\{e^{-\lambda_n t}\}$ is complete in $L_p[R_+]$, $1 \leq p < 2$.

Corollary: Let $\epsilon > 0$ and

$$\sum_{n=0}^{\infty} \left(\frac{\Re(\lambda_n)}{1 + |\lambda_n|^2} \right)^{1+\epsilon} = \infty \tag{9}$$

then the system $\{e^{-\lambda_n t}\}$ is complete in $L_p[R_+]$, $1 \leq p < 2$.

Proof: Take $\phi(t) = e^{-\epsilon t}$. Of course, we cannot let ϵ tend to zero, since

$$\lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon t} dt = \infty \tag{10}$$

The corollary of Theorem 4 is important since it states that the 'almost' Müntz-Szász condition (9) implies completeness in $L_p[R_+]$ for $1 \leq p < 2$.

We would like to formulate a result similar to the above corollary, but in terms of the exact Müntz-Szász condition (1). We need the following

Lemma: Let the weighted $p-$norm $\|f\|_{p,w}$ be defined as

$$\|f\|_{p,w} = \left(\int_0^\infty |f(t)|^p w(t)\, dt \right)^{1/p} \qquad p \geq 1 \tag{11}$$

where $w(t) > 0$ is a probability density over $[0, \infty]$, i.e. $\int_0^\infty w(t)dt = 1$. Then $\|f\|_{p,w}$ is an increasing function of p, that is

$$\|f\|_{p,w} \geq \|f\|_{q,w} \quad \text{for} \quad p \geq q \tag{12}$$

Proof: It is straightforward to show that the derivative with respect to p of the logarithm of the weighted $p-$norm can be written as

$$\frac{\partial}{\partial p} \ln \|f\|_{p,w} = \frac{1}{p^2} K(r(t), w(t)) \tag{13}$$

where $r(t)$ is the probability density

$$r(t) = |f(t)|^p w(t) / \int_0^\infty |f(t)|^p w(t)\, dt \tag{14}$$

and $K(r(t), w(t))$ is the Kullback information [9]

$$K(r(t), w(t)) = \int_0^\infty r(t) \ln \left(\frac{r(t)}{w(t)} \right) dt \tag{15}$$

The fact that Kullback information is always positive or zero completes the proof.

Theorem 5: Let $\epsilon > 0$. If the system $\{e^{-\lambda_n t}\}$ is complete in $L_p[R_+]$ then the system $\{e^{-\lambda'_n t}\}$ is complete in $L_q[R_+]$, $q \leq p$, where

$$\lambda'_n = \lambda_n + \epsilon \left(\frac{1}{q} - \frac{1}{p} \right) \tag{16}$$

Proof: Consider the $L_{p,\epsilon}[R_+]$ spaces induced by the $\|f\|_{p,w}$ norms with $w(t) = \epsilon e^{-\epsilon t}$:

$$\|f\|_{p,w} = \left(\int_0^\infty |f(t)|^p \epsilon e^{-\epsilon t}\, dt \right)^{1/p} \qquad p \geq 1 \tag{17}$$

It is seen that the following chain of implications is valid:

$$\begin{array}{lll}
\#1 \;\; \{e^{-\lambda_n t}\} & \text{complete in} & L_p[R_+] \Longrightarrow \\
\#2 \;\; \{e^{-(\lambda_n - \epsilon/p)t}\} & \text{complete in} & L_{p,\epsilon}[R_+] \Longrightarrow \\
\#3 \;\; \{e^{-(\lambda_n - \epsilon/p)t}\} & \text{complete in} & L_{q,\epsilon}[R_+] \Longrightarrow \\
\#4 \;\; \{e^{-(\lambda_n - \epsilon/p + \epsilon/q)t}\} & \text{complete in} & L_q[R_+]
\end{array} \tag{18}$$

and this proves the theorem. The step #2 $\Rightarrow$ #3 follows from the Lemma, since completeness in a space with a topology induced by a given norm implies completeness in a space with a topology induced by a coarser norm [11] p. 106. Note that $\epsilon(1/q - 1/p) > 0$ when $q < p$.

Corollary: Let $\Re(\lambda_n) > \epsilon > 0$ and

$$\sum_{n=0}^{\infty} \frac{\Re(\lambda_n - \epsilon)}{1 + |\lambda_n - \epsilon|^2} = \infty \tag{19}$$

then the system $\{e^{-\lambda_n t}\}$ is complete in $L_p[R_+],\ 1 \leq p < 2$.

Proof: Straightforward.

Formula (19) as compared to formula (9) has the advantage of keeping the original form of the Müntz-Szász condition (1).

3 EQUIVALENT FORMULATIONS

The importance of the Müntz-Szász condition

$$\sum_{n=0}^{\infty} \frac{\Re(\lambda_n)}{1+|\lambda_n|^2} = \infty \tag{20}$$

has been demonstrated in the previous section. But of course, there may be equivalent formulations. For example in [10] we find the formulation

$$\sum_{n=0}^{\infty} \frac{\Re(\lambda_n)}{1+|\lambda_n - 1/2|^2} = \infty \tag{21}$$

and in [5] we find the formulation

$$\sum_{n=0}^{\infty} \left(1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right|\right) = \infty \tag{22}$$

Also, it is of interest to consider the scaled formulation

$$\sum_{n=0}^{\infty} \frac{\Re(\alpha\lambda_n)}{1+|\alpha\lambda_n|^2} = \infty \tag{23}$$

with $\alpha > 0$. In order to show that the above formulations are all equivalent, we need the following *Definition:* Two functions $f(z), g(z)$ mapping the closed right halfplane $C_+ = \{z : \Re(z) \geq 0\}$ to the closed positive real axis R_+ are C-equivalent, denoted $f(z) \equiv g(z)$, if there exist real constants $a > 0$ and $b > 0$ such that

$$af(z) \leq g(z) \leq bf(z) \quad \forall z \in C_+ \tag{24}$$

Another way to state the above definition is: $f(z) \equiv g(z)$ if and only if

$$0 < \inf_{z\in C_+} g(z)/f(z) \leq \sup_{z\in C_+} g(z)/f(z) < \infty \tag{25}$$

which has the simple interpretation that $f(z)$ and $g(z)$ are C-equivalent provided the quotient $g(z)/f(z)$ is bounded both below and above in C_+. It is easily proved that the relation $f(z) \equiv g(z)$ is an equivalence relation. Also, when $f(z) \equiv g(z)$, it is straightforward to show that the convergence or divergence of the infinite sum $\sum f(z_n)$, with z_n in C_+, is equivalent respectively with the convergence or divergence of the infinite sum $\sum g(z_n)$. Hence the Müntz-Szász condition (20) can be written as

$$\sum_{n=0}^{\infty} g(\lambda_n) = \infty \qquad g(z) \equiv \xi(z) \tag{26}$$

where

$$\xi(z) = \Re(z)/(1+|z|^2) \tag{27}$$

as defined in Theorem 4.

Applying this to the first alternative formulation (21) we obtain

$$g_1(z)/\xi(z) = (1+|z|^2)/(1+|z-1/2|^2) \tag{28}$$

Clearly $g_1(z)/\xi(z)$ remains bounded below and above over the entire complex plane, and it is seen that we may even replace the real number 1/2 in the denominator of formula (28) with any complex number.

The second alternative formulation (22) is more interesting. We have

$$g_2(z)/\xi(z) = ((1-|z-1|/|z+1|)(1+|z|^2))/\Re(z) \tag{29}$$

Since the bilinear transformation $(z-1)/(z+1)$ maps C_+ onto the closed unit disk, we can write $(z-1)/(z+1) = re^{i\theta}$ with $0 \leq r \leq 1$. Substituting this in equation (29), we obtain

$$g_2(z)/\xi(z) = 2(1+r^2)/(1+r) \tag{30}$$

Considering that $(1+r^2)/(1+r)$ attains a global minimum at $r = -1+\sqrt{2}$ we readily obtain the inequalities

$$4(\sqrt{2}-1) \leq g_2(z)/\xi(z) \leq 2 \tag{31}$$

The third alternative formulation (23) yields

$$g_3(z)/\xi(z) = \alpha(1+|z|^2)/(1+\alpha^2|z|^2) \tag{32}$$

which is also bounded below and above. More precisely we have

$$\min(\alpha, 1/\alpha) \leq g_3(z)/\xi(z) \leq \max(\alpha, 1/\alpha) \tag{33}$$

As a last illustrative example we show that the formulation (8) in Theorem 4 does not provide a C-equivalent criterion. We have

$$g_4(z) = \xi(z)\phi(-\ln(\xi(z))) \tag{34}$$

and hence $g_4(z)/\xi(z) = \phi(-\ln(\xi(z)))$. Now it is easily shown that

$$0 \leq \xi(z) \leq \frac{1}{2} \quad \forall z \in C_+ \tag{35}$$

Since $\phi(t)$ is a positive decreasing function we have

$$\phi(\infty) \leq \phi(-\ln(\xi(z))) \leq \phi(\ln 2) \quad \forall z \in C_+ \tag{36}$$

But $\phi(\infty) = 0$ by the requirement $\int_0^\infty \phi(t)\, dt < \infty$. As a consequence $g_4(z)$ and $\xi(z)$ are not C-equivalent.

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