A Generalized Möbius Transform and Arithmetic Fourier Transforms
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Abstract—A general approach to arithmetic Fourier transforms is developed. The implementation is based on the concept of killer polynomials and the solution of an arithmetic deconvolution problem pertaining to a generalized Möbius transform. This results in an extension of the Bruns procedure, valid for all prime numbers, and in an AFT that extracts directly the sine coefficients from the Fourier series.

I. INTRODUCTION

The arithmetic Fourier transform (AFT) offers a convenient method to calculate the Fourier coefficients of a periodic function. Historically it was discovered by a mathematician named Bruns [1] at the beginning of this century. A similar, but different, algorithm was developed by Tufts and Sadasiv [2] for the calculation of the Fourier coefficients of even periodic functions. This method was extended in [3] for the calculation of the Fourier coefficients of both the even and odd components of a periodic function. Finally the Bruns approach was incorporated in [4] resulting in a more computationally balanced algorithm.

However, the basis of all these algorithms is the extraction of the cosine coefficients out of the periodic function itself and out of a shifted version of the periodic function. There is, to this author's knowledge, no algorithm that extracts the sine coefficients directly.

In this paper a general solution is presented, based on a generalized Möbius transform and on the existence of certain "killer" polynomials. In fact it is shown that with each killer polynomial corresponds a definite generalized Möbius transform. This results in a Bruns-like algorithm valid for all prime numbers and in an algorithm that extracts directly the sine coefficients.

II. A GENERALIZED MÖBIUS TRANSFORM

By an arithmetical function \( h(n) \) we mean a function defined for \( n = 1, 2, \ldots \). A multiplicative arithmetical function [6] is an arithmetical function \( h(n) \) such that \( h(mn) = h(m)h(n) \) for all relatively prime numbers \( m \) and \( n \). The expression \( km \) means "\( k \) divides \( m \).

Manuscript received June 16, 1993; revised March 24, 1994. The associate editor coordinating the review of this paper and approving it for publication was Prof. Ali N. Akansu.

IEEE Log Number 9404760.

Theorem 1: Let \( f_1, f_2, f_3, \ldots \) be a sequence of real numbers and \( \alpha(n) \), \( \beta(n) \) real arithmetical functions. For the transform pair

\[
S_n = \sum_{k=1}^{\infty} \alpha(k)f_{nk}
\]

\[
f_n = \sum_{k=1}^{\infty} \beta(k)S_{nk}
\]
to be valid, it is necessary and sufficient that

\[
\sum_{l=1}^{m} \alpha(k)\beta(l) = \sum_{l=1}^{m} \alpha(k)\beta\left(\frac{m}{k}\right) = \delta_{lm}
\]

where \( \delta_{lm} = 1 \) for \( m = 1 \) and 0 for all other values of \( m \).

Proof: Substituting (1) in (2) yields

\[
f_n = \sum_{k=1}^{\infty} \beta(k) \sum_{l=1}^{\infty} \alpha(l)f_{nk}\delta_{lm} = \sum_{m=1}^{\infty} f_{nm} \sum_{k=1}^{\infty} \beta(k)\alpha(l)
\]

and the proof is complete.

Equation (3) is called the arithmetic deconvolution equation. We do not discuss the convergence of the transform pair since in practice it is used only on a truncated series i.e. \( f_n = 0 \) for \( n > M \).

If \( \alpha(n) \) is multiplicative, then it is easy to prove [5] that \( \beta(n) \) is also a multiplicative arithmetical function. Moreover, in that case, we need only to solve the arithmetic deconvolution equation for values of \( n \) that are powers of the prime numbers. In other words

\[
\beta(1) = \alpha(1) = 1
\]

\[
\sum_{k=0}^{m} \alpha(p^k)\beta(p^{m-k}) = 0 \quad \forall m \geq 1 \quad \forall p \text{ prime.}
\]

Equation (3) can also be written in a concise fashion as

\[
\tilde{\alpha}(s)\tilde{\beta}(s) = 1
\]

where \( \tilde{\alpha}(s) \) and \( \tilde{\beta}(s) \) are the Dirichlet series

\[
\tilde{\alpha}(s) = \sum_{k=1}^{\infty} \alpha(k) k^{-s}
\]

\[
\tilde{\beta}(s) = \sum_{k=1}^{\infty} \beta(k) k^{-s}
\]

If all \( \alpha(n) = 1 \) then \( \tilde{\alpha}(s) \) is the Riemann zeta function and

\[
\tilde{\beta}(s) = 1/\tilde{\alpha}(s) = 1/\zeta(s) = \sum_{k=1}^{\infty} \mu(k) k^{-s}
\]

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where \( \mu(n) \) is the celebrated Möbius function defined as
\[
\mu(1) = 1, \\
\mu(p_1 p_2 \cdots p_m) = (-1)^m \text{ for } m \text{ different prime numbers } p_i \\
\mu(n) = 0 \text{ if } n \text{ is divisible by a square } \neq 1.
\]
This corresponds with the classical Möbius transform.

We shall now solve three arithmetic deconvolution problems pertaining to the development of the arithmetic Fourier transforms in the sequel.

**Theorem 2:** If \( \alpha(n) = (-1)^n \) then
\[
\beta(n) = -\mu(n) \text{ if } n \text{ odd } \\
\beta(n) = -2^n - 1 \mu(n) \text{ if } n \text{ even}
\]
where \( q \) is the largest number such that \( 2^q | n \).

**Proof:** Note that \( \alpha(n) \) is not multiplicative. It is straightforward to show that
\[
\tilde{\alpha}(s) = \sum_{k=1}^{\infty} (-1)^k s^{-k} = \zeta(s)(2^{1-s} - 1).
\]
Hence
\[
\tilde{\beta}(s) = \frac{1}{\tilde{\alpha}(s)} = -\sum_{k=1}^{\infty} \mu(k) s^{-k} \sum_{l=0}^{\infty} 2^l 2^{-ls}.
\]
This amounts to say that
\[
\beta(n) = -\sum_{k|n} \mu(k) 2^l.
\]
For \( n \) odd, the only term in the sum is the one with \( k = n \) and \( l = 0 \) and hence
\[
\beta(n) = -\mu(n).
\]
For \( n \) even, say \( n = 2^s n' \) with \( n' \) odd, the two not necessarily vanishing terms in the sum are the ones for \( l = q, q - 1 \). Hence
\[
\beta(n) = -2^s \mu(n') - 2^{s+1} \mu(2n') \text{ and since } \mu(2n') = -\mu(n')
\]
the result follows.

**Theorem 3:** Let \( m > 1 \) be a fixed number. Define the numbers \( \alpha(n) \) as
\[
\alpha(n) = 0 \text{ if } m|n \text{ else } \alpha(n) = 1.
\]
Then
\[
\beta(n) = \mu(n) \text{ if not } m|n \text{ else } \beta(n) = \mu(n) + \mu(m \hat{n})
\]
where \( \hat{n} \) is defined by \( n = m^t \hat{n} \) and \( t \) is the largest number such that \( m^t | n \).

**Proof:** By inspection we see that
\[
\tilde{\alpha}(s) = \sum_{k=1}^{\infty} [k^{-s} - (mk)^{-s}] = \zeta(s)(1 - m^{-s}).
\]
Hence
\[
\tilde{\beta}(s) = 1/\tilde{\alpha}(s) = \sum_{k=1}^{\infty} \mu(k) s^{-k} \sum_{l=0}^{\infty} m^{-ls}
\]
and
\[
\beta(n) = \sum_{km^n=n} \mu(k).
\]
If not \( m|n \) the result is obvious, and if \( m|n \) we have
\[
\beta(n) = \sum_{l=0}^{t} \mu(ilm^t).
\]
Since \( \mu(n) = 0 \) when \( n \) contains a squared factor, the result follows.

**Corollary:** Let \( q \) be a prime number and \( t \) a positive number. Define the numbers \( \alpha(n) \) as
\[
\alpha(n) = 0 \text{ if } q^t|n \text{ else } \alpha(n) = 1.
\]
Then
\[
\beta(n) = \mu(n) \text{ if not } q^t|n.
\]
If on the other hand \( q^t|n \), there are two cases to consider:
1) \( t = 1 \). Then \( \beta(n) = 0 \).
2) \( t > 1 \). Clearly \( n = q^t n' \) with \( n' \) and \( q \) relatively prime and \( d \geq t \). Then
\[
\begin{align*}
\beta(n) &= \sum_{kq^t=n} \mu(k) \sum_{0 \leq k \leq d} \mu(n'm^d) \\
&= \mu(n') \sum_{0 \leq k \leq d} \mu(q^d-t')
\end{align*}
\]
\[
\text{for } 0 \leq t \leq d.
\]
**Proof:** This is an application of the theorem with \( m = q^t \). Note that in this case \( \alpha(n) \) and \( \beta(n) \) are multiplicative. The only difficulty arises in the case \( q^t|n \). We have
\[
\beta(n) = \sum_{kq^t=n} \mu(k) \sum_{0 \leq t \leq d} \mu(q^d-t')
\]
in virtue of the multiplicativity of the Möbius function. For \( t = 1 \) we have \( \beta(n) = \mu(n') \mu(1) + \mu(q) = 0 \) and for \( t > 1 \) the result follows easily by inspecting and deleting all terms with an argument containing a squared factor.

A very interesting corollary of the corollary is the existence, for all prime numbers \( p \), of a generalized Möbius transform
\[
S_n = \sum_{k \neq i} f_{nk}
\]
\[
f_n = \sum_{k \neq i} \mu(k)Snk.
\]
Let \( a(n) \) be defined as

\[
\begin{align*}
    a(n) &= 0 \text{ if } n \equiv 71 \pmod{3} \quad (32a) \\
    a(n) &= 1 \text{ if } n \equiv 1 \pmod{3} \quad (32b) \\
    a(n) &= -1 \text{ if } n \equiv 2 \pmod{3}. \quad (32c)
\end{align*}
\]

Let \( V_1(n) \) be the number of distinct prime factors of \( n \) of the form \( 3w + 1 \). Then \( a(71) \) is given by

\[
\begin{align*}
    P(1) &= 1 \quad (33a) \\
    P(n) &= 0 \text{ if } n \equiv 31 \pmod{3} \quad (33b) \\
    B(n) &= 0, \text{ if } n \text{ is divisible by a square } (33c) \\
    P(n) &= -1, \text{ if } V_1(n) \text{ is odd } \quad (33d) \\
    P(n) &= 1, \text{ if } V_1(n) \text{ is even}. \quad (33e)
\end{align*}
\]

**Proof:** We first show that \( a(n) \) is a multiplicative arithmetical function. Suppose that \( m \) and \( n \) are relatively prime. If \( 3 \mid mn \) then one of the factors is divisible by 3 and hence \( a(mn) = a(m)a(n) = 0 \). If the product \( mn \) is of the form \( 3w + 1 \) then both \( m \) and \( n \) must either be of the form \( 3u + 1 \) or of the form \( 3v + 2 \) and again \( a(mn) = a(m)a(n) \). If the product \( mn \) is of the form \( 3w + 2 \) then one of \( m, n \) must be of the form \( 3u + 1 \) and the other of the form \( 3v + 2 \) and again \( a(mn) = a(m)a(n) \).

Hence we only have to solve the arithmetic deconvolution equation for powers of the prime numbers. For \( p = 3 \) we clearly have

\[
\beta(3^j) = 0 \quad \forall j > 0. \quad (34)
\]

For \( p \neq 3 \) we distinguish two cases:

1) \( p \equiv 1 \pmod{3} \). Then all odd powers of \( p \) are of the form \( 3v + 1 \) and equation (6) can be written as

\[
\sum_{k=0}^{t} \beta(p^k) = 0 \quad \forall t > 0. \quad (35)
\]

This has the solution

\[
\beta(p) = -1, \quad \beta(p^t) = 0 \quad \forall t > 1. \quad (36)
\]

2) \( p \equiv 2 \pmod{3} \). Then all odd powers of \( p \) are of the form \( 3v + 2 \) and all even powers are of the form \( 3v + 1 \) and (6) yields

\[
\sum_{k=0}^{t} \beta(p^k)(-1)^{t-k} = 0 \quad \forall t > 0. \quad (37)
\]

This has the solution

\[
\beta(p) = 1, \quad \beta(p^t) = 0 \quad \forall t > 1. \quad (38)
\]

This completes the proof, taking into account the multiplicativity of \( \beta(n) \). We will refer to this multiplicative arithmetical function as \( \mu(3n) \) since like \( \mu(n) \), it maps the natural numbers on the set \( \{-1, 0, 1\} \).
2) It is not possible to recover both the sine coefficients
and the cosine coefficients in one transform. As a
consequence the trigonometric polynomials should be
chosen such that they "kill" either the cosine or the
sine coefficients. Hence we impose one of the following
conditions

\[ R2. \quad T_n(2\pi k/p_n) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, p_n - 1 \] (48a)

or \( T_n(2\pi k/p_n) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, p_n - 1. \) (48b)

In the first case we call the polynomial \( A_n(z) \) a sine
killer, and in the second case a cosine killer. Note
that the missing coefficients can be recovered in both
cases by applying the transform to the shifted function
\( F(\theta + \eta) \) for two values of \( \eta \) [3], [4].

3) Most of the coefficients \( T_n(2\pi k/qn) \)—in the case of a
sine killer—or \( T_n(2\pi k/qm) \)—in the case of a cosine
killer—should vanish except for certain values of \( k \)
belonging to a set of multiples of \( m \). In other words

\[ R3. \quad T_n(2\pi k/qm) = 0 \quad \text{or} \quad \hat{T}_n(2\pi k/qm) = 0 \]

with \( 1 \leq r \leq q \) and \( 0 \leq b_1 < b_2 < \cdots < b_r < q. \)

4) Last but not least, we should have

\[ R4. \quad \text{The values} \quad T_n(2\pi b_k/q) = \xi_k \quad \text{or} \quad \hat{T}_n(2\pi b_k/q) = \hat{\xi}_k \quad \text{are not dependent on} \quad n. \]

This then leads to the following formulas, susceptible to be
inverted by a generalized Möbius transform

\[ S_n = C_n = \sum_{k=1}^{r} \xi_k \sum_{l=0}^{\infty} f_{n}[l+b_k] \] (50)

the case of a sine killer, and

\[ S_n = \hat{C}_n = \sum_{k=1}^{r} \hat{\xi}_k \sum_{l=0}^{\infty} g_{n}[l+b_k] \] (51)

in the case of a cosine killer. Equations (50) and (51) have the
form of a generalized Möbius transform as in Theorem 1
if one takes—for a sine killer

\[ \alpha(n) = \xi_k \quad \text{if} \quad n \quad \text{(mod} \quad q) = b_k \quad \text{else} \quad \alpha(n) = 0 \] (52)

and similarly for a cosine killer.

**Theorem 5:** There exist polynomials \( A_n(z) \) with real
coefficients of the form

\[ A_n(z) = \frac{1}{n} [z^n - 1]/(z^q - 1)] \Phi(z) \] (53)

where \( \Phi(z) \) is a polynomial of degree \( q - 1 \) that satisfy all
requirements R1-R4.

**Proof:** R1 is obvious. Clearly

\[ A_n(e^{2\pi i k/n}) = T_n(2\pi k/n) + i\hat{T}_n(2\pi k/nq) = 0 \] (54)

for all values of \( k \) that are not multiples of \( n. \) This satisfies
R2-R3 in part. For values of \( k \) that are multiples of \( n, \) say
\( k = bn \) we have

\[ A_n(e^{2\pi i b/q}) = T_n(2\pi b/q) + i\hat{T}_n(2\pi b/q) = \Phi(e^{2\pi i b/q}). \] (55)

Clearly R4 is satisfied. To satisfy the rest of R2-R3 we assign
values \( \Phi(e^{2\pi i b/q}) = \kappa_b \) for \( b = 0, 1, \ldots, q - 1. \) Hence the set
\( B = \{b_1, b_2, \ldots, b_r\} \) is also the set \( \{b\kappa_b \neq 0\}. \) For R2-R3 to
be completely satisfied, we have to investigate whether it is
possible to have all \( \kappa_b \) either real or purely imaginary. By the
Lagrange interpolation formula [5] we have

\[ \Phi(z) = \frac{1}{q} \sum_{k=0}^{q-1} \kappa_k e^{2\pi i b/q}(z^q - 1)/(z - e^{2\pi i b/q}). \] (56)

It is seen that for \( \Phi(z) \) to be a polynomial with real
coefficients, the following conditions need to hold:

1) \( \kappa_0 \) is real,
2) if \( q \) is even \( \kappa_q/2 \) is real,
3) \( \kappa_b = \kappa_{q-b} \) for all other \( b. \)

For a sine killer this means that \( B \) can contain the singletons
\( \{0\}, \{q/2\} \) (if \( q \) is even) and possibly doubletons \( \{b, q-b\} \)
with

\[ \kappa_b = \xi_b = \xi_{q-b} = \xi_{q-b} \quad \text{for} \quad b \neq 0, q/2. \] (57)

For a cosine killer \( B \) can contain only doubletons \( \{b, q-b\} \)
with

\[ \kappa_b = i\hat{\xi}_b = -i\hat{\xi}_{q-b}, \quad \text{for} \quad b \neq 0, q/2. \] (58)

The proof is complete. Note that there are no cosine killers
for \( q \leq 3. \)

In order to show what this means in practice we give the
following examples, which include the original AFT, the Bruns
AFT, a generalization of the Bruns approach in terms of a
selected prime number and a cosine killer of order 2.

**Example 1:** The original AFT [2], [3]. Here \( \Phi(z) \equiv 1 \) and
\( q = 1. \) Hence

\[ A_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} \xi_k \] (59)

We have \( r = 1, b_1 = 0, \xi_1 = 1. \) and hence

\[ S_n = \sum_{l=1}^{\infty} f_{nl}. \] (60)

The inversion formula is given by

\[ f_n = \sum_{l=1}^{\infty} \mu(l)(S_{nl} - f_0). \] (61)
Example 2: The Bruns AFT \([1, 4]\), Here \(\Phi(z) = (1 - z)/2\) and \(q = 2\). Hence
\[
A_n(z) = \frac{1}{2^n} \sum_{k=0}^{2n-1} (-z)^k.
\]  
(62)

We have \(r = 1, b_1 = 1, \xi_1 = 1, \) and hence
\[
S_n = \sum_{l=0}^{\infty} f_n(2l+1).
\]  
(63)

By the corollary of Theorem 3 the inversion formula is given by
\[
f_n = \sum_{l=0}^{\infty} \mu(l) S_{nl}.
\]  
(64)

Example 3: Here we take \(\Phi(z) = z\) and \(q = 2\). Hence
\[
A_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} z^{2k+1}.
\]  
(65)

We have \(r = 2, b_1 = 0, b_2 = 1, \xi_1 = 1, \xi_2 = -1.\) Hence
\[
S_n - f_0 = \sum_{l=1}^{n-1} (-1)^l f_{nl}.
\]  
(66)

The inversion coefficients are given as in Theorem 2. Note that, in this author's view, this does not seem a very promising approach, as there are inversion coefficients of order of magnitude \(O(2^n)\).

Example 4: A generalization of the Bruns procedure. Let \(q > 1\) be any number
\[
\Phi(z) = 1 - \frac{1}{q}(z^q - 1)/(z - 1)
\]  
(67)

\[
A_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} z^{qk} - \frac{1}{qn} \sum_{k=0}^{n-1} z^k.
\]  
(68)

Note that for \(q = 2\) this is the Bruns polynomial. We have
\[
r = q - 1, \quad b_k = k, \quad \xi_k = 11 \leq k \leq r
\]  
(69)

Hence
\[
S_n = \sum_{l=1}^{q-1} \sum_{k=0}^{\infty} f_n(ql+k) = \sum_{\text{not } q|k} f_{nk}.
\]  
(70)

the order \(q\) of the transform is a prime number then by the corollary of Theorem 3 the inversion formula is simply
\[
f_n = \sum_{\text{not } q|k} \mu(k) S_{nk}.
\]  
(71)

Example 5: The cosine killer of order 3. Here \(\Phi(z) = -\frac{1}{\sqrt{3}} z(1 - z), q = 3\) and
\[
A_n(z) = \frac{1}{n\sqrt{3}} \sum_{k=0}^{n-1} [z^{3k+1} - z^{3k+2}].
\]  
(72)

We have
\[
r = 2, \quad b_1 = 1, \quad b_2 = 2, \quad \xi_1 = 1, \quad \xi_2 = -1.
\]  
(73)

Hence
\[
S_n = \sum_{l=0}^{\infty} [g_n(3l+1) - g_n(3l+2)].
\]  
(74)

The inversion formula is given in Theorem 4
\[
g_n = \sum_{k=1}^{\infty} \mu_3(k) S_{nk}.
\]  
(75)

Note that
\[
S_n = \frac{1}{n\sqrt{3}} \sum_{k=0}^{n-1} k = 0
\]
\[
\times \left[F\left(\frac{2\pi}{n}(k + \frac{1}{3})\right) - F\left(\frac{2\pi}{n}(k + \frac{2}{3})\right)\right].
\]  
(76)

IV. CONCLUSION

A general procedure to develop arithmetic Fourier transforms has been presented in this paper. It is based on the concepts of a generalized Möbius transform and killer polynomials. The results include a generalized Bruns procedure that is easily computable if the order \(q\) is a prime number, and a transform of order 3 that extracts directly the sine coefficients of the Fourier series. But this is not exhaustive. For any killer polynomial a generalized Möbius transform pair can be developed. The only problem is to solve the associated arithmetic deconvolution problem.

REFERENCES


Luc Knockaert (M'81) received the Physical Engineer, Telecommunications Engineer, and Ph.D. degrees in Applied Sciences from the State University of Ghent, Belgium, in 1974, 1977, and 1987, respectively. His current interest is the application of algebraic, number-theoretic, and statistical methods in signal processing and identification.