STRICT PASSIVITY AND MODEL REDUCTION

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INTRODUCTORY OUTLINE

• GENERALITIES
• STRICT PASSIVITY
• TESTS AND ENFORCEMENT
• PROJECTION BASED MOR
• STRICT PASSIVITY AND MOR
• MARKOV-KRYLOV MOR
• BANDLIMITED MOR
STATE SPACE FORMAT

\[ \dot{x} = Ax + Bu \]
\[ y = L^T x + Du \]
Multiport, i.e., \( \text{size}(L) = \text{size}(B) \), and minimal realization

TRANSFER FUNCTION

\[ y = H(s)u \]
\[ H(s) = L^T(sI - A)^{-1}B + D \]
STRICT PASSIVITY

\[ H(i\omega) + H(i\omega)^H > 0 \quad \forall \omega \]

SCATTERING REPRESENTATION

\[ y = a + b, \quad u = a - b, \quad b = S(s)a \]
\[ H(s) = [I - S(s)]^{-1}[I + S(s)] \]
\[ S(s) = [H(s) + I]^{-1}[H(s) - I] \]

\[ I - S(i\omega)^H S(i\omega) > 0 \quad \forall \omega \]
i.e., \( \|S\|_\infty < 1 \)
STRICT PASSIVITY (ctd.)

Positive-real lemma : linear matrix inequality (LMI)

$$\exists P = P^T > 0 : \begin{bmatrix} A^T P + PA & PB - L \\ B^T P - L^T & -D - D^T \end{bmatrix} < 0$$

i.e. $$\exists P = P^T > 0$$, nonsingular $$Q$$ and rectangular $$W$$ :

$$A^T P + PA = -QQ^T < 0$$
$$PB - L = -QW$$
$$D + D^T > W^TW \geq 0$$
STRICT PASSIVITY (ctd.)

Eliminating $Q$ and $W$ yields

$$D + D^T > -(L - PB)^T (A^T P + PA)^{-1} (L - PB)$$

Putting $G = -PA$ implies $G + G^T = QQ^T > 0$

taking $R = PB$ we have

$$D + D^T > (L - R)^T (G + G^T)^{-1} (L - R)$$

Hence all strictly passive $H(s)$ can be written in the descriptor state space format

$$H(s) = L^T (sP + G)^{-1} R + \frac{1}{2} (L - R)^T (G + G^T)^{-1} (L - R) + D_1$$

with $D_1 + D_1^T > 0$, $P = P^T > 0$, $G + G^T > 0$. 

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A STRICTLY PASSIVE RANDOM GENERATOR

The MATLAB code is:

\[
P = \text{randn}(n); P = P \times P' + \text{epsil} \times \text{eye}(n);
\]

\[
L = \text{randn}(n, p); R = \text{randn}(n, p);
\]

\[
G = \text{randn}(n); G = G \times G' + \text{epsil} \times \text{eye}(n);
\]

\[
D0 = 0.5 \times (L - R)' \times G \backslash (L - R);
\]

\[
Z = \text{randn}(n); Z = Z - Z'; G = G + Z;
\]

\[
D = \text{randn}(p); D = D \times D';
\]

\[
Z = \text{randn}(p); Z = Z - Z'; D = D + Z + D0;
\]

\[
sys = \text{dss}(-G, R, L', D, P);
\]

\[
\text{epsil} \text{ is a small positive number, and the command } \text{sys} = \text{dss}(A, B, C, D, E)
\]

creates a descriptor system with transfer function \[ H(s) = C(sE - A)^{-1}B + D. \]
We can test strict passivity by solving the algebraic Riccati equation:

\[ A^T X + X A + (L - X B) (D + D^T)^{-1} (L - X B)^T = 0 \]

and verifying that \( X = X^T \) is a stabilizing solution, i.e., \( X < 0 \).

Alternatively, one can transform \( H(s) \) to scattering form \( S(s) \) by means of

\[
\begin{align*}
\dot{x} &= \left[ A - B(D + I)^{-1} L^T \right] x + 2B(D + I)^{-1} a \\
b &= (D + I)^{-1} L^T x + (D + I)^{-1}(D - I)a
\end{align*}
\]

and test whether \( \| S \|_\infty < 1 \).
PASSIVITY ENFORCEMENT

- Eigenvalue approach: Gustavsen-Semlyen
- Hamiltonian methods: Grivet-Talocia, Boyd-Balakrishnan-Kabamba
- Nevanlinna-Pick interpolation: Coelho-Phillips-Silveira
- Optimization methods: Balakrishnan-Kashyap, Coelho-Phillips-Silveira
PROJECTION-BASED MODEL REDUCTION

Descriptor state space model
\[
P\dot{x} + Gx = Bu \\
y = L^T x + Du
\]
and reduced model
\[
\tilde{P}\dot{x} + \tilde{G}x = \tilde{B}u \\
y = \tilde{L}^T x + Du
\]
solely dependent on a rectangular matrix \( U \) such that
\[
\tilde{L} = U^T L \quad \tilde{B} = U^T B \\
\tilde{P} = U^T PU \quad \tilde{G} = U^T GU
\]
often with \( U^T U = I \), i.e., \( U \) is column-orthogonal.
Recall that

\[ H(s) = L^T(sP + G)^{-1}B + \frac{1}{2}(L - B)^T(G + G^T)^{-1}(L - B) + D_1 \]

Projection-based model order reduction

\[ \tilde{G} = U^TGU \quad \tilde{P} = U^TPU \quad \tilde{L} = U^TL \quad \tilde{B} = U^TB \]

with \( U \) full rank rectangular. Then

\[ H_2(s) = \tilde{L}^T(s\tilde{P} + \tilde{G})^{-1}\tilde{B} + \frac{1}{2}(\tilde{L} - \tilde{B})^T(\tilde{G} + \tilde{G}^T)^{-1}(\tilde{L} - \tilde{B}) + D_1 \]

is strictly passive and

\[ (\tilde{L} - \tilde{B})^T(\tilde{G} + \tilde{G}^T)^{-1}(\tilde{L} - \tilde{B}) \leq (L - B)^T(G + G^T)^{-1}(L - B) \]

Hence

\[ H_r(s) = \tilde{L}^T(s\tilde{P} + \tilde{G})^{-1}\tilde{B} + \frac{1}{2}(L - B)^T(G + G^T)^{-1}(L - B) + D_1 \]

is strictly passive.
HOW TO FIND A SUITABLE $U$ #1?

Laurent-Taylor expansion of $H(s)$ in the vicinity of $s = \infty$

$$H(s) = L^T (sP + G)^{-1} B + D$$

$$= D + \sum_{k=0}^{\infty} (-1)^k s^{-k-1} L^T (P^{-1}G)^k P^{-1} B$$

With $P^{-1}G = \Omega = -A$ and $P^{-1}B = R$

$$H(s) = \sum_{k=-1}^{\infty} (-1)^k s^{-k-1} M_k$$

The coefficients $M_k = L^T \Omega^k R$ and $M_{-1} = -D$

are the Markov moments of $H(s)$ at $s = \infty$. 
How to find a suitable $U$ #1? (ctd.)

Krylov matrix

$$\mathcal{K} = [R, \Omega R, \Omega^2 R, \ldots, \Omega^{q-1} R]$$

orthonormal basis for the columns of $\mathcal{K}$

equivalent with the 'economy-size' SVD

$$\mathcal{K} = U \Sigma V^T$$

where $U^T U = I$.

The new Markov moments are given by

$$\tilde{\mathcal{M}}_{-1} = -D \quad \tilde{\mathcal{M}}_k = \tilde{L}^T \tilde{\Omega}^k \tilde{R}$$

We can prove that $\tilde{\mathcal{M}}_k = \mathcal{M}_k$ for $k = 0, 1, \ldots, q - 1$. 
BILINEAR TRANSFORMATION

Instead of $s = \infty \ (1/s)$ or $s = 0 \ (s)$
in general: bilinear transformation

$s = \frac{\alpha u + \beta}{\gamma u + \delta} \quad \alpha \delta - \beta \gamma \neq 0$

Transfer function in the $u$—domain is

$H(s) = (\gamma u + \delta) L^T [u(\alpha P + \gamma G) + (\beta P + \delta G)]^{-1} B + D$

With $\hat{R} = (\alpha P + \gamma G)^{-1} B \quad \hat{\Omega} = (\alpha P + \gamma G)^{-1} (\beta P + \delta G)$

we obtain another, say $\hat{U}$ matrix, but strict passivity is guaranteed.
Suppose we have an orthonormal Hilbert space basis \( \{ \psi_n(s) \} \) with respect to some scalar product. Expand

\[
Z(s)^{-1} B = \sum_{k=0}^{q-1} T_k \psi_k(s) + R_q(s)
\]

where the (regular) matrix pencil \( Z(s) = sP + G \) and \( R_q(s) \) is a remainder term.

Writing the projection matrix \( K_q \) and its 'economy-size' SVD as

\[
K_q = [T_0, T_1, \ldots, T_{q-1}] = U \Sigma V^T
\]

yields the column-orthogonal matrix \( U \). The orthogonal projector \( P_r = UU^T \) leaves \( K_q \) invariant, i.e. \( P_r K_q = K_q \)
HOW TO FIND A SUITABLE $U$ # 2? (ctd.)

Hence

$$U^T B = \sum_{k=0}^{q-1} U^T Z(s) U U^T T_k \psi_k(s) + U^T Z(s) R_q(s)$$

$$U (U^T Z(s) U)^{-1} U^T B = \sum_{k=0}^{q-1} T_k \psi_k(s) + Q(s) R_q(s)$$

$$Q(s) = U (U^T Z(s) U)^{-1} U^T Z(s)$$

Note that $Q(s)$ is idempotent, i.e. $Q(s)^2 = Q(s)$
CONSEQUENCES FOR ROM

\[
H_r(s) = \left( L^T U (U^T Z(s) U)^{-1} U^T B + D \right)
\]

\[
H_r(s) = \left( L^T U (s U^T PU + U^T GU)^{-1} U^T B + D \right)
\]

\[
H_r(s) = \sum_{k=0}^{q-1} L^T T_k \psi_k(s) + L^T Q(s) R_q(s) + D
\]

\[
H(s) = \sum_{k=0}^{q-1} L^T T_k \psi_k(s) + L^T R_q(s) + D
\]

\[
H(s) - H_r(s) = L^T (I - Q(s)) R_q(s)
\]

If the remainder term \( R_q(s) \) is small, then \( H_r(s) \approx H(s) \).
LAGUERRE-KAUTZ BASIS

The simple Laguerre basis

$$\phi_n(s) = \sqrt{2\gamma} \frac{(s - \gamma)^n}{(s + \gamma)^{n+1}} \quad n = 0, 1, \ldots$$

is complete orthonormal with respect to the scalar product

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(i\omega) \overline{g(i\omega)} \, d\omega$$

The same can be said about the two-pole Kautz basis

$$\phi_{2n}(s) = \sqrt{2\gamma} (s + \sqrt{\gamma^2 + \sigma^2}) \frac{((s - \gamma)^2 + \sigma^2)^n}{((s + \gamma)^2 + \sigma^2)^{n+1}} \quad n = 0, 1, \ldots$$

$$\phi_{2n+1}(s) = \sqrt{2\gamma} (s - \sqrt{\gamma^2 + \sigma^2}) \frac{((s - \gamma)^2 + \sigma^2)^n}{((s + \gamma)^2 + \sigma^2)^{n+1}} \quad n = 0, 1, \ldots$$
BANDLIMITED LAGUERRE-KAUTZ BASIS

We have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_n(i\omega)\phi_m(i\omega) \, d\omega = \delta_{n,m}$$

With $B$ the symmetric narrowband support $B = [-\beta, -\alpha] \cup [\alpha, \beta]$, take the rational frequency coordinate transform $\nu \in B \rightarrow \omega \in \mathbb{R}$:

$$\omega = \zeta(\nu) \overset{\text{def}}{=} \frac{\beta^2}{\nu} \frac{\nu^2 - \alpha^2}{\beta^2 - \nu^2} \quad \alpha \leq |\nu| \leq \beta$$
BANDLIMITED LAGUERRE-KAUTZ BASIS (ctd.)

Then

\[
\frac{1}{2\pi} \int_B \psi_n(i\omega)\overline{\psi_m(i\omega)} \, d\omega = \delta_{n,m}
\]

where \( \psi_n(s) = \tau(s) \phi_n(\eta(s)) \) with

\[
\eta(s) = \frac{\beta^2}{s} \frac{s^2 + \alpha^2}{s^2 + \beta^2}, \quad \tau(s) = \beta \frac{s^2 + s\sqrt{\beta^2 + 2\alpha\beta - 3\alpha^2 + \alpha\beta}}{s(s^2 + \beta^2)}
\]
CALCULATING THE T-COEFFICIENTS

Due to bandlimited orthonormality, the matrices $T_k$ are obtained by means of

$$T_k = \frac{1}{2\pi} \int_{B} Z(i\omega)^{-1} B \psi_k(i\omega) \, d\omega$$

Opting for $M$ quadrature points $\alpha \leq \omega_l \leq \beta$ with positive weights $b_l$, we obtain the approximation

$$T_k \approx \frac{1}{\pi} \Re \left\{ \sum_{l=1}^{M} b_l \psi_k(i\omega_l) \right\}$$

where the matrices $W_l$ are the solutions of the linear equations $Z(i\omega_l)W_l = B$. 
BANDLIMITED LAGUERRE-KAUTZ ROM ALGORITHM

1 Select $q$, the bandwidth $[\alpha, \beta]$ and the bandlimited Laguerre-Kautz basis $\{\psi_n(s)\}$
2 Select $M$ quadrature points $\omega_k$ in $[\alpha, \beta]$ with appropriate weights $b_k$
3 Solve the linear equations to obtain the $W_k$
4 Calculate the matrices $T_k$ by means of the quadrature rule
5 Construct the projection matrix $K_q$ and compute its 'economy-size' SVD
6 Perform $\tilde{L} = U^T L$, $\tilde{B} = U^T B$ and $\tilde{P} = U^T P U$, $\tilde{G} = U^T G U$,
7 The reduced order model is $H_r(s) = \tilde{L}^T \left(s\tilde{P} + \tilde{G}\right)^{-1} \tilde{B} + D$
QUESTIONS ?