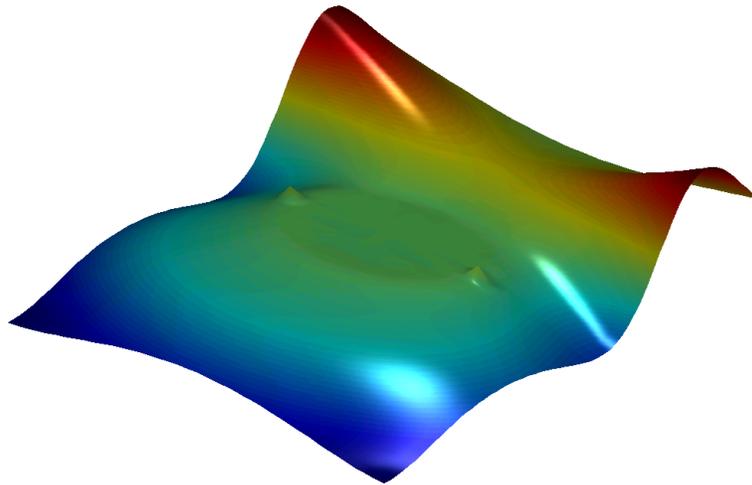


Numerical Assignment: the March-on-in-Time Algorithm

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1 Introduction

In a lot of applications the spectrum of the signal is very smallbanded, i.e. only a limited number of harmonic signals appear. These situations can be efficiently simulated by solving Maxwell's equations in the frequency domain for this limited amount of frequencies. However, for some applications knowledge of the behavior over a broad spectral band is required. Consider for instance the propagation of a bit through the conductors on a PCB or the reflection of a short radar pulse at a certain target. These situations would require simulations over the entire frequency band. Very often this is unacceptable because it is highly computationally intensive. Alternatively, these simulations could be done in the time domain. The signal is very often a pulse. This pulse has a spectrum that contains the entire relevant frequency range. After a Fourier Transform one obtains information about the system at all

these frequencies. Even though these simulations require a more advanced solver than their frequency domain counterpart, they are often preferred because of their higher efficiency. In this project you are going to implement such a 2D time domain solver for simulation of scattering at perfectly electrically conducting objects. The resulting program can then be used to analyze a number of simple problems.

2 The wave equation in the 2D TM case

The initial calculations take place in the frequency domain for the sake of simplicity. We start with Maxwell's equations in free space in the frequency domain:

$$\nabla \cdot \mathbf{d}(\mathbf{r}) = -\frac{1}{j\omega} \nabla \cdot \mathbf{j}(\mathbf{r}) \quad (1)$$

$$\nabla \times \mathbf{e}(\mathbf{r}) = -j\omega \mathbf{b}(\mathbf{r}) \quad (2)$$

$$\nabla \cdot \mathbf{b}(\mathbf{r}) = 0 \quad (3)$$

$$\nabla \times \mathbf{h}(\mathbf{r}) = j\omega \mathbf{d}(\mathbf{r}) + \mathbf{j}(\mathbf{r}) \quad (4)$$

with the constitutive equations:

$$\mathbf{d}(\mathbf{r}) = \varepsilon \mathbf{e}(\mathbf{r}) \quad (5)$$

$$\mathbf{b}(\mathbf{r}) = \mu \mathbf{h}(\mathbf{r}) \quad (6)$$

In a conductor we have, additionally, Ohm's law: $\mathbf{j}(\mathbf{r}) = \sigma \mathbf{e}(\mathbf{r})$. For very good conductors the conductivity is extremely high (for instance copper with $\sigma \approx 10^7$ Siemens / meter). A physical current density can only be caused by an electric field that is very near to zero. An idealized model for this situation is the perfectly electrical conductor (PEC). In such a conductor all fields are identically zero and currents can only flow at the surface of such a conductor.

We wish to solve these equations in geometries that are invariant in the z -direction. Therefore, we can search for solutions that are independent of the z -direction. When such a solution is substituted in the curl equations and then separated into longitudinal and transversal components, we obtain the following equations:

$$\nabla_t \times e_z(\boldsymbol{\rho}) \mathbf{u}_z = -j\omega \mu \mathbf{h}_t(\boldsymbol{\rho}) \quad (7)$$

$$\nabla_t \times \mathbf{e}_t(\boldsymbol{\rho}) = -j\omega \mu h_z(\boldsymbol{\rho}) \mathbf{u}_z \quad (8)$$

$$\nabla_t \times h_z(\boldsymbol{\rho}) \mathbf{u}_z = j\omega \varepsilon \mathbf{e}_t(\boldsymbol{\rho}) + \mathbf{j}_t(\boldsymbol{\rho}) \quad (9)$$

$$\nabla_t \times \mathbf{h}_t(\boldsymbol{\rho}) = j\omega \varepsilon e_z(\boldsymbol{\rho}) \mathbf{u}_z + j_z(\boldsymbol{\rho}) \mathbf{u}_z \quad (10)$$

with: $\boldsymbol{\rho} = x\mathbf{u}_x + y\mathbf{u}_y$, $\mathbf{e}_t = e_x\mathbf{u}_x + e_y\mathbf{u}_y$, $\mathbf{h}_t = h_x\mathbf{u}_x + h_y\mathbf{u}_y$, $\mathbf{j}_t = j_x\mathbf{u}_x + j_y\mathbf{u}_y$, $\nabla_t = \partial_x \mathbf{u}_x + \partial_y \mathbf{u}_y$. With this assumption, Maxwell's equations decompose into two

sets of equations: the equations (7) and (10) contain only the unknowns e_z and \mathbf{h}_t (TM equations) while the equations (8) and (9) contain only the unknowns h_z and \mathbf{e}_t (TE problem).

In the following, we focus solely on the TM problem. We eliminate \mathbf{h}_t by substituting equation (7) into equations (10). That way we obtain the following equation for the electric field:

$$\nabla_t (\nabla_t \cdot e_z(\boldsymbol{\rho}) \mathbf{u}_z) - \nabla_t^2 e_z(\boldsymbol{\rho}) \mathbf{u}_z - k^2 e_z(\boldsymbol{\rho}) \mathbf{u}_z = -j\omega\mu j_z(\boldsymbol{\rho}) \quad (11)$$

with $k^2 = \omega^2\epsilon\mu$. The divergence in the first term is zero because ∇_t and \mathbf{u}_z are orthogonal. The following scalar Helmholtz equation for the longitudinal e_z field is obtained:

$$\nabla_t^2 e_z(\boldsymbol{\rho}) + k^2 e_z(\boldsymbol{\rho}) = j\omega\mu j_z(\boldsymbol{\rho}) \quad (12)$$

We obtain the time domain equation by application of an inverse Fourier Transform:

$$\nabla_t^2 E_z(\boldsymbol{\rho}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E_z(\boldsymbol{\rho}, t) = \mu \frac{\partial}{\partial t} J_z(\boldsymbol{\rho}, t) \quad (13)$$

When solving such linear differential equations we first search for a Green function that is the solution of:

$$\nabla_t^2 G(\boldsymbol{\rho}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(\boldsymbol{\rho}, t) = -\delta(\boldsymbol{\rho})\delta(t) \quad (14)$$

The Green function G is the field caused by an elementary line source at position $\boldsymbol{\rho} = 0$ that sends a delta pulse at a time $t = 0$. In the following we will omit the index z to simplify the notation.

3 The time domain 2D Green function

In principle we can proceed as follows: first we solve the equation of the 2D frequency domain Green function and afterwards apply an inverse Fourier Transform to obtain the time domain Green function. This transformation can be calculated by using the residue theorem from complex analysis.

A more straightforward way, which additionally leads to a deeper physical understanding, is starting from the 3D Green function. We know that the frequency domain 3D Green function is given by:

$$G_{3D}(\mathbf{r}, \omega) = \frac{e^{-j\omega r/c}}{4\pi r}. \quad (15)$$

with $r = \sqrt{\rho^2 + z^2}$. This function has a very simple inverse Fourier Transform:

$$G_{3D}(\mathbf{r}, t) = \frac{\delta(t - r/c)}{4\pi r}. \quad (16)$$

A two dimensional source can be seen as an infinitely long line source in 3D. For calculation of the 2D Green function it is therefore sufficient to integrate the 3D Green function along the z-axis:

$$G_{2D}(\boldsymbol{\rho}, t) = \int_{-\infty}^{+\infty} \frac{\delta(t - r/c)}{4\pi r} dz. \quad (17)$$

We use a lemma about integrals and integrands containing Dirac distributions.

Theorem 1 *The integral*

$$\int_{-\infty}^{+\infty} g(z)\delta(f(z))dz$$

equals

$$\sum_i \frac{g(z_i)}{|f'(z_i)|}$$

where the summation runs over all real zeros of $f(z)$.

We have $f(z) = t - \sqrt{\rho^2 + z^2}/c$ en $g(z) = \frac{1}{4\pi\sqrt{\rho^2 + z^2}}$. The two zeros of $f(z)$ are $z = \pm\sqrt{(ct)^2 - \rho^2}$ when $ct > \rho$. In the other case there are no zeros and the result after integration is zero. Finally we obtain:

$$G_{2D}(\boldsymbol{\rho}, t) = \int_{-\infty}^{+\infty} \frac{\delta(t - r/c)}{4\pi r} = \frac{1}{2\pi} \frac{H(t - \rho/c)}{\sqrt{t^2 - \rho^2/c^2}}. \quad (18)$$

with H the Heaviside step function. Remember that the Green function gives the field caused by a Dirac pulse in both time and space. The field caused by a general current distribution along a line C can be calculated by a convolution with this current distribution:

$$E_z^s(\boldsymbol{\rho}, t) = -\frac{\mu}{2\pi} \int_C \frac{H(t - |\boldsymbol{\rho} - \boldsymbol{\rho}'|/c)}{\sqrt{t^2 - |\boldsymbol{\rho} - \boldsymbol{\rho}'|^2/c^2}} * \partial_t J_z(\boldsymbol{\rho}', t) dl' \quad (19)$$

where the asterix indicates temporal convolution. From now on, we will omit de subscript z for convenience.

4 The boundary integral equation

The problem we wish to solve is the following: an electromagnetic wave illuminates a very good (perfect) conductor (PEC). We want to know the total field. This field E consists of two parts: the incoming field and the scattered field. The incoming field E^i is the field in absence of the conductor. The scattered field E^s is the difference between the total field and the incoming field. This scattered field is caused by a yet

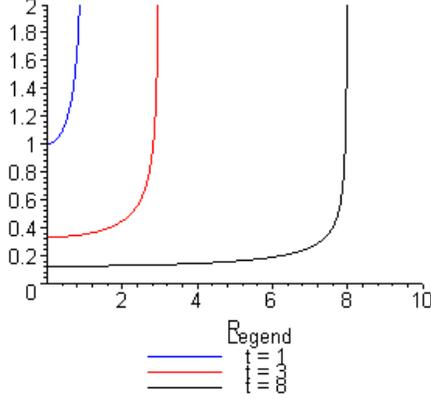


Figure 1: Green's function vs distance to the origin at $t = 1$, $t = 3$, en $t = 8$.

unknown current distribution $J(\boldsymbol{\rho}, t)$ on the boundary of the conductor and can be computed by convolution with the fundamental solution of the 2D wave equation:

$$E^s(\boldsymbol{\rho}, t) = -\frac{\mu}{2\pi} \int_C \frac{H(t - |\boldsymbol{\rho} - \boldsymbol{\rho}'|/c)}{\sqrt{t^2 - |\boldsymbol{\rho} - \boldsymbol{\rho}'|^2/c^2}} * \partial_t J(\boldsymbol{\rho}', t) dl' \quad (20)$$

where the integral is computed over the boundary of the conductor. As described above the electric field in the PEC is equal to zero. The boundary condition states that the electric field just outside the PEC has to be zero.

$$\lim_{\boldsymbol{\rho} \rightarrow C} (E^i(\boldsymbol{\rho}, t) + E^s(\boldsymbol{\rho}, t)) = 0, \quad \forall t > 0. \quad (21)$$

We use this condition as an equation to determine the unknown current density. If we use the integral representation for the scattered field, the boundary condition becomes:

$$E^i(\boldsymbol{\rho}, t) - \frac{\mu}{2\pi} \int_C \frac{H(t - |\boldsymbol{\rho} - \boldsymbol{\rho}'|/c)}{\sqrt{t^2 - |\boldsymbol{\rho} - \boldsymbol{\rho}'|^2/c^2}} * \partial_t J(\boldsymbol{\rho}', t) dl' = 0 \quad \forall t > 0, \forall \boldsymbol{\rho} \in C. \quad (22)$$

The unknown current distribution must satisfy this integral equation. In the next section we explain how this equation can be solved approximately through a discretization procedure. For simplicity, we don't solve the equations for $J(\boldsymbol{\rho}, t)$, but for the auxiliary quantity $U(\boldsymbol{\rho}, t) = \mu \frac{\partial}{\partial t} J(\boldsymbol{\rho}, t)$.

5 Discretization

To solve the integral equation (22) numerically, we have to perform a discretization. This is because we now have an infinite number of unknowns, namely the value

of $U(\boldsymbol{\rho}, t)$ at every point of the surface of the scatterer, at all positive moments in time. With a computer, however, we can only solve problems with a finite number of unknowns. Therefore we are going to approximate the function $U(\boldsymbol{\rho}, t)$ by a function with a finite number of unknowns and approximate the equation by a linear system of finite dimension. This is done by means of the following procedure:

- We approximate the curve that describes the boundary of the scatterer by a piecewise linear curve, i.e. the boundary of the scatterer is approximated by N_S segments.
- We approximate the unknown $U(\boldsymbol{\rho}, t)$ by a linear combination of products of N_S spatial and N_T temporal basis functions:

$$U(\boldsymbol{\rho}, t) \approx \sum_{n=1}^{N_S} \sum_{i=1}^{N_T} U_{i,n} f_n(\boldsymbol{\rho}) T_i(t)$$

- We substitute this approximation in the equation (22) and evaluate the equation at equidistant moments in time $t_j = j\Delta t, j = 1, \dots, N_T$ and in the centers of the N_S segments $\boldsymbol{\rho}_m, m = 1, \dots, N_S$.

The first discretization steps are illustrated in the figures 2, 3 and 4. As temporal basis functions we use pulses $T_i(t) = T(t - i\Delta t)$ with

$$T(t) = \begin{cases} 1, & -\Delta t < t < 0 \\ 0, & \text{elsewhere.} \end{cases} \quad (23)$$

We choose pulses for the spatial basis functions as well. The basis function $f_n(\boldsymbol{\rho})$ has the value 1 on segment n and zero on all others. Applying the third step in the discretisation procedure yields

$$E^i(\boldsymbol{\rho}_m, j\Delta t) - \frac{1}{2\pi} \sum_{n=1}^{N_S} \sum_{i=1}^{N_T} U_{i,n} \int_C \frac{H(t - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|/c)}{\sqrt{t^2 - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|^2/c^2}} * T_i(t) \Big|_{t=j\Delta t} f_n(\boldsymbol{\rho}') dl' \quad (24)$$

Causality, included in the equations by the Heaviside function, allows us to do the summation over i only up to j . Expanded, the temporal convolution looks like:

$$\frac{H(t - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|/c)}{\sqrt{t^2 - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|^2/c^2}} * T_i(t) \Big|_{t=j\Delta t} = \int \frac{H(t' - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|/c)}{\sqrt{t'^2 - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|^2/c^2}} T(j\Delta t - i\Delta t - t') dt' \quad (25)$$

In this form, it is obvious that the convolution depends only on the difference between i and j . Therefore, we introduce the new temporal summation index $k = j - i$:

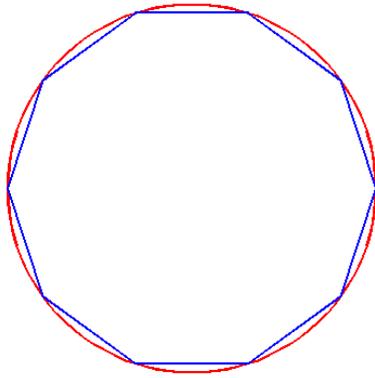


Figure 2: Approximation of the boundary of the scatterer by linear segments.

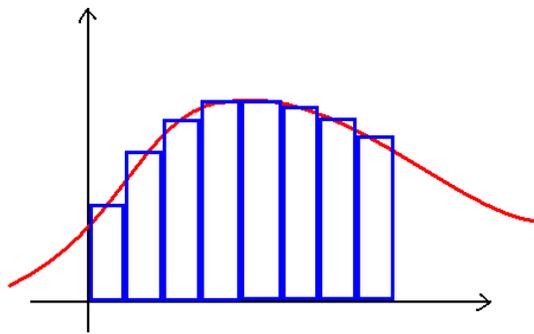


Figure 3: Approximation of the temporal dimension by pulses.

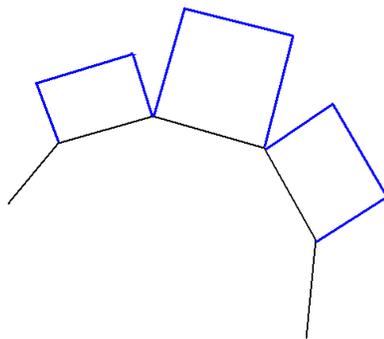


Figure 4: Approximation of the spatial dimension by pulses.

$$E^i(\boldsymbol{\rho}_m, j\Delta t) - \frac{1}{2\pi} \sum_{n=1}^{N_S} \sum_{k=0}^{j-1} U_{j-k,n} \int_C \frac{H(t - |\boldsymbol{\rho} - \boldsymbol{\rho}'|/c)}{\sqrt{t^2 - |\boldsymbol{\rho} - \boldsymbol{\rho}'|^2/c^2}} * T(t) \Big|_{t=k\Delta t} f_n(\boldsymbol{\rho}') dl' \quad (26)$$

or in matrix notation:

$$\mathbf{V}_j + \sum_{k=0}^{j-1} \mathbf{Z}_k \cdot \mathbf{U}_{j-k} = 0, \quad j = 1, \dots, N_T \quad (27)$$

with $(\mathbf{V}_j)_m = E^i(\boldsymbol{\rho}_m, j\Delta t)$, $(\mathbf{U}_i)_n = U_{i,n}$ and

$$(\mathbf{Z}_k)_{m,n} = -\frac{1}{2\pi} \int_{C_n} \frac{H(t - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|/c)}{\sqrt{t^2 - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|^2/c^2}} * T(t) \Big|_{t=k\Delta t} f_n(\boldsymbol{\rho}') dl' \quad (28)$$

with C_m segment m . For $j = 1$ this system reduces to

$$\mathbf{V}_1 + \mathbf{Z}_0 \cdot \mathbf{U}_1 = 0. \quad (29)$$

This equation can be solved directly for \mathbf{U}_1 . For $j = 2$ one obtains a system that contains \mathbf{U}_1 en \mathbf{U}_2 . This system can easily be solved for \mathbf{U}_2 , because you know \mathbf{U}_1 by now. This way, you march on in time, until you know the solution for all moments in time. This explains the name of the algorithm. An example of the solution that is obtained in this manner is shown in figure 5.

All that remains now is the calculation of the integrals. We start with the temporal convolution:

$$\int \frac{H(t' - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|/c)}{\sqrt{t'^2 - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|^2/c^2}} T(k\Delta t - t') dt' = \int_a^b dt' \frac{1}{\sqrt{t'^2 - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|^2/c^2}} \quad (30)$$

with, considering the supports of the Heaviside function and the temporal basis functions, $a = \max(k\Delta t, |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|/c)$ and $b = \max((k+1)\Delta t, |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|/c)$. This integration can be done analytically and the result is:

$$F(k, \boldsymbol{\rho}_m, \boldsymbol{\rho}') = \log \frac{b + \sqrt{b^2 - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|^2/c^2}}{a + \sqrt{a^2 - |\boldsymbol{\rho}_m - \boldsymbol{\rho}'|^2/c^2}}. \quad (31)$$

In figure 6 this function has been evaluated as a function of $|\boldsymbol{\rho}_m - \boldsymbol{\rho}'|/c$ for $t = 0$, $t = 3$, and $t = 8$. To calculate the elements of \mathbf{Z}_k , we need to do another spatial integration. These are approximated numerically by means of a Gaussian quadrature rule of order N_G :

$$(\mathbf{Z}_k)_{m,n} = -\frac{l_n}{2\pi} \sum_{g=1}^{N_G} w_g F(k, \boldsymbol{\rho}_m, \boldsymbol{\rho}_{n,g}) \quad (32)$$

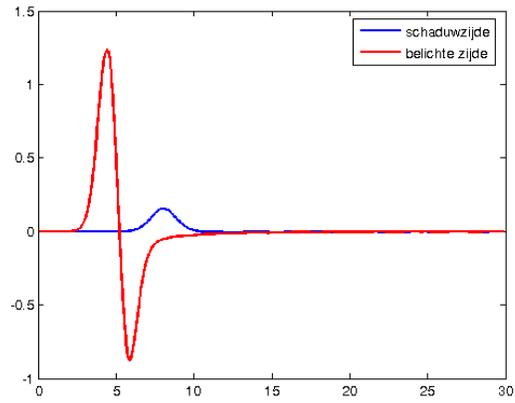


Figure 5: The auxiliary current $U(\rho, t)$ on the surface of a cylinder illuminated by a plane wave.

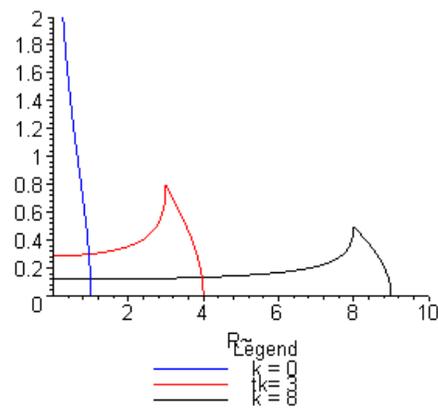


Figure 6: Convolution of the Green function with the temporal basis functions.

with l_n the length of segment n , w_g the weights of a Gaussian quadrature rule on the interval $[0, 1]$ and $\boldsymbol{\rho}_{n,g}$ the point on segment n with line coordinate x_g , with x_g the abscis values that belong with the Gauss-rule.

We need to tackle a small problem. In figure 6 it is shown that for $k = 0$ the integrand we wish to handle with the Gauss-rule has a singularity for $\boldsymbol{\rho}_m = \boldsymbol{\rho}'$. This singularity lies within the integration boundaries for $\boldsymbol{\rho}'$ when $m = n$. In this case we are forced to do the spatial integration analytically. Fortunately, this is possible. The result is:

$$(\mathbf{Z}_0)_{m,m} = -\frac{L_m}{2\pi} \log \frac{2c\Delta t + \sqrt{(2c\Delta t)^2 - L_m^2}}{L_m} - \frac{c\Delta t}{\pi} \arctan \frac{L_m}{\sqrt{(2c\Delta t)^2 - L_m^2}} \quad (33)$$

with $L_m = \min(l_m, 2c\Delta t)$.

6 Assignment

6.1 Validation by means of a simple example

In a first step you have to validate the code. This can be done by calculating the time domain scattering at a cylinder. Use an incoming wave

$$E^i(\boldsymbol{\rho}, t) = \frac{4}{T\sqrt{\pi}} e^{-\gamma^2} \quad (34)$$

with $\gamma = \frac{4}{T}(c(t - t_0) - x)$. T determines the width of the pulse; the shorter the pulse, the broader its spectrum. A short pulse demands a sufficiently short Δt and a sufficiently fine discretization of the scatterer. Let your code solve this situation; the result is a solution $U(\boldsymbol{\rho}, t) = \mu \partial_t J(\boldsymbol{\rho}, t)$. After a Fourier transformation you find the spectral content of this solution $u(\boldsymbol{\rho}, \omega) = j\omega \mu j(\boldsymbol{\rho}, \omega)$. This is the response to the Fourier transformation of the incoming field. We have:

$$E^i(\boldsymbol{\rho}, t) = \frac{1}{2\pi} \int d\omega A(\omega) e^{i(\omega t - kx)} \quad (35)$$

with $A(\omega) = e^{-j\omega t_0 - (\frac{T\omega}{8c})^2} / c$. To know the response to a plane wave with frequency ω and unit amplitude you proceed as follows:

$$j_0(\boldsymbol{\rho}, \omega) = \frac{\mathcal{F}(U)}{j\omega A(\omega)\mu}. \quad (36)$$

This current can be directly compared to the analytical solution for a PEC cylinder. In figure 7 you see an example of such a validation. Because the method has only

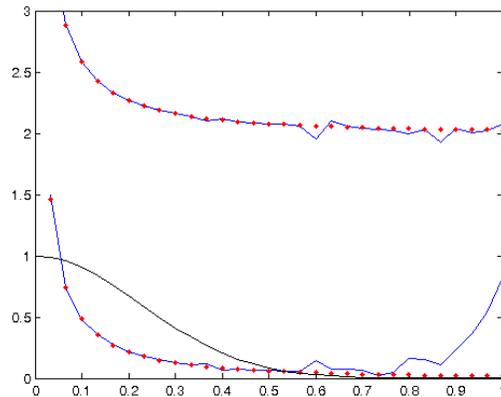


Figure 7: Comparison of the normalised current ηJ [in units Volt] at the illuminated and shadow side of a cylinder with unit radius as a function of the normalised frequency ω/c [in units per meter]. The spectrum of the excitation is plotted in black.

limited accuracy and the incoming field has a finite spectrum, the results will only agree in a limited frequency range.

In summary:

- Implement the march-on-in-time algorithm as described above. Think carefully about what data structures you will be using to describe the scatterer geometry.
- Implement a function that allows you to plot the electric field and the induced current resulting from the scattering of a time-harmonic plane wave by a circular cylinder.
- Reproduce Fig. 7. Discuss how your simulation parameters determine the frequency range over which you can expect good agreement (i.e. over which your method is accurate).
- The method to solve the exterior scattering problem for a circular cylinder and a plane incident wave can be found in one of the appendices to this document.
- Carefully choose the parameter t_0 in the incident wave to make sure all fields at the scatterer's surface are zero (up to some tolerance) at the start of the simulation.

6.2 Computation of resonance frequencies of a cylindrical cavity

In this assignment you will excite the cylinder from the inside. To do this, you need to use an electric field that is a solution of the Maxwell equations and that is radiated by a source configuration within the cylinder. More explicitly, you are going to use as incident field $E^i(\boldsymbol{\rho}, t)$ the electric field radiated by a line source at the origin:

$$J(\boldsymbol{\rho}, t) = \frac{4}{T\sqrt{\pi}} e^{-(4c\frac{t-t_0}{T})^2} \frac{\delta(\rho)}{2\pi\rho}. \quad (37)$$

- Compute the field $E^i(\boldsymbol{\rho}, t)$ radiated by (37), using the 2D time-domain Green's function. The integral representation cannot be integrated analytically. You will have to resort to numerical quadrature. The integrand of the integral, however, contains a mild singularity ($\mathcal{O}(1/\sqrt{t})$). This can be removed by applying a suitable substitution of the integration variable. The new integral representation has a smooth integrand and can be very accurately approximated by Gauss-Legendre quadrature.
- Which cavity modes do you expect this incident wave will excite? Compute the corresponding resonance frequencies by solving the sourceless TM Maxwell equations in a cylindrical cavity.
- Perform a Fourier transformation on the numerical solution of the MOT system. Normalize this spectrum using the spectrum of the incident wave as detailed in last assignment. What resonance peaks can you resolve? Does this agree with your expectations?
- Repeat the above points for an off-center source at $\boldsymbol{\rho} = \frac{1}{2}\mathbf{u}_x$. Which modes do you expect now to be excited? Can you find the corresponding peaks in the spectrum of the MOT solution?

6.3 Creative assignment

Invent one or more configurations of scatterers and solve the integral equation for this configuration. Make sure that at least one of your examples is physically or technically relevant. For instance, browse the internet for interesting problems. Extract conclusions from your simulations and try where possible to discuss the accuracy.

7 Report

You need to write a report of this project. The report must contain the email addresses of the authors. It should not exceed 15 pages. Follow this structure:

- A short summary of the problem.
- Description of the implementation. Include only small relevant code snippets in your report and accompany these snippets with an explanation as to why they are relevant. No full code listing should be included in the report.
- The solutions to the assignments in the previous section.

You can implement this assignment in either Matlab, Python, or Julia. Include in your report clear instructions on how to reproduce the results and figures you use in the report. Test your code on multiple machines to ensure it can be run during assessment of your work.

It is highly recommended to work together in a *private* (!) git repository. This facilitates combining work from multiple team members and keeping track of code development throughout the project duration.

A Methods provided

A couple of matlab functions are provided. A first routine can be used to calculate the Gauss nodes and weights. The syntax is:

```
[x,w] = lgwt(N,a,b).
```

Here N is the number of Gauss nodes and a and b are the upper and lower limit of the integration interval. After execution, the array x contains the abscis values and the array w the corresponding weights. If f is the array that contains the function values of the integrandum in the abscis values x the integral can be calculated by means of the command

```
sum( w .* f ).
```

A second function allows you to do the Fourier Transform of a function. If the array f contains the samples of a function in equidistant abscis values $t_j = j\Delta t, j = 0, \dots, N_T - 1$, the Fourier Transform can be calculated with

```
[v,F] = fourierTransform(delta_t,f,dim).
```

with dim the dimension along which the Fourier transform should be taken. So, if you want to take the Fourier transform along the rows of an array, you set dim to 2, while if you wish to transform along the columns you set dim to 1. If you leave out the argument, the transformation will be along the rows by default, which may lead to weird results if you work with column vectors. After execution v contains the frequencies at which the Fourier Transform is known and F is the array with the value of the Fourier Transform at these frequencies. To obtain accurate results, you must take Δt smaller than the Nyquist sample interval, the signal must be zero for $t = 0$ and sufficiently damped out for $t = (N_T - 1)\Delta t$.

B Analytical solution for circular cylinder

Consider the incident plane wave

$$e_z^i(\boldsymbol{\rho}) = e^{jkx}, \quad (38)$$

The current induced on a PEC surface is solution to the following integral equation:

$$e_z^i(\boldsymbol{\rho}) = j\omega\mu \int_c G_k(|\boldsymbol{\rho} - \boldsymbol{\rho}'|) j_z(\boldsymbol{\rho}') d\boldsymbol{c}', \quad \forall \boldsymbol{\rho} \in c. \quad (39)$$

In this section, we will solve this integral equation, starting from the Maxwell equations in the exterior domain, complemented with a suitable set of boundary conditions:

$$\nabla^2 e_z(\boldsymbol{\rho}) + k^2 e_z(\boldsymbol{\rho}) = 0 : \quad \forall \boldsymbol{\rho} \text{ outside of the PEC}, \quad (40)$$

$$e_z(\boldsymbol{\rho}) = 0 : \quad \forall \boldsymbol{\rho} \text{ on boundary of and inside the PECs.} \quad (41)$$

This problem, supplemented with the requirement that the scattered field must be radiating to infinity (and not the other way around) is equivalent to the integral equation (39). Equation (40) can be solved by means of separation of variables (in a cylindrical coordinate system), with the following result:

$$e_z(\boldsymbol{\rho}) = \sum_{n=-\infty}^{\infty} a_n J_n(k\rho) e^{jn\phi} + b_n H_n^{(2)}(k\rho) e^{jn\phi}, \quad (42)$$

where $\boldsymbol{\rho} = \rho [\cos \phi, \sin \phi]$. We will now write both the incoming and scattered electric field in a form similar to (42).

Since the incoming field is given by $e_z^i(\boldsymbol{\rho}) = e^{jkx}$, it can be written in a series of type (42) by appealing to the Jacobi-Anger expansion:

$$e^{jkx} = e^{jk\rho \cos \phi} = \sum_{n=-\infty}^{\infty} j^n J_n(k\rho) e^{jn\phi}. \quad (43)$$

The scattered field outside of the PEC circular cylinder will be written as

$$e_z^s(\boldsymbol{\rho}) = \sum_{n=-\infty}^{\infty} b_n H_n^{(2)}(k\rho) e^{jn\phi}. \quad (44)$$

The reason why the a_n coefficients are zero in the expansion of the scattered field is that

$$J_n(k\rho) = \frac{1}{2} [H_n^{(2)}(k\rho) + H_n^{(1)}(k\rho)], \quad (45)$$

i.e. the field $J_n(k\rho) e^{jn\phi}$ is actually a superposition of a wave coming from infinity to the origin (i.e. $H_n^{(1)}(k\rho) e^{jn\phi}$) and a wave propagating from the origin to infinity (i.e. $H_n^{(2)}(k\rho) e^{jn\phi}$). Since the scattered field of the cylinder should not have a part coming from infinity, the a_n coefficients are set to zero for the representation of the scattered field.

Finally, the PEC boundary conditions at $\rho = a$ can be enforced as follows

$$\sum_{n=-\infty}^{\infty} j^n J_n(ka) e^{jn\phi} + \sum_{n=-\infty}^{\infty} b_n H_n^{(2)}(ka) e^{jn\phi} = 0, \forall \phi. \quad (46)$$

Multiplying this equation with $e^{jm\phi}$ and integrating over $\phi \in [0, 2\pi]$ yields

$$b_m = -j^m \frac{J_m(ka)}{H_m^{(2)}(ka)}, \quad (47)$$

which completes the analytical solution

$$e_z(\boldsymbol{\rho}) = \sum_{n=-\infty}^{\infty} j^n \left[J_n(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) \right] e^{jn\phi}. \quad (48)$$

To evaluate this series, it is useful to know that the Bessel functions $J_n(z)$ and $H_n^{(2)}(z)$ can be computed by means of the commands `besselj(n, z)` and `besselh(n, 2, z)` respectively. Also, the series needs to be truncated at some finite n , which requires some numerical experimentation. The rule of thumb that Bessel functions start to decay exponentially as soon as their index exceeds their argument can be used as a starting point for this exploration.

To find the induced current use (Faraday's law) that

$$\mathbf{h}_t = -\frac{1}{j\omega\mu} \nabla_t \times e_z \mathbf{u}_z \quad (49)$$

and (by the jump/continuity conditions at the PEC surface)

$$j_z \mathbf{u}_z = \mathbf{n} \times \mathbf{h}_t, \quad (50)$$

where \mathbf{n} is the unit normal on the contour, directed in the exterior domain. Upon some further vector algebra we arrive at

$$j_z = \frac{1}{j\omega\mu} \left. \frac{\partial e_z}{\partial \rho} \right|_{\rho=a} \quad (51)$$

In other words, up to a factor, the induced current is the normal (and for a circular scatterer this means radial) derivative of the total current at the boundary of the scatterer. Hint: the final expression can be simplified drastically by leveraging the Wronskian relationship $J_n(z)H_n'^{(2)}(z) - J_n'(z)H_n^{(2)}(z) = -\frac{2j}{\pi z}$.