The identification of a space-dependent load source in anisotropic thermoelastic systems

K. Van Bockstal\textsuperscript{a}, M. Slodička\textsuperscript{a} and L. Marin\textsuperscript{b}

\textsuperscript{a} Research Group NaM\textsuperscript{2}, Department of Mathematical Analysis, Ghent University
\textsuperscript{b} Department of Mathematics, Faculty of Mathematics and Computer Science, University of Bucharest
\textsuperscript{b} Institute of Solid Mechanics, Romanian Academy

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Outline

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Three types of thermoelasticity

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$: isotropic and homogeneous thermoelastic body
- $\Gamma = \partial \Omega$: Lipschitz continuous boundary
- $T$: final time
- **Coupled thermoelastic system** [Muñoz Rivera and Qin, 2002]: specific formulas are used in the study of thermoelasticity to describe how objects change in shape (displacement vector $u$) with changes in temperature $\theta$ from the reference value $T_0 > 0$ (in Kelvin)

\[
\begin{align*}
\rho \partial_{tt} u - \alpha \Delta u - \beta \nabla (\nabla \cdot u) + \gamma \nabla \theta &= p & \text{in } \Omega \times (0, T) \\
\rho C_s \partial_t \theta - \kappa \Delta \theta - K \ast \Delta \theta + T_0 \gamma \nabla \cdot \partial_t u &= h & \text{in } \Omega \times (0, T)
\end{align*}
\]

- $p$: load (body force) vector; $h$: heat source
- The Lamé parameters $\alpha$ and $\beta$, the mass density $\rho$, the specific heat $C_s$, the coupling (absorbing) coefficient $\gamma$ and the thermal coefficient $\kappa$ are assumed to be positive constants
- The sign ‘$\ast$’ denotes the convolution product

\[
(K \ast \theta)(x, t) := \int_0^t K(t - s)\theta(x, s)ds, \quad (x, t) \in \Omega \times (0, T)
\]
Three types of thermoelasticity:

- **type-I**: $K = 0$ and $\kappa \neq 0$:
  \[
  \rho C_s \partial_t \theta - \kappa \Delta \theta = h
  \]

- **type-II**: $K \neq 0$ and $\kappa = 0$:
  \[
  \rho C_s \partial_t \theta - K \Delta \theta = h
  \]

- **type-III**: $K \neq 0$ and $\kappa \neq 0$:
  \[
  \rho C_s \partial_t \theta - \kappa \Delta \theta - K \Delta \theta = h
  \]

Inverse source problems for (an-)isotropic thermoelasticity are studied.
[Bellassoued and Yamamoto, 2011] investigated an inverse heat source problem for type-I thermoelasticity: they determine $h(x)$ by measuring $u|_{\omega \times (0,T)}$ and $\theta(\cdot, t_0)$, where $\omega$ is a subdomain of $\Omega$ such that $\Gamma \subset \partial \omega$ and $t_0 \in (0, T)$.

[Wu and Liu, 2012] studied an inverse source problem of determining $p(x)$ for type-II thermoelasticity from a displacement measurement $u|_{\omega \times (0,T)}$.

Using a Carleman estimate, a Hölder stability for the inverse source problem is proved in both contributions, which implies the uniqueness of a solution to the inverse source problem.

Gap: no numerical scheme is provided to recover the unknown source.
**Problem (A)**

*Can we find a unique* $p(x)$ *and/or* $h(x)$ *from the additional final in time measurements*

$$u(x, T) = \xi_T(x) \text{ and/or } \theta(x, T) = \zeta_T(x)$$

*for all types of thermoelasticity and can we provide a numerical scheme?*

**Goal:** The way of retrieving the unknown source is not by the minimization of a certain cost functional (which is typical for IPs), but by using an alternative technique
Solution (Problem (A))

Up to now, using our approach, it is possible to recover \( p(x) \) uniquely for all types of thermoelasticity from the additional final in time measurement (the condition of final overdetermination)

\[
  u(x, T) = \xi_T(x),
\]

in the presence of a damping term \( g(\partial_t u) \) in the hyperbolic equation of the thermoelastic system, i.e.

\[
\begin{aligned}
  \rho \partial_{tt} u + g(\partial_t u) - \alpha \Delta u - \beta \nabla (\nabla \cdot u) + \gamma \nabla \theta &= p(x) \quad \text{in } \Omega \times (0, T); \\
  \rho C_s \partial_t \theta - \kappa \Delta \theta - K \Delta \theta + T_0 \gamma \nabla \cdot \partial_t u &= 0 \quad \text{in } \Omega \times (0, T); \\
  u(x, t) &= 0 \quad \text{on } \Gamma \times (0, T); \\
  \theta(x, t) &= 0 \quad \text{on } \Gamma \times (0, T); \\
  u(x, 0) &= \partial_t u(x, 0) = 0, \quad \theta(x, 0) = 0 \quad \text{in } \Omega,
\end{aligned}
\]

▶ A damping term in thermoelastic systems is also considered in [Qin, 2008, Chapter 9], [Kirane and Tatar, 2001], [Oliveira and Charão, 2008], ...

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The results can be extended to anisotropic thermoelastic systems

\[
\begin{aligned}
\varrho(x) \partial_{tt} u + g(\partial_t u) + L^e u + \text{div}(B(x) \theta) &= p(x) + r, & (x,t) \in \Omega \times (0,T), \\
\varrho(x) C_s(x) \partial_t \theta - \nabla \cdot (K(x) \nabla \theta) - (K \ast \Delta \theta) + T_0 B(x) : \nabla \partial_t u &= h, & (x,t) \in \Omega \times (0,T), \\
u(x,t) &= 0, & (x,t) \in \Gamma \times (0,T), \\
\theta(x,t) &= 0, & (x,t) \in \Gamma \times (0,T),
\end{aligned}
\]

together with the initial conditions

\[
u(x,0) = 0, \quad \partial_t u(x,0) = 0, \quad \theta(x,0) = 0, \quad x \in \Omega.
\]

As before, the goal is to determine \(p(x)\) from

\[
u_T(x) := u(x, T) = \xi_T(x), \quad x \in \Omega.
\]

Overview results (in both papers):

- A variational approach is used, which implies uniqueness for all types of thermoelasticity if \( g : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is strictly monotone increasing and \( K \) is strongly positive definite
- if \( g \) is linear (i.e. \( g = gI \) with \( g > 0 \)), then
  - A stable iterative algorithm is proposed to recover the unknown vector source \( p \) by extending the iterative procedure of [Johansson and Lesnic, 2007] for the heat equation to thermoelastic systems, but without using an adjoint problem
  - It is possible to consider the case of non-homogeneous Dirichlet boundary conditions and initial conditions
  - Also additional given source terms can be considered

- In the following: more details are given for isotropic thermoelasticity of type-III
Mathematical analysis

**Theorem (Well-posedness of the direct problem (given general $p$))**

Assume that $p : (0, T) \rightarrow L^2(\Omega)$ belong to $L^2 \left( (0, T), L^2(\Omega) \right)$, $\bar{u}_0(x) \in H^1(\Omega)$, $\bar{u}_1(x) \in L^2(\Omega)$ and $\bar{\theta}_0 \in H^1(\Omega)$. Assume that any of the following conditions holds for the kernel $K : (0, T) \rightarrow \mathbb{R}$:

1. $K'(t) \not\equiv 0$ and $(-1)^j K^{(j)}(t) \geq 0$, $t > 0$, $j = 0, 1, 2$, i.e. $K$ is strongly positive definite;

2. $K \in L^1(0, T)$ s.t. $\int_0^T |K(t)| \, dt \leq \kappa$;

3. $\exists C > 0$ s.t. $\max_{t \in [0, T]} |K(t)| \leq C$.

Then, the variational problem has a unique solution $(u, \theta)$ such that

$u \in C \left( [0, T], L^2(\Omega) \right) \cap L^2 \left( (0, T), H^1_0(\Omega) \right)$, $\partial_t u \in C \left( [0, T], L^2(\Omega) \right)$, $\partial_{tt} u \in L^2 \left( (0, T), H^1_0(\Omega)^* \right)$, $\theta \in C \left( [0, T], L^2(\Omega) \right) \cap L^2 \left( (0, T), H^1_0(\Omega) \right)$ and $\partial_t \theta \in L^2 \left( (0, T), H^1_0(\Omega)^* \right)$.

Moreover, when $\bar{u}_0(x) = 0$, $\bar{u}_1(x) = 0$, $\bar{\theta}_0 = 0$, $h = 0$ and $p = p(x)$, the following estimate holds

$$\max_{t \in [0, T]} \left\{ \| \partial_t u(t) \|^2 + \| u(t) \|^2_{H^1_0(\Omega)} + \| \theta(t) \|^2 \right\} + \int_0^T \| \nabla \theta(t) \|^2 \, dt \leq C \| p \|^2.$$  

▶ See [Lions and Magenes, 1972] and [Van Bockstal and Marin, 2017, Theorem 4.1]
Coupled variational formulation: find \( \langle u(t), \theta(t), p \rangle \in H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \) such that \( u(x, T) = \xi_T(x) \) and
\[
\begin{align*}
\rho (\partial_{tt} u, \varphi) + (g(\partial_t u), \varphi) + \alpha (\nabla u, \nabla \varphi) + \beta (\nabla \cdot u, \nabla \cdot \varphi) + \gamma (\nabla \theta, \varphi) &= (p, \varphi), \\
\rho C_s (\partial_t \theta, \psi) + \kappa (\nabla \theta, \nabla \psi) + (k \ast \nabla \theta, \nabla \psi) - \gamma T_0 (\partial_t u, \nabla \psi) &= 0,
\end{align*}
\]
for all \( \varphi \in H^1_0(\Omega) \) and \( \psi \in H^1_0(\Omega) \) and a.a. \( t \in (0, T] \).

Theorem (Uniqueness)

Let \( \langle u_1, \theta_1, p_1 \rangle \) and \( \langle u_2, \theta_2, p_2 \rangle \) satisfy the thermoelastic system. Set \( u = u_1 - u_2, p = p_1 - p_2 \) and \( \theta = \theta_1 - \theta_2 \) such that \( u(x, 0) = 0, u(x, T) = 0, \partial_t u(x, 0) = 0 \) and \( \theta(x, 0) = 0 \). Then \( p = 0 \) a.e. in \( \Omega \) and \( \langle u, \theta \rangle = \langle 0, 0 \rangle \) a.e. in \( \Omega \times (0, T) \).

- Subtract, equation by equation, the variational formulation corresponding with the different solutions
- We want to add up both resulting equation such that the mixed term is cancelled out
- A good choice of the test functions is needed:
\[
\varphi = \partial_t u(t) \quad \text{and} \quad \psi = \frac{\theta(t)}{T_0}
\]
Sketch of the proof of uniqueness for type-III thermoelasticity

- Another trick: integrate in time over \((0, T)\) such that

\[
\int_{\Omega} \int_{0}^{T} p(x) \cdot \partial_t u(x, t) \, dt = \int_{\Omega} [p(x) \cdot u(x, T) - p(x) \cdot u(x, 0)] = 0
\]

- We obtain that

\[
\frac{\rho}{2} \| \partial_t u(T) \|^2 + \int_{0}^{T} (g(\partial_t u_1) - g(\partial_t u_2), \partial_t u_1 - \partial_t u_2) \\
+ \frac{\rho C_s}{2T_0} \| \theta(T) \|^2 + \frac{\kappa}{T_0} \int_{0}^{T} \| \nabla \theta \|^2 + \frac{1}{T_0} \int_{0}^{T} (K \ast \nabla \theta, \nabla \theta) = 0
\]

- We make distinction based on the different assumptions on \(K\)
Sketch of the proof of uniqueness for type-III thermoelasticity

Uniqueness for a Positive Definite Convolution Kernel I

- Assume that the twice differentiable function $K: (0, T] \rightarrow \mathbb{R}$ satisfies
  \[
  K'(t) \neq 0 \quad \text{and} \quad (-1)^j K^{(j)}(t) \geq 0, \quad t > 0, \quad j = 0, 1, 2,
  \]
i.e. $K$ is strongly positive definite
  \[
  \int_0^T \phi(t)(K \ast \phi)(t)dt \geq C_0 \int_0^T (K \ast \phi)^2(t)dt, \quad \forall T > 0, \forall \phi \in L^1_{\text{loc}}(\Omega)
  \]
- This implies
  \[
  \int_0^T \left( g(\partial_t u_1) - g(\partial_t u_2), \partial_t u_1 - \partial_t u_2 \right) + \int_0^T \|\nabla \theta\|^2 \leq 0
  \]
Sketch of the proof of uniqueness for type-III thermoelasticity

Uniqueness for a Positive Definite Convolution Kernel II

- Assume \( g \) componentwise strictly monotone increasing. Then \( u_t = 0 \) a.e. in \( \Omega \times (0, T) \). Therefore,

\[
    u(x, 0) = 0 \implies u(x, t) = 0 \text{ a.e. in } \Omega \times (0, T)
\]

- \( \theta = 0 \) on \( \partial \Omega \) \( \implies \theta = 0 \) a.e. in \( \Omega \times (0, T) \)

- This implies that

\[
    (p, \varphi) = 0, \quad \forall \varphi \in H^1_0(\Omega).
\]

From this, we conclude that \( p = 0 \) in \( L^2(\Omega) \)

- Examples:
  - E.g. \( K(t) = t^{-\alpha}, \, t \in (0, T] \), with \( 0 < \alpha < 1 \) (singular kernel)
  - E.g. \( K(t) = \exp(-t), \, t \in [0, T] \)
Sketch of the proof of uniqueness for type-III thermoelasticity

Uniqueness for $K \in L^1(0, T)$ s.t. $\int_0^T |K(t)| \, dt \leq \kappa$. 

Young's inequality for convolutions:

$$\|f * g\|_r \leq \|f\|_p \|g\|_q, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad 1 \leq p, q, r \leq \infty. \quad (1)$$

Applying this inequality, one obtains

$$\left| \int_0^T ((K \ast \nabla \theta)(t), \nabla \theta(t)) \, dt \right| = \left| \int_\Omega \int_0^T (K \ast \nabla \theta)(x, t) \nabla \theta(x, t) \, dt \, dx \right|$$

$$\leq \int_\Omega \left( \int_0^T (K \ast \nabla \theta)^2(x, t) \, dt \right) \sqrt{\int_0^T \nabla \theta(x, t)^2 \, dt} \, dx \leq (1)$$

$$\leq \left( \int_\Omega \left( \int_0^T |K(t)| \, dt \right)^2 \right) \sqrt{\int_0^T \nabla \theta(x, t)^2 \, dt} \, dx \leq \left( \int_0^T |K(t)| \, dt \right) \int_0^T \|\nabla \theta(t)\|^2 \, dt,$$
Sketch of the proof of uniqueness for type-III thermoelasticity

Uniqueness for a bounded convolution kernel

\[
\left| \int_0^T \left( \int_0^t K(t-s) \nabla \theta(s) \, ds, \nabla \theta(t) \right) \, dt \right| \\
\leq C\varepsilon \int_0^T \left\| \int_0^t K(t-s) \nabla \theta(s) \, ds \right\|^2 \, dt + \varepsilon \int_0^T \| \nabla \theta(t) \|^2 \, dt \\
\leq C\varepsilon \int_0^T \left( \int_0^t |K(t-s)| \| \nabla \theta(s) \| \, ds \right)^2 \, dt + \varepsilon \int_0^T \| \nabla \theta(t) \|^2 \, dt \\
\leq C\varepsilon \int_0^T \left( \int_0^t |K(t-s)|^2 \, ds \right) \left( \int_0^t \| \nabla \theta(s) \|^2 \, ds \right) \, dt + \varepsilon \int_0^T \| \nabla \theta(t) \|^2 \, dt \\
\leq C\varepsilon \int_0^T \left( \int_0^t \| \nabla \theta(s) \|^2 \, ds \right) \, dt + \varepsilon \int_0^T \| \nabla \theta(t) \|^2 \, dt.
\]

Fixing \( \varepsilon \) sufficiently small and applying Grönwall’s lemma implies that \( u = p = 0 \) and \( \theta = 0 \).
Algorithm for finding the source term if $g$ is linear

(i) Choose an initial guess $p_0 \in L^2(\Omega)$. Let $\langle u_0, \theta_0 \rangle$ be the solution to the thermoelastic system with $p = p_0$

(ii) Assume that $p_k$ and $\langle u_k, \theta_k \rangle$ have been constructed. Let $\langle w_k, \eta_k \rangle$ solve the thermoelastic system with $p(x) = u_k(x, T) - \xi_T(x)$

(iii) Define

$$p_{k+1}(x) = p_k(x) - \omega w_k(x, T), \quad x \in \Omega$$

where $\omega > 0$ (relaxation parameter), and let $\langle u_{k+1}, \theta_{k+1} \rangle$ solve the thermoelastic system with $p = p_{k+1}$

(iv) The procedure continues by repeating steps (ii) and (iii) until a desired level of accuracy is achieved (see next slide)

- This is a Landweber-Fridmann iteration scheme [Fridman, 1956].
- The proof of convergence can be found in [Van Bockstal and Slodička, 2015, Theorem 3.3] for isotropic materials and in [Van Bockstal and Marin, 2017, Theorem 4.2]
For linear systems

Stopping criterion

- **Morozov’s discrepancy principle** is used [Morozov, 1966]
- The case is considered when there is some error in the additional measurement, i.e.
  \[ \| \xi_T - \xi_T^e \| \leq e, \]
  where \( e(\tilde{e}) \) depends on the noise level with magnitude \( \tilde{e} > 0 \)
- The solutions \( p_k^e, u_k^e \) and \( \theta_k^e \) at iteration \( k \) are obtained by using the algorithm
- The discrepancy principle suggests to finish the iterations at the smallest index \( k = k(e, \omega) \) for which
  \[ E_{k,u_T} = \left\| u_k^e(\cdot, T) - \tilde{\xi}_T^e \right\| \leq e \]
Numerical experiment: setting

- 1D linear model for isotropic type-I ($K = 0$) and type-III thermoelasticity is considered
- $\Omega = [0, 1], \ T = 1$
- Copper alloy: shear modulus $G = 4.8 \times 10^{10} \text{N/m}^2$, Poisson’s ratio $\nu = 0.34$, $\alpha_T = 16.5 \times 10^{-6} \text{1/K}$, $\kappa = 401 \text{W/mK}$, $\rho = 8960 \text{kg/m}^3$ and $C_s = 385 \text{J/kgK}$
- $g = 2 \times 10^8$, $T_0 = 293\text{K}$
- $\alpha = \mu$, $\beta = \mu + \lambda$ with $\lambda = \frac{2\nu G}{1 - 2\nu}$ and $\mu = G$
- Three choices for the convolution kernel are made, namely $K = 0$, $K = \exp(-t)$ and $K = 1/\sqrt{t}$
Numerical experiment: setting

- The forward coupled problems in this procedure are discretized in time according to the backward Euler method with timestep 0.0005.
- At each time-step, the resulting elliptic coupled problems are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization. A fixed uniform mesh consisting of 200 intervals is used.
- The finite element library DOLFIN [Logg and Wells, 2010, Logg et al., 2012b] from the FEniCS project [Logg et al., 2012a] is used.
Exact solution

\[ u(x, t) = (1 + t)^2 x(x - 1)^2 \quad \text{and} \quad \theta(x, t) = (1 + t)x(1 - x)^2 \]

\[ p_1(x) = 10x(1 - x) \]

**Figure:** The exact source \( p_1 \) and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement, for various convolution kernels, namely (a) \( K = 0 \), (b) \( K = \exp(-t) \), and (c) \( K = 1/\sqrt{t} \). The relaxation parameter \( \omega = 10 \).
Results of numerical experiments

\[ p_2(x) = \exp \left( -20(x - 0.5)^2 \right) \]

\[ p_2(x) = \exp \left( -20(x - 0.5)^2 \right) \]

Figure: The exact source \( p_2 \) and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement, for various convolution kernels, namely (a) \( K = 0 \), (b) \( K = \exp(-t) \), and (c) \( K = 1/\sqrt{t} \). The relaxation parameter \( \omega = 10 \).
### Results of numerical experiments

**Table**: The stopping iteration number $\tilde{k} = k(e(\tilde{e}), 10)$ and the CPU time (mins), obtained for the experiments with the unknown sources $p_1$ and $p_2$.

<table>
<thead>
<tr>
<th>$\tilde{e}$</th>
<th>$1%$</th>
<th>$5%$</th>
<th>$10%$</th>
<th>$0.5%$</th>
<th>$1%$</th>
<th>$3%$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tilde{k}$</td>
<td>time</td>
<td>$\tilde{k}$</td>
<td>time</td>
<td>$\tilde{k}$</td>
<td>time</td>
</tr>
<tr>
<td>$K = 0$</td>
<td>136</td>
<td>94.7</td>
<td>11</td>
<td>8.2</td>
<td>9</td>
<td>6.3</td>
</tr>
<tr>
<td>$K = \exp(-t)$</td>
<td>133</td>
<td>138.7</td>
<td>9</td>
<td>10.4</td>
<td>9</td>
<td>9.9</td>
</tr>
<tr>
<td>$K = 1/\sqrt{t}$</td>
<td>142</td>
<td>144.3</td>
<td>10</td>
<td>11</td>
<td>8</td>
<td>8.9</td>
</tr>
</tbody>
</table>

Following experiments:

\[
p_3(x) = \begin{cases} 
0 & 0 \leq x < \frac{1}{3} \\
6x - 2 & \frac{1}{3} \leq x \leq \frac{1}{2} \\
4 - 6x & \frac{1}{2} \leq x \leq \frac{2}{3} \\
0 & \frac{2}{3} \leq x \leq 1
\end{cases}, \quad p_4(x) = \begin{cases} 
x(0.5 - x)(1 - x) & 0 \leq x \leq \frac{1}{2} \\
x(x - 0.5)(1 - x) & \frac{1}{2} \leq x \leq 1
\end{cases},
\]

\[
p_5(x) = \begin{cases} 
0 & 0 \leq x < \frac{1}{3} \\
1 & \frac{1}{3} \leq x \leq \frac{2}{3} \\
0 & \frac{2}{3} \leq x \leq 1
\end{cases}, \quad p_6(x) = 10x(x - 1)^2
\]
Figure: The exact sources $p_3$, $p_4$ and $p_5$ and its numerical approximations for $\bar{\epsilon} = 0\%$ (a,c,e) and for different noise levels (b,d,f). The relaxation parameter $\omega = 10$. 
Other relaxation parameter

Figure: The exact source $p_4$ and its numerical approximations for $\omega = 2$ (a) and for $\omega = 20$ (b).

- The results for small noise are similar to the results obtained when $\omega = 10$.
Results of numerical experiments

**Figure:** The exact source $p_5$ and its numerical approximations for $\tilde{\varepsilon} = 1\%$ (a) and for $\tilde{\varepsilon} = 3\%$ (b) for different values of $g$. The relaxation parameter $\omega = 10$.

**Figure:** The exact source $p_2$ and its numerical approximations for $\tilde{\varepsilon} = 3\%$ for different values of $g$. The relaxation parameter $\omega = 10$. 
Results of numerical experiments

**Figure:** The non-symmetric exact source $p_6$ and its numerical approximations (using $\varepsilon = 3\%$) for different initial guesses: 0 (a), $6.44x - 12.27x^2 + 5.83x^3$ (b), $9.68x - 18.46x^2 + 8.78x^3$ (c) and $12.88x - 24.54x^2 + 11.65x^3$ (d). The relaxation parameter $\omega = 10$. 
Figure: The non-symmetric exact source $p_6$ and its numerical approximations for $T = 0.2$ (a) and $T = 0.5$ (b). The relaxation parameter $\omega = 10$. 
Conclusion

- It is possible to recover uniquely an unknown vector source in all types of damped thermoelastic systems when an additional final in time measurement of the displacement is measured.
- A numerical algorithm in a linear case gives accurate shape recovery.
- The algorithm is sensitive to the amount of noise added to the data.
- There is a certain limitation of the method with respect to the recovery of non-symmetric sources.
Future research

- More numerical experiments (e.g. influence of the parameter $g$ on the results)
- Testing different stopping criteria (up to now, no better results)
- What if $g$ is nonlinear?
- Other inverse problems for thermoelasticity, e.g. the recovery of time-dependent sources, convolution kernel
- Goal: with numerical scheme!
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