The well-posedness of a mathematical model for an intermediate state between type-I and type-II superconductivity

K. Van Bockstal and M. Slodička

Ghent University
Department of Mathematical Analysis
Numerical Analysis and Mathematical Modelling Research Group

Outline

Introduction
- Type-I versus type-II superconductivity
- Macroscopic models for type-I superconductors
- Macroscopic models for type-II superconductors
- Intermediate state between type-I and type-II superconductivity

Macroscopic model for an intermediate state between type-I and type-II superconductors

Mathematical Analysis
- Useful estimates
- Variational formulation
- Time discretization: existence of a solution

Conclusion and further research
Features of superconductivity

▶ Kammerlingh Onnes (1911): perfect conductivity

For various cooled down materials the electrical resistance not only decreases with temperature, but also has a sudden drop at some critical absolute temperature $T_c$.

▶ Meissner and Ochsenfeld (1933): perfect diamagnetism
  ⇒ i.e. expulsion of the magnetic induction $B$

▶ Kammerlingh Onnes (1914): threshold field
  ⇒ restore the normal state through the application of a large magnetic field

▶ A way to classify superconductors: type-I and type-II
Similar behaviour for a very weak external magnetic field when the temperature $T < T_c$ is fixed.

As the external magnetic field becomes stronger it turns out that two possibilities can happen ⇒ phase diagram in the $T-H$ plane.

- **Type-I (a):** the $B$ field remains zero inside the superconductor until suddenly, as the critical field $H_c$ is reached, the superconductivity is destroyed.

- **Type-II (b):** a mixed state occurs in addition to the superconductive and the normal state (two different critical fields).

What are the macroscopic models which are used in the modelling of type-I and type-II superconductors?
Macroscopic models for type-I superconductors

**Type-I**

- $\Omega \subset \mathbb{R}^3$: bounded Lipschitz domain, $\nu$ unit normal vector on $\Gamma$

- The quasi-static Maxwell equations for linear materials are considered

\[
\nabla \times \mathbf{H} = \mathbf{J} \quad \text{Ampère’s law} \quad \mathbf{H} \quad \text{magnetic field}
\]
\[
\nabla \times \mathbf{E} = -\mu \partial_t \mathbf{H} \quad \text{Faraday’s law} \quad \mathbf{E} \quad \text{electric field} \quad \mu > 0 \quad \text{magnetic permeability}
\]
\[
\nabla \cdot \mathbf{H}_0 = 0 \quad \mathbf{J} \quad \text{current density}
\]

- London and London (1935): a macroscopic description of type-I superconductors involves a **two-fluid model**

\[
\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s
\]
\[
\mathbf{J}_n = \sigma \mathbf{E}
\]
\[
\nabla \times \mathbf{H} = \sigma \mathbf{E} + \mathbf{J}_s \quad \mathbf{J}_n \quad \text{normal current density}
\]
\[
\nabla \times \mathbf{E} = -\mu \partial_t \mathbf{H} \quad \mathbf{J}_s \quad \text{superconducting current density}
\]
\[
\nabla \cdot \mathbf{H}_0 = 0 \quad \sigma \quad \text{conductivity of normal electrons}
\]

- Below the critical temperature $T_c$, the current consists of **superconducting electrons and normal electrons**
Macroscopic models for type-I superconductors

London equations (1935) ⇒ local law for $J_s$

\[
\partial_t J_s = \Lambda^{-1} E \\
\nabla \times J_s = -\Lambda^{-1} B \\
\Lambda = \frac{m_e}{n_s e^2}
\]

$n_s$ density of superelectrons

$m_e$ mass of an electron

$-e$ electric charge of an electron

⇒ Correct description of two basic properties of superconductors:

perfect conductivity and perfect diamagnetism (Meissner effect)

\[
\nabla \cdot B = 0 \Rightarrow \exists A \text{ such that } B = \nabla \times A \text{ and } \nabla \cdot A = 0 \\
\n\nabla \times J_s = -\Lambda^{-1} B \Rightarrow J_s(x, t) = -\Lambda^{-1} A(x, t), \quad (x, t) \in \Omega \times (0, T)
\]
Generalization of London and London: nonlocal laws

[Pippard, 1953]

\[ J_{s,p}(x, t) = \int_{\Omega} Q(x - x') A(x', t) \, dx', \quad (x, t) \in \Omega \times (0, T) \]

with

\[ Q(x - x') A(x', t) = -\tilde{C} \frac{x - x'}{|x - x'|^4} \left[ A(x', t) \cdot (x - x') \right] \exp \left( -\frac{|x - x'|}{r_0} \right), \]

\[ \tilde{C} := \frac{3}{4\pi \xi_0 \Lambda} > 0, \quad r_0 := \frac{\xi_0 l}{\xi_0 + l} \]

\( \xi_0 \) the coherence length of the material, \( l \) is the mean free path
Macroscopic models for type-I superconductors

[Eringen, 1984]

\[
J_{s,e}(x, t) = \int_\Omega \sigma_0(|x - x'|)(x - x') \times H(x', t) \, dx' = -(K_0 \ast H)(x, t),
\]

\((x, t) \in \Omega \times (0, T)\)

with

\[
\sigma_0(s) = \begin{cases} 
\frac{\tilde{C}}{2s^2} \exp\left(-\frac{s}{r_0}\right) & s < r_0; \\
0 & s \geq r_0
\end{cases}
\]
Pippard’s nonlocal law fails to explain the vanishing of electrical resistance.

It is possible to recover from Eringen’s law the London equations and the form given by Pippard:

\[ J_s = J_{s,e} = -\mathcal{K}_0 \star H \]

\[
\begin{align*}
\nabla \times \mathbf{H} &= \sigma \mathbf{E} + J_s \\
\nabla \times \mathbf{E} &= -\mu \partial_t \mathbf{H}
\end{align*}
\]

Taking the curl of Ampère’s law result in

\[
\sigma \mu \partial_t \mathbf{H} + \nabla \times \nabla \times \mathbf{H} + \nabla \times (\mathcal{K}_0 \star \mathbf{H}) = 0
\]

Well-posedness is studied into detail in [Slodička and Van Bockstal, 2014].

Also the error estimates for two time-discrete schemes (an implicit and a semi-implicit) based on backward Euler method are derived in [Slodička and Van Bockstal, 2014].
Macroscopic models for type-II superconductors

**Type-II**

- Dependency between current density $J$ and the electric field $E$

- **Ohm’s law** for non-superconducting metal (dashed)

- **Bean’s critical-state model** for type-II superconductors (fine dashed): current either flows at the critical level $J_c$ or not at all $\Rightarrow$ not fully applicable

- The **power law** by Rhyner for type-II superconductors (continuous)

\[ E = \sigma_c^{-n} |J|^{n-1} J, \quad n \in (7, 1000) \]
Take the curl of the power law and use Faraday’s law
⇒ nonlinear and degenerate partial differential equation for the magnetic field

\[ \mu \partial_t H + \sigma_c^{-n} \nabla \times (|\nabla \times H|^{n-1} \nabla \times H) = 0 \]

Studied by: [Barrett and Prigozhin, 2000, Yin et al., 2002, Prigozhin and Sokolovsky, 2004, Wei and Yin, 2005]
Intermediate state between type-I and type-II superconductivity

- The classification into type-I and type-II is **insufficient** for multiband superconductors [Babaev and Speight, 2005]
- This are superconductors with **several superconducting components**
- The material ‘magnesium diboride’ combines the characteristics of both types [Nagamatsu et al., 2001]
- New kind of superconductor: **type-1.5 superconductors** [Moshchalkov et al., 2009, Babaev et al., 2012]
- Allows coexistence of various properties of type-I and type-II superconductors

**Problem**

*Is it possible to derive macroscopic models for an intermediate state between type-I and type-II superconductors?*
### Type-I:

\[
\mu \partial_t H + \sigma^{-1} \nabla \times \nabla \times H + \sigma^{-1} \nabla \times (K_0 \ast H) = 0
\]

### Type-II (\( n \in (7, 1000) \)):

\[
\mu \partial_t H + \sigma_c^{-n} \nabla \times (|\nabla \times H|^{n-1} \nabla \times H) = 0
\]

By introducing a real parameter \( \beta \geq 1 \) and a real function \( f(\beta) \), we propose to combine both equations to

\[
\mu \partial_t H + \sigma^{-1} f(\beta) \nabla \times \nabla \times H + \sigma_c^{-\beta} g(\beta) \nabla \times (|\nabla \times H|^{\beta-1} \nabla \times H) + \sigma^{-1} f(\beta) \nabla \times (K_0 \ast H) = 0
\]

with

- \( f \in C([1, \infty)) \) monotonically decreasing, \( f(1) = 1 \) and \( 0 \leq f(\beta) \leq 1 \) for \( \beta > 1 \)
- \( f \) equals zero or is very small for \( \beta > 7 \)
- \( g(\beta) := 1 - f(\beta) \)

**Intermediate state:** \( 1 < \beta \leq 7 \)
It is assumed (for simplicity) that $\mu = \sigma = \sigma_c = 1$

The aim of this paper is to address the well-posedness of the following problem for $\beta \geq 1$:

$$
\begin{align*}
\partial_t H + f(\beta) \nabla \times \nabla \times H + g(\beta) \nabla \times \left( |\nabla \times H|^{\beta-1} \nabla \times H \right) \\
+ f(\beta) \nabla \times (K_0 \ast H) &= F \quad \text{in } \Omega \times (0, T); \\
H \times ν &= 0 \quad \text{on } \Gamma \times (0, T); \\
H(x, 0) &= H_0 \quad \text{in } \Omega;
\end{align*}
$$

to design a numerical scheme for computations and to derive error estimates for the time discretization.

Some possible choices for $f$:

$$
f(\beta) = \begin{cases} 
\frac{(-1)^\alpha}{6^\alpha} (\beta - 7)^\alpha & 1 \leq \beta \leq 7, \\
0 & \beta > 7
\end{cases}, \quad \alpha \in \mathbb{N}
$$

$$
f(\beta) = \exp(-k\beta), \quad k \in \mathbb{R}^+
$$

Focus on mathematical analysis, not on implementation.
Using spherical coordinates one can deduce that

\[ \sigma_0(|x|)x \in L^p(\Omega) \text{ for } p \in [1, 3) : \]

\[ \int_{\Omega} |\sigma_0(|x|)x|^p \, dx \leq \int_{B(0,r_0)} \frac{C}{|x|^{2p}} \left| \exp\left(-\frac{|x|}{r_0}\right) \right|^p |x|^p \, dx \]

\[ \leq C \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\theta) d\theta \int_0^{r_0} r^{2-p} \, dr \leq C \left[ \frac{r^{3-p}}{3-p} \right]_{r_0} < \infty \]

\[ |J_s(x, t)| = |(K_0 \ast H)(x, t)| \leq C(q) \|H(t)\|_q \text{ for } q > \frac{3}{2}, \quad \forall x \in \Omega: \]

\[ |J_s(x, t)| = \left| \int_{\Omega} \sigma_0 \left( |x - x'| \right) (x - x') \times H(x', t) \, dx' \right| \leq \int_{\Omega} |\sigma_0 \left( |x - x'| \right) (x - x')| |H(x', t)| \, dx' \]

\[ \leq \sqrt[p]{\int_{\Omega} \sigma_0 \left( |x - x'| \right) (x - x')|^p \, dx'} \sqrt[q]{\int_{\Omega} |H(x', t)|^q \, dx'} \leq C \|H(t)\|_q \]

\[ \square \]

For instance, it holds that

\[ (K_0 \ast h, \nabla \times h) \leq C_\varepsilon \|h\|^2 + \varepsilon \|\nabla \times h\|^2, \quad \forall h \in H(\text{curl}, \Omega) \]
The suitable choice for the space of test functions is

\[ V_0 = \{ \varphi \in L^2(\Omega) : \nabla \times \varphi \in L^{\beta+1}(\Omega) \text{ and } \varphi \times \nu = 0 \text{ on } \Gamma \} \subset H_0(\text{curl}, \Omega). \]

This is a closed subspace of the space

\[ V = \{ \varphi \in L^2(\Omega) : \nabla \times \varphi \in L^{\beta+1}(\Omega) \} \subset H(\text{curl}, \Omega), \]

and is endowed with the same graph norm

\[ \| \varphi \|_V = \| \varphi \|_{V_0} = \| \varphi \|_{L^2(\Omega)} + \| \nabla \times \varphi \|_{L^{\beta+1}(\Omega)}. \]

**Definition**

Let \( \beta \geq 1, \ H_0 \in V \) and \( F \in L^2 \left( (0, T), L^2(\Omega) \right) \). The variational formulation of (14) reads as:

find \( H \in C \left( [0, T], L^2(\Omega) \right) \) with \( \nabla \times H \in L^{\beta+1} \left( (0, T), L^{\beta+1}(\Omega) \right) \) and \( \partial_t H \in L^2 \left( [0, T], L^2(\Omega) \right) \) such that

\[
(\partial_t H(t), \varphi) + f(\beta) (\nabla \times H(t), \nabla \times \varphi) + g(\beta) \left( |\nabla \times H(t)|^{\beta-1} \nabla \times H(t), \nabla \times \varphi \right) + f(\beta) (K_0 * H(t), \nabla \times \varphi) = (F(t), \varphi), \quad \forall \varphi \in V_0,
\]

for a.e. \( t \in [0, T] \).
Lemma (reflexivity)

The vector spaces $\mathbb{V}$ and $\mathbb{V}_0$ are reflexive Banach spaces.

Lemma (monotonicity)

Let $\beta \geq 1$. There exists a positive constant $C_0(\beta) = \frac{1}{4 \cdot 12} \cdot 1^{\frac{1}{2}}$ such that for any $H_1, H_2 \in \mathbb{V}$ hold

\[
\left( |\nabla \times H_1|^{\beta - 1} \nabla \times H_1 - |\nabla \times H_2|^{\beta - 1} \nabla \times H_2, \nabla \times (H_1 - H_2) \right) \geq C_0(\beta) \| \nabla \times (H_1 - H_2) \|^{\beta + 1}_{L^{\beta + 1}(\Omega)}.
\]

Theorem (uniqueness)

The problem (14) admits at most one solution $\partial_t H \in L^2 ([0, T], L^2(\Omega))$ with $\nabla \times H \in L^{\beta + 1} ((0, T), L^{\beta + 1}(\Omega))$ if $H_0 \in L^2(\Omega)$. 
Proof uniqueness:
Assume that we have two solutions $H_1$ and $H_2$. Set $H = H_1 - H_2$. Subtract the variational equation for $H = H_1$ from for $H = H_2$, set $\varphi = H$ into the resulting equation and integrating in time for $t \in (0, T)$:

$$\|H(t)\|^2 + f(\beta) \int_0^t \|\nabla \times H\|^2 + g(\beta) C_0 \int_0^t \|\nabla \times H\|_{L^{\beta+1}(\Omega)}^{\beta+1} \leq -f(\beta) \int_0^t (K_0 \ast H, \nabla \times H) \leq C_\varepsilon \int_0^t \|H\|^2 + \varepsilon \int_0^t \|\nabla \times H\|^2.$$

We consider four cases:

- $\beta = 1$: then $f(\beta) = 1$ and $g(\beta) = 0$. Fixing a sufficiently small positive $\varepsilon$ and applying the Grönwall argument, we get that $H = 0$ a.e. in $Q_T$;

- $1 < \beta < 7$: then $f$ and $g$ are strict positive $\Rightarrow H = 0$ a.e. in $Q_T$;

- $\beta \geq 7$ and $f(\beta) = 0$ for $\beta \geq 7$: $H = 0$ a.e. in $Q_T$;

- $\beta \geq 7$ and $f(\beta) > 0$ for $\beta \geq 7$ but very small: analogously as the case $1 < \beta < 7$. 
Numerical scheme to approximate the solution

- **Rothe’s method** [Kačur, 1985]: divide \([0, T]\) into \(n \in \mathbb{N}\) equidistant subintervals \((t_{i-1}, t_i)\) for \(t_i = i\tau\), where \(\tau = T/n < 1\) and for any function \(z\)

\[
  z_i \approx z(t_i) \text{ and } \partial_t z(t_i) \approx \delta z_i := \frac{z_i - z_{i-1}}{\tau}
\]

- **Convolution explicitly** (from the previous time step):

\[
\begin{aligned}
  (\delta h_i, \varphi) + f(\beta) (\nabla \times h_i, \nabla \times \varphi) \\
  + g(\beta) (|\nabla \times h_i|^{\beta-1} \nabla \times h_i, \nabla \times \varphi) \\
  = (f_i, \varphi) - f(\beta) (K_0 \ast h_{i-1}, \nabla \times \varphi) \;
  \text{ and } \\
  h_0 = H_0
\end{aligned}
\]

- **Monotone operator theory** [Vainberg, 1973]:

**Theorem (uniqueness on a single time step)**

Assume \(H_0 \in L^2(\Omega)\) and \(F \in L^2((0, T), L^2(\Omega))\). Then there exists a \(\tau_0 > 0\) such that the variational problem has a unique solution for any \(i = 1, \ldots, n\) and any \(\tau < \tau_0\).
Convergence: a priori estimates as uniform bounds

Suppose that \( F \in L^2((0, T), L^2(\Omega)) \)

(i) Let \( H_0 \in L^2(\Omega) \). Then, there exists a positive constant \( C \) such that

\[
\max_{1 \leq i \leq n} \| h_i \|^2 + \sum_{i=1}^{n} \| h_i - h_{i-1} \|^2 + \sum_{i=1}^{n} \| \nabla \times h_i \|_{L^{\beta+1}(\Omega)} \tau \leq C
\]

for all \( \tau < \tau_0 \).

(ii) If \( \nabla \cdot H_0 = 0 = \nabla \cdot f_i \) then \( \nabla \cdot h_i = 0 \) for all \( i = 1, \ldots, n \).

(iii) If \( H_0 \in V \) then

\[
\max_{1 \leq i \leq n} \| \nabla \times h_i \|_{L^{\beta+1}(\Omega)}^{\beta+1} + \sum_{i=1}^{n} \| \delta h_i \|^2 \tau \leq C
\]

for all \( \tau < \tau_0 \).
Time discretization: existence of a solution

- $H_n$: piecewise linear in time spline of the solutions $h_i, i = 1, \ldots, n$
- $\overline{H}_n$: piecewise constant in time spline of the solutions $h_i, i = 1, \ldots, n$
- The variational formulation on a single timestep can be rewritten on the whole time frame as

$$
(\partial_t H_n(t), \varphi) + f(\beta) \left( \nabla \times \overline{H}_n(t), \nabla \times \varphi \right) + g(\beta) \left( |\nabla \times \overline{H}_n(t)|^{\beta-1} \nabla \times \overline{H}_n(t), \nabla \times \varphi \right)
$$

$$
= (\overline{F}_n(t), \varphi) - f(\beta) \left( K_0 \star \overline{H}_n(t - \tau), \nabla \times \varphi \right).
$$

- Convergence of the sequences $H_n$ and $\overline{H}_n$ to the unique weak solution is proved if $\tau \to 0$ or $n \to \infty$
Main ideas of the proof:

- **Compact embedding** [Palatucci et al., 2013, Lemma 10]:

\[
H^{\frac{1}{2}}(\Omega) \hookrightarrow L^2(\Omega) \cong L^2(\Omega)^* \hookrightarrow H^{-1}_0(\text{curl}, \Omega)
\]

implies [Kačur, 1985]

\[
H_n \to H \text{ in } C([0, T], L^2(\Omega))
\]

and

\[
\bar{H}_n \to H \text{ in } L^2([0, T], L^2(\Omega))
\]

- **Minty-Browder’s trick** for the convergence of the nonlinear term
- **\(H\)** is the weak solution of the problem

---

**Theorem (Existence solution)**

Let \(H_0 \in \mathbf{V}\) and \(F \in L^2((0, T), L^2(\Omega))\). Assume that \(\nabla \cdot H_0 = 0 = \nabla \cdot F(t)\) for any time \(t \in [0, T]\). Then there exists a weak solution \(H \in C([0, T], L^2(\Omega))\) with \(\partial_t H \in L^2((0, T), L^2(\Omega))\).
Error estimates for the time discretization

**Theorem (Error)**

Suppose that \( F \in \text{Lip}([0, T], L^2(\Omega)) \). If \( H_0 \in V \) then

\[
\max_{t \in [0, T]} \| H_n(t) - H(t) \|^2 + \int_0^T \| \nabla \times [H_n - H] \|_{L^{\beta + 1}(\Omega)}^{\beta + 1} \leq C_T.
\]

Please note that the positive constant \( C \) in this estimates is of the form \( Ce^{CT} \).
Conclusion:

- Macroscopic model for an intermediate state between type-I and type-II superconductivity is proposed
- Well-posedness is proved
- Numerical scheme for calculations is provided

Future research:

- Numerical implementation
- Comparison with available results about neither type-I nor type-II superconductors
References I

Type-1.5 superconductivity in multiband systems: Magnetic response, broken symmetries and microscopic theory - a brief overview.

*Physica C: Superconductivity, 479(0):2 – 14.*
Proceedings of {VORTEX} {VII} Conference.

Semi-meissner state and neither type-I nor type-II superconductivity in multicomponent superconductors.


Bean’s critical-state model as the $p \to \infty$ limit of an evolutionary $p$-laplacian equation.

References II

Electrodynamics of memory-dependent nonlocal elastic continua.

*Method of Rothe in evolution equations*, volume 80 of *Teubner Texte zur Mathematik*.
Teubner, Leipzig.

Moshchalkov, V., Menghini, M., Nishio, T., Chen, Q. H., Silhanek, A. V., Dao, V. H.,
Type-1.5 superconductivity.

Superconductivity at 39K in magnesium diboride.
References III

Local and global minimizers for a variational energy involving a fractional norm.

Pippard, A. B. (1953).
An experimental and theoretical study of the relation between magnetic field and current in a superconductor.

Ac losses in type-II superconductors induced by nonuniform fluctuations of external magnetic field.
References IV

A nonlocal parabolic model for type-I superconductors.
*Numerical Methods for Partial Differential Equations*, pages n/a–n/a.


Numerical solutions to bean’s critical-state model for type-II superconductors.

A degenerate evolution system modeling bean’s critical-state type-II superconductors.
*Discrete and Continuous Dynamical Systems, 8*:781–794.