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The well-posedness of a mathematical model for an intermediate state between type-I and type-II superconductivity

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Outline

Introduction
- Type-I versus type-II superconductivity
- Macroscopic models for type-I superconductors
- Macroscopic models for type-II superconductors
- Intermediate state between type-I and type-II superconductivity

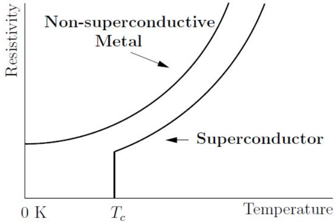
Macroscopic model for an intermediate state between type-I and type-II superconductors

Mathematical Analysis
- Usefull estimates
- Variational formulation
- Time discretization: existence of a solution

Conclusion and further research

Features of superconductivity

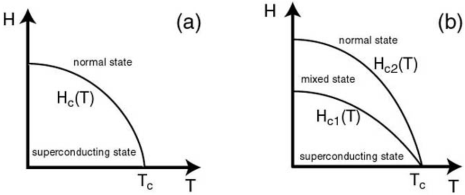
- Kammerlingh Onnes (1911): perfect conductivity



For various cooled down materials the electrical resistance not only decreases with temperature, but also has a sudden drop at some critical absolute temperature T_c

- Meissner and Ochsenfeld (1933): perfect diamagnetism
  $\Rightarrow$ i.e. expulsion of the magnetic induction $\boldsymbol{B}$
- Kammerlingh Onnes (1914): threshold field
  $\Rightarrow$ restore the normal state through the application of a large magnetic field
- A way to classify superconductors: type-I and type-II

- Similar behaviour for a very weak external magnetic field when the temperature $T < T_c$ is fixed
- As the external magnetic field becomes stronger it turns out that two possibilities can happen $\Rightarrow$ phase diagram in the T-H plane



- Type-I (a): the $\boldsymbol{B}$ field remains zero inside the superconductor until suddenly, as the critical field H_c is reached, the superconductivity is destroyed
- Type-II (b): a mixed state occurs in addition to the superconductive and the normal state (two different critical fields)
- What are the macroscopic models which are used in the modelling of type-I and type-II superconductors?

Type-I

- $\Omega \subset \mathbb{R}^3$: bounded Lipschitz domain, $\boldsymbol{\nu}$ unit normal vector on Γ
- The quasi-static Maxwell equations for linear materials are considered

$\nabla \times \boldsymbol{H} = \boldsymbol{J}$	Ampère's law	$\boldsymbol{H}$	magnetic field		
$\nabla \times \boldsymbol{E} = -\mu \partial_t \boldsymbol{H}$	Faraday's law	$\boldsymbol{E}$	electric field	$\mu > 0$	magnetic permeability
$\nabla \cdot \boldsymbol{H}_0 = 0$		$\boldsymbol{J}$	current density		

- London and London (1935): a macroscopic description of type-I superconductors involves a two-fluid model

$$\boldsymbol{J} = \boldsymbol{J}_n + \boldsymbol{J}_s$$

$$\boldsymbol{J}_n = \sigma \boldsymbol{E} \qquad \text{Ohm's law}$$

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \boldsymbol{J}_s$$
$$\nabla \times \boldsymbol{E} = -\mu \partial_t \boldsymbol{H}$$
$$\nabla \cdot \boldsymbol{H}_0 = 0$$

$\boldsymbol{J}_n$	normal current density
$\boldsymbol{J}_s$	superconducting current density
σ	conductivity of normal electrons

- Below the critical temperature T_c, the current consists of superconducting electrons and normal electrons

London equations (1935) $\Rightarrow$ local law for $\boldsymbol{J}_s$

$$\partial_t \boldsymbol{J}_s = \Lambda^{-1}\boldsymbol{E}$$
$$\nabla \times \boldsymbol{J}_s = -\Lambda^{-1}\boldsymbol{B}$$
$$\Lambda = \frac{m_e}{n_s e^2}$$

n_s	density of superelectrons
m_e	mass of an electron
$-e$	electric charge of an electron

$\Rightarrow$ Correct description of two basic properties of superconductors:

perfect conductivity and perfect diamagnetism (Meissner effect)

$$\nabla \cdot \boldsymbol{B} = 0 \Rightarrow \exists \boldsymbol{A} \text{ such that } \boldsymbol{B} = \nabla \times \boldsymbol{A} \text{ and } \nabla \cdot \boldsymbol{A} = 0$$

$$\nabla \times \boldsymbol{J}_s = -\Lambda^{-1}\boldsymbol{B} \quad \Rightarrow \quad \boldsymbol{J}_s(\boldsymbol{x},t) = -\Lambda^{-1}\boldsymbol{A}(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T)$$

Generalization of London and London: nonlocal laws

[Pippard, 1953]

$$\boldsymbol{J}_{s,p}(\boldsymbol{x},t) = \int_\Omega Q(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{A}(\boldsymbol{x}',t)\,\mathrm{d}\boldsymbol{x}', \qquad (\boldsymbol{x},t) \in \Omega \times (0,T)$$

with

$$Q(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{A}(\boldsymbol{x}',t) = -\widetilde{C}\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^4}\left[\boldsymbol{A}(\boldsymbol{x}',t)\cdot(\boldsymbol{x}-\boldsymbol{x}')\right]\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{r_0}\right),$$

$$\widetilde{C} := \frac{3}{4\pi\xi_0\Lambda} > 0, \qquad r_0 := \frac{\xi_0 l}{\xi_0 + l}$$

ξ_0 the coherence length of the material, l is the mean free path

[Eringen, 1984]

$$\boldsymbol{J}_{s,e}(\boldsymbol{x},t) = \int_\Omega \sigma_0\left(|\boldsymbol{x}-\boldsymbol{x}'|\right)\left(\boldsymbol{x}-\boldsymbol{x}'\right) \times \boldsymbol{H}(\boldsymbol{x}',t)\,\mathrm{d}\boldsymbol{x}' =: -(\mathcal{K}_0 \star \boldsymbol{H})(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T)$$

with

$$\sigma_0(s) = \begin{cases} \frac{\widetilde{C}}{2s^2}\exp\left(-\frac{s}{r_0}\right) & s < r_0; \\ 0 & s \geqslant r_0 \end{cases}$$

- Pippard's nonlocal law fails to explain the vanishing of electrical resistance
- It is possible to recover from Eringen's law the London equations and the form given by Pippard

$$\Rightarrow \boldsymbol{J}_s = \boldsymbol{J}_{s,e} = -\mathcal{K}_0 \star \boldsymbol{H} \quad \text{in} \quad \begin{cases} \nabla \times \boldsymbol{H} &= \sigma \boldsymbol{E} + \boldsymbol{J}_s \\ \nabla \times \boldsymbol{E} &= -\mu \partial_t \boldsymbol{H} \end{cases}$$

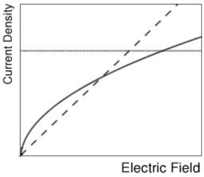
- Taking the curl of Ampère's law result in

$$\boxed{\sigma\mu\partial_t \boldsymbol{H} + \nabla \times \nabla \times \boldsymbol{H} + \nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) = \boldsymbol{0}}$$

- Well-posedness is studied into detail in [Slodička and Van Bockstal, 2014].
- Also the error estimates for two time-discrete schemes (an implicit and a semi-implicit) based on backward Euler method are derived in [Slodička and Van Bockstal, 2014].

Type-II

- Dependency between current density $\boldsymbol{J}$ and the electric field $\boldsymbol{E}$



- Ohm's law for non-superconducting metal (dashed)
- Bean's critical-state model for type-II superconductors (fine dashed): current either flows at the critical level $\boldsymbol{J}_c$ or not at all $\Rightarrow$ not fully applicable
- The power law by Rhyner for type-II superconductors (continuous)

$$\boldsymbol{E} = \sigma_c^{-n} |\boldsymbol{J}|^{n-1} \boldsymbol{J}, \qquad n \in (7, 1000)$$

- Take the curl of the power law and use Faraday's law
  $\Rightarrow$ nonlinear and degenerate partial differential equation for the magnetic field

$$\boxed{\mu \partial_t \boldsymbol{H} + \sigma_c^{-n} \nabla \times \left(|\nabla \times \boldsymbol{H}|^{n-1} \nabla \times \boldsymbol{H} \right) = \mathbf{0}}$$

- Studied by: [Barrett and Prigozhin, 2000, Yin et al., 2002, Prigozhin and Sokolovsky, 2004, Wei and Yin, 2005]

- The classification into type-I and type-II is insufficient for multiband superconductors [Babaev and Speight, 2005]
- This are superconductors with several superconducting components
- The material 'magnesium diboride' combines the characteristics of both types [Nagamatsu et al., 2001]
- New kind of superconductor: type-1.5 superconductors [Moshchalkov et al., 2009, Babaev et al., 2012]
- Allows coexistence of various properties of type-I and type-II superconductors

Problem

Is it possible to derive macroscopic models for an intermediate state between type-I and type-II superconductors?

- Type-I:

$$\mu\partial_t \boldsymbol{H} + \sigma^{-1}\nabla \times \nabla \times \boldsymbol{H} + \sigma^{-1}\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) = \boldsymbol{0}$$

- Type-II ($n \in (7, 1000)$):

$$\mu\partial_t \boldsymbol{H} + \sigma_c^{-n}\nabla \times \left(|\nabla \times \boldsymbol{H}|^{n-1}\nabla \times \boldsymbol{H}\right) = \boldsymbol{0}$$

- By introducing a real parameter $\beta \geqslant 1$ and a real function $f(\beta)$, we propose to combine both equations to

$$\mu\partial_t \boldsymbol{H} + \sigma^{-1}f(\beta)\nabla \times \nabla \times \boldsymbol{H} + \sigma_c^{-\beta}g(\beta)\nabla \times \left(|\nabla \times \boldsymbol{H}|^{\beta-1}\nabla \times \boldsymbol{H}\right) + \sigma^{-1}f(\beta)\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) = \boldsymbol{0}$$

  with
  - $f \in C([1, \infty))$ monotonically decreasing, $f(1) = 1$ and $0 \leqslant f(\beta) \leqslant 1$ for $\beta > 1$
  - f equals zero or is very small for $\beta > 7$
  - $g(\beta) := 1 - f(\beta)$
- Intermediate state: $1 < \beta \leqslant 7$

- It is assumed (for simplicity) that $\mu = \sigma = \sigma_c = 1$
- The aim of this paper is to address the well-posedness of the following problem for $\beta \geqslant 1$:

$$\begin{cases} \partial_t \boldsymbol{H} + f(\beta)\nabla \times \nabla \times \boldsymbol{H} + g(\beta)\nabla \times \left(|\nabla \times \boldsymbol{H}|^{\beta-1}\nabla \times \boldsymbol{H}\right) & \\ \qquad\qquad + f(\beta)\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) = \boldsymbol{F} & \text{in } \Omega \times (0, T); \\ \boldsymbol{H} \times \boldsymbol{\nu} = \boldsymbol{0} & \text{on } \Gamma \times (0, T); \\ \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0 & \text{in } \Omega; \end{cases}$$

  to design a numerical scheme for computations and to derive error estimates for the time discretization
- Some possible choices for f:
  - $$f(\beta) = \begin{cases} \frac{(-1)^\alpha}{6^\alpha}(\beta - 7)^\alpha & 1 \leqslant \beta \leqslant 7 \\ 0 & \beta > 7 \end{cases}, \quad \alpha \in \mathbb{N}$$
  - $$f(\beta) = \exp(-k\beta), \quad k \in \mathbb{R}^+$$
- Focus on mathematical analysis, not on implementation.

Using spherical coordinates one can deduce that

- $\sigma_0(|\boldsymbol{x}|)\boldsymbol{x} \in \mathbf{L}^p(\Omega)$ for $p \in [1,3)$:

$$\int_\Omega |\sigma_0(|\boldsymbol{x}|)\boldsymbol{x}|^p \, \mathrm{d}\boldsymbol{x} \leqslant \int_{B(\mathbf{0},r_0)} \frac{C}{|\boldsymbol{x}|^{2p}} \left| \exp\left(-\frac{|\boldsymbol{x}|}{r_0}\right)\right|^p |\boldsymbol{x}|^p \, \mathrm{d}\boldsymbol{x}$$

$$\leqslant C \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin(\theta)\mathrm{d}\theta \int_0^{r_0} r^{2-p} \mathrm{d}r \leqslant C \left[\frac{r^{3-p}}{3-p}\right]_0^{r_0} < \infty$$

- $|\boldsymbol{J}_s(\boldsymbol{x},t)| = |(\mathcal{K}_0 \star \boldsymbol{H})(\boldsymbol{x},t)| \leqslant C(q) \|\boldsymbol{H}(t)\|_q$ for $q > \frac{3}{2}$, $\quad \forall \boldsymbol{x} \in \Omega$:

$$|\boldsymbol{J}_s(\boldsymbol{x},t)| = \left| \int_\Omega \sigma_0\left(\left|\boldsymbol{x}-\boldsymbol{x}'\right|\right)(\boldsymbol{x}-\boldsymbol{x}') \times \boldsymbol{H}(\boldsymbol{x}',t) \, \mathrm{d}\boldsymbol{x}' \right| \leqslant \int_\Omega \left|\sigma_0\left(\left|\boldsymbol{x}-\boldsymbol{x}'\right|\right)(\boldsymbol{x}-\boldsymbol{x}')\right| \left|\boldsymbol{H}(\boldsymbol{x}',t)\right| \, \mathrm{d}\boldsymbol{x}'$$

$$\leqslant \sqrt[p]{\int_\Omega \left|\sigma_0\left(\left|\boldsymbol{x}-\boldsymbol{x}'\right|\right)(\boldsymbol{x}-\boldsymbol{x}')\right|^p \mathrm{d}\boldsymbol{x}'} \sqrt[q]{\int_\Omega |\boldsymbol{H}(\boldsymbol{x}',t)|^q \, \mathrm{d}\boldsymbol{x}'} \leqslant C \|\boldsymbol{H}(t)\|_q$$

- For instance, it holds that

$$\boxed{(\mathcal{K}_0 \star \boldsymbol{h}, \nabla \times \boldsymbol{h}) \leqslant C_\varepsilon \|\boldsymbol{h}\|^2 + \varepsilon \|\nabla \times \boldsymbol{h}\|^2, \quad \forall \boldsymbol{h} \in \mathbf{H}(\mathbf{curl}, \Omega)}$$

The suitable choise for the space of test functions is

$$\mathbf{V}_0 = \left\{ \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) \,:\, \nabla \times \boldsymbol{\varphi} \in \mathbf{L}^{\beta+1}(\Omega) \text{ and } \boldsymbol{\varphi} \times \boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma \right\} \subset \mathbf{H}_0(\mathbf{curl}\,, \Omega).$$

This is a closed subspace of the space

$$\mathbf{V} = \left\{ \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) \,:\, \nabla \times \boldsymbol{\varphi} \in \mathbf{L}^{\beta+1}(\Omega) \right\} \subset \mathbf{H}(\mathbf{curl}\,, \Omega),$$

and is endowed with the same graph norm

$$\|\boldsymbol{\varphi}\|_{\mathbf{V}} = \|\boldsymbol{\varphi}\|_{\mathbf{V}_0} = \|\boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \times \boldsymbol{\varphi}\|_{\mathbf{L}^{\beta+1}(\Omega)}\,.$$

Definition

Let $\beta \geqslant 1$, $\boldsymbol{H}_0 \in \mathbf{V}$ and $\boldsymbol{F} \in L^2\left((0, T), \mathbf{L}^2(\Omega)\right)$. The variational formulation of (14) reads as: find $\boldsymbol{H} \in C\left([0, T], \mathbf{L}^2(\Omega)\right)$ with $\nabla \times \boldsymbol{H} \in L^{\beta+1}\left((0, T), \mathbf{L}^{\beta+1}(\Omega)\right)$ and $\partial_t \boldsymbol{H} \in L^2\left([0, T], \mathbf{L}^2(\Omega)\right)$ such that

$$(\partial_t \boldsymbol{H}(t), \boldsymbol{\varphi}) + f(\beta)\,(\nabla \times \boldsymbol{H}(t), \nabla \times \boldsymbol{\varphi}) + g(\beta)\left(|\nabla \times \boldsymbol{H}(t)|^{\beta-1}\nabla \times \boldsymbol{H}(t), \nabla \times \boldsymbol{\varphi}\right)$$
$$+ f(\beta)\,(\mathcal{K}_0 \star \boldsymbol{H}(t), \nabla \times \boldsymbol{\varphi}) = (\boldsymbol{F}(t), \boldsymbol{\varphi})\,, \quad \forall \boldsymbol{\varphi} \in \mathbf{V}_0,$$

for a.e. $t \in [0, T]$.

Lemma (reflexivity)

The vector spaces $\mathbf{V}$ *and* $\mathbf{V}_0$ *are reflexive Banach spaces.*

Lemma (monotonicity)

Let $\beta \geqslant 1$*. There exists a positive constant* $C_0(\beta) = \frac{1}{4 \cdot 12^{\frac{\beta+1}{2}}}$ *such that for any* $\boldsymbol{H}_1, \boldsymbol{H}_2 \in \mathbf{V}$ *hold*

$$\left(|\nabla \times \boldsymbol{H}_1|^{\beta-1}\nabla \times \boldsymbol{H}_1 - |\nabla \times \boldsymbol{H}_2|^{\beta-1}\nabla \times \boldsymbol{H}_2, \nabla \times (\boldsymbol{H}_1 - \boldsymbol{H}_2)\right) \geqslant C_0(\beta) \left\|\nabla \times (\boldsymbol{H}_1 - \boldsymbol{H}_2)\right\|^{\beta+1}_{\mathbf{L}^{\beta+1}(\Omega)}.$$

Theorem (uniqueness)

The problem (14) admits at most one solution $\partial_t \boldsymbol{H} \in L^2\left([0, T], \mathbf{L}^2(\Omega)\right)$ *with* $\nabla \times \boldsymbol{H} \in L^{\beta+1}\left((0, T), \mathbf{L}^{\beta+1}(\Omega)\right)$ *if* $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$.

Proof uniqueness:

Assume that we have two solutions $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. Set $\boldsymbol{H} = \boldsymbol{H}_1 - \boldsymbol{H}_2$. Subtract the variational equation for $\boldsymbol{H} = \boldsymbol{H}_1$ from for $\boldsymbol{H} = \boldsymbol{H}_2$, set $\varphi = \boldsymbol{H}$ into the resulting equation and integrating in time for $t \in (0, T)$:

$$\|\boldsymbol{H}(t)\|^2 + f(\beta)\int_0^t \|\nabla \times \boldsymbol{H}\|^2 + g(\beta) C_0 \int_0^t \|\nabla \times \boldsymbol{H}\|^{\beta+1}_{\mathbf{L}^{\beta+1}(\Omega)}$$
$$\leqslant -f(\beta)\int_0^t (\mathcal{K}_0 \star \boldsymbol{H}, \nabla \times \boldsymbol{H}) \leqslant C_\varepsilon \int_0^t \|\boldsymbol{H}\|^2 + \varepsilon \int_0^t \|\nabla \times \boldsymbol{H}\|^2.$$

We consider four cases:

- $\beta = 1$: then $f(\beta) = 1$ and $g(\beta) = 0$. Fixing a sufficiently small positive ε and applying the Grönwall argument, we get that $\boldsymbol{H} = \mathbf{0}$ a.e. in Q_T;
- $1 < \beta < 7$: then f and g are strict positive $\Rightarrow \boldsymbol{H} = \mathbf{0}$ a.e. in Q_T;
- $\beta \geqslant 7$ and $f(\beta) = 0$ for $\beta \geqslant 7$: $\boldsymbol{H} = \mathbf{0}$ a.e. in Q_T;
- $\beta \geqslant 7$ and $f(\beta) > 0$ for $\beta \geqslant 7$ but very small: analogously as the case $1 < \beta < 7$.

Numerical scheme to approximate the solution

- Rothe's method [Kačur, 1985]: divide $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals (t_{i-1}, t_i) for $t_i = i\tau$, where $\tau = T/n < 1$ and for any function z

$$z_i \approx z(t_i) \text{ and } \quad \partial_t z(t_i) \approx \delta z_i := \frac{z_i - z_{i-1}}{\tau}$$

- Convolution explicitly (from the previous time step):

$$\begin{cases} (\delta \boldsymbol{h}_i, \boldsymbol{\varphi}) + f(\beta)\,(\nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) & \\ +g(\beta)\left(|\nabla \times \boldsymbol{h}_i|^{\beta-1} \nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}\right) & = (\boldsymbol{f}_i, \boldsymbol{\varphi}) - f(\beta)\,(\mathcal{K}_0 \star \boldsymbol{h}_{i-1}, \nabla \times \boldsymbol{\varphi})\,; \\ \boldsymbol{h}_0 & = \boldsymbol{H}_0 \end{cases}$$

- Monotone operator theory [Vainberg, 1973]:

Theorem (uniqueness on a single time step)

Assume $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{F} \in L^2\left((0, T), \mathbf{L}^2(\Omega)\right)$. Then there exists a $\tau_0 > 0$ such that the variational problem has a unique solution for any $i = 1, \ldots, n$ and any $\tau < \tau_0$.

Convergence: a priori estimates as uniform bounds

Suppose that $\boldsymbol{F} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right)$

(i) Let $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$. Then, there exists a positive constant C such that

$$\max_{1 \leqslant i \leqslant n} \|\boldsymbol{h}_i\|^2 + \sum_{i=1}^{n} \|\boldsymbol{h}_i - \boldsymbol{h}_{i-1}\|^2 + \sum_{i=1}^{n} \|\nabla \times \boldsymbol{h}_i\|_{\mathbf{L}^{\beta+1}(\Omega)}^{\beta+1} \tau \leqslant C$$

for all $\tau < \tau_0$.

(ii) If $\nabla \cdot \boldsymbol{H}_0 = 0 = \nabla \cdot \boldsymbol{f}_i$ then $\nabla \cdot \boldsymbol{h}_i = 0$ for all $i = 1, \ldots, n$.

(iii) If $\boldsymbol{H}_0 \in \mathbf{V}$ then

$$\max_{1 \leqslant i \leqslant n} \|\nabla \times \boldsymbol{h}_i\|_{\mathbf{L}^{\beta+1}(\Omega)}^{\beta+1} + \sum_{i=1}^{n} \|\delta \boldsymbol{h}_i\|^2 \tau \leqslant C$$

for all $\tau < \tau_0$.

- $\boldsymbol{H}_n$: piecewise linear in time spline of the solutions $\boldsymbol{h}_i, i = 1, \ldots, n$
- $\overline{\boldsymbol{H}}_n$: piecewise constant in time spline of the solutions $\boldsymbol{h}_i, i = 1, \ldots, n$
- The variational formulaton on a single timestep can be rewritten on the whole time frame as

$$\begin{aligned}(\partial_t \boldsymbol{H}_n(t), \boldsymbol{\varphi}) + f(\beta)\left(\nabla \times \overline{\boldsymbol{H}}_n(t), \nabla \times \boldsymbol{\varphi}\right) \\ + g(\beta)\left(|\nabla \times \overline{\boldsymbol{H}}_n(t)|^{\beta-1}\nabla \times \overline{\boldsymbol{H}}_n(t), \nabla \times \boldsymbol{\varphi}\right) \\ = \left(\overline{\boldsymbol{F}}_n(t), \boldsymbol{\varphi}\right) - f(\beta)\left(\mathcal{K}_0 \star \overline{\boldsymbol{H}}_n(t-\tau), \nabla \times \boldsymbol{\varphi}\right).\end{aligned}$$

- Convergence of the sequences $\boldsymbol{H}_n$ and $\overline{\boldsymbol{H}}_n$ to the unique weak solution is proved if $\tau \to 0$ or $n \to \infty$

- Main ideas of the proof:
  - Compact embedding [Palatucci et al., 2013, Lemma 10]:

$$\mathbf{H}^{\frac{1}{2}}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega) \cong \mathbf{L}^2(\Omega)^* \hookrightarrow \mathbf{H}_0^{-1}(\mathbf{curl}\,,\Omega)$$

    implies [Kačur, 1985]

$$\boldsymbol{H}_n \to \boldsymbol{H} \text{ in } C\left([0,T],\mathbf{L}^2(\Omega)\right)$$

    and

$$\overline{\boldsymbol{H}}_n \to \boldsymbol{H} \text{ in } L^2\left([0,T],\mathbf{L}^2(\Omega)\right)$$

  - Minty-Browder's trick for the convergence of the nonlinear term
  - $\boldsymbol{H}$ is the weak solution of the problem

Theorem (Existence solution)

Let $\boldsymbol{H}_0 \in \mathbf{V}$ and $\boldsymbol{F} \in L^2\left((0,T),\mathbf{L}^2(\Omega)\right)$. Assume that $\nabla \cdot \boldsymbol{H}_0 = 0 = \nabla \cdot \boldsymbol{F}(t)$ for any time $t \in [0,T]$. Then there exists a weak solution $\boldsymbol{H} \in C\left([0,T],\mathbf{L}^2(\Omega)\right)$ with $\partial_t \boldsymbol{H} \in L^2\left((0,T),\mathbf{L}^2(\Omega)\right)$.

Error estimates for the time discretization

Theorem (Error)

Suppose that $\boldsymbol{F} \in \mathrm{Lip}([0, T], \mathbf{L}^2(\Omega))$. *If* $\boldsymbol{H}_0 \in \mathbf{V}$ *then*

$$\max_{t\in[0,T]} \|\boldsymbol{H}_n(t) - \boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla \times [\overline{\boldsymbol{H}}_n - \boldsymbol{H}]\|_{\mathbf{L}^{\beta+1}(\Omega)}^{\beta+1} \leqslant C\tau.$$

Please note that the positive constant C *in this estimates is of the form* Ce^{CT}.

Conclusion:

- Macroscopic model for an intermediate state between type-I and type-II superconductivity is proposed
- Well-posedness is proved
- Numerical scheme for calculations is provided

Future research:

- Numerical implementation
- Comparison with available results about neither type-I nor type-II superconductors

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