The identification of a space-dependent load source in isotropic thermoelastic systems: numerical algorithm and experiments

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- $\Omega \subset \mathbb{R}^d, d \in \{1, 2, 3\}$: isotropic and homogeneous thermoelastic body
- $\Gamma = \partial\Omega$: Lipschitz continuous boundary
- T: final time
- Coupled thermoelastic system [Muñoz Rivera and Qin, 2002]: specific formulas are used in the study of thermoelasticity to describe how objects change in shape (displacement vector $\mathbf{u}$) with changes in temperature θ from the reference value $T_0 > 0$ (in Kelvin)

$$\begin{cases} \rho\partial_{tt}\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\left(\nabla\cdot\mathbf{u}\right) + \gamma\nabla\theta & = \mathbf{p} \qquad \text{in } \Omega \times (0, T) \\ \rho C_s\partial_t\theta - \kappa\Delta\theta - K * \Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} & = h \qquad \text{in } \Omega \times (0, T) \end{cases}$$

- $\mathbf{p}$: load (body force) vector; h: heat source
- The Lamé parameters α and β, the mass density ρ, the specific heat C_s, the coupling (absorbing) coefficient γ and the thermal coefficient κ are assumed to be positive constants
- The sign '$*$' denotes the convolution product

$$\left(K * \theta\right)(\mathbf{x}, t) := \int_0^t K(t-s)\theta(\mathbf{x}, s)\mathrm{d}s, \qquad (\mathbf{x}, t) \in \Omega \times (0, T)$$

Types of thermoelasticity

$$\begin{cases} \rho\partial_{tt}\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\left(\nabla\cdot\mathbf{u}\right) + \gamma\nabla\theta & = \mathbf{p} \quad \text{in } \Omega\times(0,T); \\ \rho C_s\partial_t\theta - \kappa\Delta\theta - K * \Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} & = h \quad \text{in } \Omega\times(0,T); \\ \mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_0(\mathbf{x}), \quad \partial_t\mathbf{u}(\mathbf{x},0) = \overline{\mathbf{u}}_1(\mathbf{x}), \quad \theta(\mathbf{x},0) & = \overline{\theta}_0(\mathbf{x}) \quad \text{in } \Omega \end{cases}$$

Three types of thermoelasticity:

- type-I: $K = 0$ and $\kappa \neq 0$:

$$\rho C_s\partial_t\theta - \kappa\Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} = h$$

- type-II: $K \neq 0$ and $\kappa = 0$:

$$\rho C_s\partial_t\theta - K * \Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} = h$$

- type-III: $K \neq 0$ and $\kappa \neq 0$:

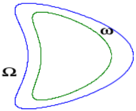
$$\rho C_s\partial_t\theta - \kappa\Delta\theta - K * \Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} = h$$

Inverse source problems for isotropic thermoelasticity are studied

- [Bellassoued and Yamamoto, 2011] investigated an inverse heat source problem for type-I thermoelasticity: they determine $h(\mathbf{x})$ by measuring

$$\mathbf{u}_{|\omega\times(0,T)} \text{ and } \theta(\cdot, t_0),$$

where ω is a subdomain of Ω such that $\Gamma \subset \partial\omega$ and $t_0 \in (0, T)$



- [Wu and Liu, 2012] studied an inverse source problem of determining $\mathbf{p}(\mathbf{x})$ for type-II thermoelasticity from a displacement measurement

$$\mathbf{u}_{|\omega\times(0,T)}$$

- Using a Carleman estimate, a Hölder stability for the inverse source problem is proved in both contributions, which implies the uniqueness of a solution to the inverse source problem
- Gap: no numerical scheme is provided to recover the unknown source

Problem (A)

Can we find a unique $\mathbf{p}(\mathbf{x})$ *and/or* $h(\mathbf{x})$ *from the additional final in time measurements*

$$\mathbf{u}(\mathbf{x}, T) = \xi_T(\mathbf{x}) \text{ and/or } \theta(\mathbf{x}, T) = \zeta_T(\mathbf{x})$$

for all types of thermoelasticity and can we provide a numerical scheme?

- Goal: The way of retrieving the unknown source is not by the minimization of a certain cost functional (which is typical for IPs), but by using an alternative technique
- The forward problem is well-posed
- Note that these inverse problems are ill-posed in the sense that small errors present in the measurement can give rise to large errors into the solutions (unbounded, oscillatory solutions)

Solution (Problem (A))

Up to now, using our approach, it is possible to recover $\mathbf{p}(\mathbf{x})$ *uniquely for all types of thermoelasticity from the additional final in time measurement (the condition of final overdetermination)*

$$\mathbf{u}(\mathbf{x}, T) = \boldsymbol{\xi}_T(\mathbf{x}),$$

in the presence of a damping term $\mathbf{g}\left(\partial_t \mathbf{u}\right)$ *in the hyperbolic equation of the thermoelastic system, i.e.*

$$\begin{cases} \rho\partial_{tt}\mathbf{u} + \mathbf{g}\left(\partial_t \mathbf{u}\right) - \alpha\Delta\mathbf{u} - \beta\nabla\left(\nabla\cdot\mathbf{u}\right) + \gamma\nabla\theta = \mathbf{p}(\mathbf{x}) & \text{in } \Omega\times(0,T); \\ \rho C_s\partial_t\theta - \kappa\Delta\theta - K * \Delta\theta + T_0\gamma\nabla\cdot\partial_t\mathbf{u} = 0 & \text{in } \Omega\times(0,T); \\ \mathbf{u}(\mathbf{x},t) = \mathbf{0} & \text{on } \Gamma\times(0,T); \\ \theta(\mathbf{x},t) = 0 & \text{on } \Gamma\times(0,T); \\ \mathbf{u}(\mathbf{x},0) = \partial_t\mathbf{u}(\mathbf{x},0) = \mathbf{0}, \quad \theta(\mathbf{x},0) = 0 & \text{in } \Omega, \end{cases}$$

- A damping term in thermoelastic systems is also considered in [Qin, 2008, Chapter 9], [Kirane and Tatar, 2001], [Oliveira and Charão, 2008],...

Van Bockstal, K. and Slodička, M., *Recovery of a space-dependent vector source in thermoelastic systems*, Inverse Problems Sci. Eng., 2015, 23(6), pp. 956–968

The results can be extended to anisotropic thermoelastic systems

$$\begin{cases} \varrho(\mathbf{x})\partial_{tt}\mathbf{u} + \mathbf{g}\left(\partial_t \mathbf{u}\right) + \mathcal{L}^{\mathrm{e}}\mathbf{u} + \mathrm{div}(\mathbb{B}(\mathbf{x})\theta) = \mathbf{p}(\mathbf{x}) + \mathbf{r}, & (\mathbf{x},t) \in \Omega \times (0,T), \\ \varrho(\mathbf{x})C_s(\mathbf{x})\partial_t\theta - \nabla\cdot(\mathbb{K}(\mathbf{x})\nabla\theta) - (K * \Delta\theta) + T_0\mathbb{B}(\mathbf{x}) : \nabla\partial_t\mathbf{u} = h, & (\mathbf{x},t) \in \Omega \times (0,T), \\ \mathbf{u}(\mathbf{x},t) = \mathbf{0}, & (\mathbf{x},t) \in \Gamma \times (0,T), \\ \theta(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Gamma \times (0,T), \end{cases}$$

together with the initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0}, \quad \partial_t\mathbf{u}(\mathbf{x},0) = \mathbf{0}, \quad \theta(\mathbf{x},0) = 0, \quad \mathbf{x} \in \Omega.$$

As before, the goal is to determine $\mathbf{p}(\mathbf{x})$ from

$$\mathbf{u}_T(\mathbf{x}) := \mathbf{u}(\mathbf{x},T) = \boldsymbol{\xi}_T(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

Van Bockstal, K. and Marin, L., *Recovery of a space-dependent vector source in anisotropic thermoelastic systems*, Computer Methods in Applied Mechanics and Engineering, 2017, 321, pp. 269–293

Overview results

- A variational approach is used, which implies uniqueness for all types of thermoelasticity if $\mathbf{g} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is strictly monotone increasing and K is strongly positive definite
- if $\mathbf{g}$ is linear (i.e. $\mathbf{g} = g\mathbf{I}$ with $g > 0$), then
  - A stable iterative algorithm is proposed to recover the unknown vector source $\mathbf{p}$ by extending the iterative procedure of [Johansson and Lesnic, 2007] for the heat equation to thermoelastic systems, but without using an adjoint problem
  - It is possible to consider the case of non-homogeneous Dirichlet boundary conditions and initial conditions
  - Also additional given source terms can be considered

Algorithm for finding the source term if $\mathbf{g}$ is linear

(*i*) Choose an initial guess $\mathbf{p}_0 \in \mathbf{L}^2(\Omega)$. Let $\langle \mathbf{u}_0, \theta_0 \rangle$ be the solution to the thermoelastic system with $\mathbf{p} = \mathbf{p}_0$

(*ii*) Assume that $\mathbf{p}_k$ and $\langle \mathbf{u}_k, \theta_k \rangle$ have been constructed. Let $\langle \mathbf{w}_k, \eta_k \rangle$ solve the thermoelastic system with $\mathbf{p}(\mathbf{x}) = \mathbf{u}_k(\mathbf{x}, T) - \boldsymbol{\xi}_T(\mathbf{x})$

(*iii*) Define

$$\mathbf{p}_{k+1}(\mathbf{x}) = \mathbf{p}_k(\mathbf{x}) - \omega \mathbf{w}_k(\mathbf{x}, T), \quad \mathbf{x} \in \Omega$$

where $\omega > 0$ (relaxation parameter), and let $\langle \mathbf{u}_{k+1}, \theta_{k+1} \rangle$ solve the thermoelastic system with $\mathbf{p} = \mathbf{p}_{k+1}$

(*iv*) The procedure continues by repeating steps (*ii*) and (*iii*) until a desired level of accuracy is achieved (see next slide)

- This is a Landweber-Fridmann iteration scheme [Fridman, 1956].
- The proof of convergence can be found in [Van Bockstal and Slodička, 2015, Theorem 3.3] for isotropic materials and in [Van Bockstal and Marin, 2017, Theorem 4.2] for anisotropic materials

Stopping criterion

- Morozov's discrepancy principle is used [Morozov, 1966]
- The case is considered when there is some error in the additional measurement, i.e.
$$\|\boldsymbol{\xi}_T - \boldsymbol{\xi}_T^e\| \leqslant e,$$
where $e(\tilde{e})$ depends on the noise level with magnitude $\tilde{e} > 0$
- The solutions $\mathbf{p}_k^e, \mathbf{u}_k^e$ and θ_k^e at iteration k are obtained by using the algorithm
- The discrepancy principle suggests to finish the iterations at the smallest index $k = k(e, \omega)$ for which
$$E_{k,\mathbf{u}_T} = \left\| \mathbf{u}_k^e(\cdot, T) - \widetilde{\boldsymbol{\xi}}_T^e \right\| \leqslant e$$

Numerical experiment: setting

- 1D linear model for isotropic type-I ($K = 0$) and type-III thermoelasticity is considered
- $\Omega = [0, 1]$, $T = 1$
- copper alloy: shear modulus $G = 4.8 \times 10^{10}\,\mathrm{N}/\mathrm{m}^2$, Poisson's ratio $\nu = 0.34$, $\alpha_T = 16.5 \times 10^{-6}\,1/\mathrm{K}$, $\kappa = 401\,\mathrm{W}/\mathrm{mK}$, $\rho = 8960\,\mathrm{kg}/\mathrm{m}^3$ and $C_s = 385\,\mathrm{J}/\mathrm{kgK}$
- $g = 2 \times 10^8$, $T_0 = 293\mathrm{K}$
- $\alpha = \mu$, $\beta = \mu + \lambda$ with $\lambda = \dfrac{2\nu G}{1 - 2\nu}$ and $\mu = G$
- Three choices for the convolution kernel are made, namely $K = 0$, $K = \exp(-t)$ and $K = 1/\sqrt{t}$

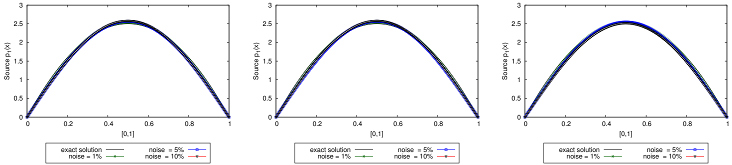
Numerical experiment: setting

- The forward coupled problems in this procedure are discretized in time according to the backward Euler method with timestep 0.0005
- At each time-step, the resulting elliptic coupled problems are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization. A fixed uniform mesh consisting of 200 intervals is used
- FEniCS PROJECT The finite element library DOLFIN [Logg and Wells, 2010, Logg et al., 2012b] from the FEniCS project [Logg et al., 2012a] is used

Exact solution

$$u(x,t) = (1+t)^2 x(x-1)^2 \quad \text{and} \quad \theta(x,t) = (1+t)x(1-x)^2$$

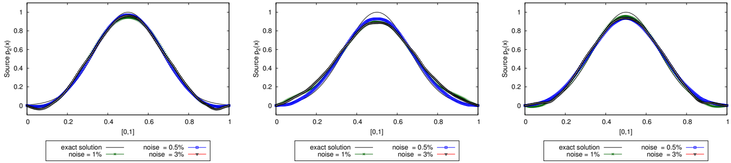
$$p_1(x) = 10x(1-x)$$



(a) (b) (c)

Figure: The exact source p_1 and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement, for various convolution kernels, namely (a) $K = 0$, (b) $K = \exp(-t)$, and (c) $K = 1/\sqrt{t}$. The relaxation parameter $\omega = 10$.

$$p_2(x) = \exp\left(-20(x-0.5)^2\right)$$



(a) (b) (c)

Figure: The exact source p_2 and its corresponding numerical solution, retrieved using various levels of noise in the additional measurement, for various convolution kernels, namely (a) $K = 0$, (b) $K = \exp(-t)$, and (c) $K = 1/\sqrt{t}$. The relaxation parameter $\omega = 10$.

Table: The stopping iteration number $\tilde{k} = k(e(\tilde{e}), 10)$ and the CPU time (mins), obtained for the experiments with the unknown sources p_1 and p_2.

$\tilde{e}$	p_1 1%		p_1 5%		p_1 10%		p_2 0.5%		p_2 1%		p_2 3%	
	$\tilde{k}$	time	$\tilde{k}$	time	$\tilde{k}$	time	$\tilde{k}$	time	$\tilde{k}$	time	$\tilde{k}$	time
$K = 0$	136	94.7	11	8.2	9	6.3	387	327.4	386	327.2	172	60.7
$K = \exp(-t)$	133	138.7	9	10.4	9	9.9	503	538.1	321	416.2	177	111.2
$K = 1/\sqrt{t}$	142	144.3	10	11	8	8.9	491	532.6	390	468.4	206	183.4

Following experiments ($K \equiv 0$):

$$p_3(x) = \begin{cases} 0 & 0 \leqslant x \leqslant \frac{1}{3} \\ 6x - 2 & \frac{1}{3} \leqslant x \leqslant \frac{1}{2} \\ 4 - 6x & \frac{1}{2} \leqslant x \leqslant \frac{2}{3} \\ 0 & \frac{2}{3} \leqslant x \leqslant 1 \end{cases}, \qquad p_4(x) = \begin{cases} x(0.5 - x)(1 - x) & 0 \leqslant x \leqslant \frac{1}{2} \\ x(x - 0.5)(1 - x) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases},$$

$$p_5(x) = \begin{cases} 0 & 0 \leqslant x < \frac{1}{3} \\ 1 & \frac{1}{3} \leqslant x \leqslant \frac{2}{3} \\ 0 & \frac{2}{3} < x \leqslant 1 \end{cases}, \qquad p_6(x) = 10x(x - 1)^2$$

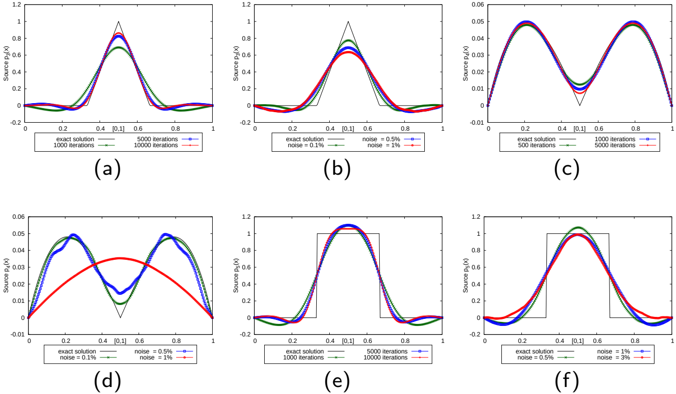
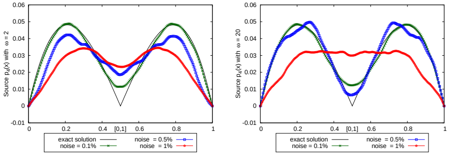


Figure: The exact sources p_3, p_4 and p_5 and its numerical approximations for $\tilde{e} = 0\%$ (a,c,e) and for different noise levels (b,d,f). The relaxation parameter $\omega = 10$.

Other relaxation parameter

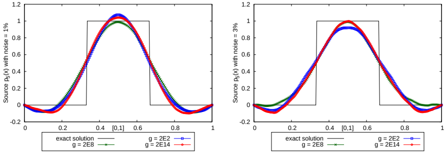


(a) (b)

Figure: The exact source p_4 and its numerical approximations for $\omega = 2$ (a) and for $\omega = 20$ (b).

- The results for small noise are similar to the results obtained when $\omega = 10$

Results of numerical experiments



(a) (b)

Figure: The exact source p_5 and its numerical approximations for $\tilde{e} = 1\%$ (a) and for $\tilde{e} = 3\%$ (b) for different values of g. The relaxation parameter $\omega = 10$.

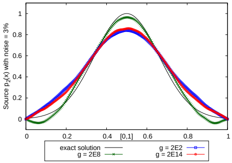


Figure: The exact source p_2 and its numerical approximations for $\tilde{e} = 3\%$ for different values of g. The relaxation parameter $\omega = 10$.

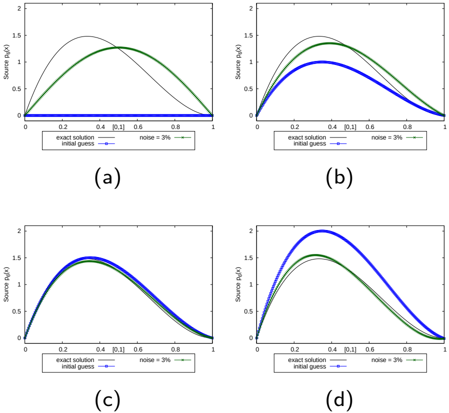
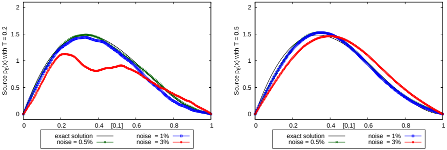


Figure: The non-symmetric exact source p_6 and its numerical approximations (using $\tilde{e} = 3\%$) for different initial guesses: 0 (a), $6.44x - 12.27x^2 + 5.83x^3$ (b), $9.68x - 18.46x^2 + 8.78x^3$ (c) and $12.88x - 24.54x^2 + 11.65x^3$ (d). The relaxation parameter $\omega = 10$.



(a) (b)

Figure: The non-symmetric exact source p_6 and its numerical approximations for $T = 0.2$ (a) and $T = 0.5$ (b). The relaxation parameter $\omega = 10$.

Conclusion

- It is possible to recover uniquely an unknown vector source in all types of damped thermoelastic systems when an additional final in time measurement of the displacement is measured
- A numerical algorithm in a linear case gives accurate shape recovery
- The algorithm is sensitive to the amount of noise added to the data
- There is a certain limitation of the method with respect to the recovery of non-symmetric sources

Future research

- More numerical experiments (e.g. influence of the parameter g on the results)
- Testing different stopping criteria (up to now, no better results)
- What if $\mathbf{g}$ is nonlinear?
- Other inverse problems for thermoelasticity, e.g. the recovery of time-dependent sources, convolution kernel
- Goal: with numerical scheme!

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