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The semi-discretization of a nonlocal parabolic and hyperbolic model for type-I superconductors

K. Van Bockstal and M. Slodička

Ghent University
Department of Mathematical Analysis
Numerical Analysis and Mathematical Modelling Research Group

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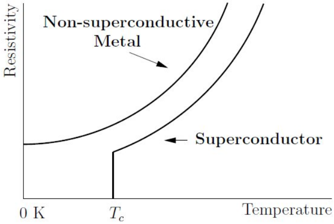
Outline

Features of superconductivity

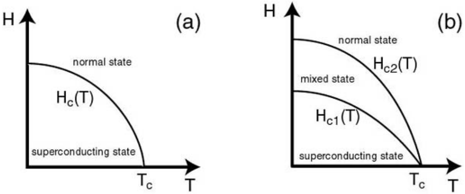
- Kammerlingh Onnes (1911): perfect conductivity



For various cooled down materials the electrical resistance not only decreases with temperature, but also has a sudden drop at some critical absolute temperature T_c

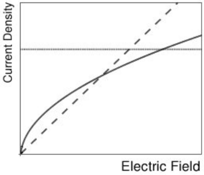
- Meissner and Ochsenfeld (1933): perfect diamagnetism
  $\Rightarrow$ i.e. expulsion of the magnetic induction $\boldsymbol{B}$
- Kammerlingh Onnes (1914): threshold field
  $\Rightarrow$ restore the normal state through the application of a large magnetic field
- A way to classify superconductors: type-I and type-II

- Similar behaviour for a very weak external magnetic field when the temperature $T < T_c$ is fixed
- As the external magnetic field becomes stronger it turns out that two possibilities can happen $\Rightarrow$ phase diagram in the T-H plane



- Type-I (a): the $\boldsymbol{B}$ field remains zero inside the superconductor until suddenly, as the critical field H_c is reached, the superconductivity is destroyed
- Type-II (b): a mixed state occurs in addition to the superconductive and the normal state (two different critical fields)
- Main topic: macroscopic models for type-I superconductors
- What are the macroscopic models which are used in the modelling of type-II superconductors?

- Dependency between current density $\boldsymbol{J}$ and the electric field $\boldsymbol{E}$



- Ohm's law for non-superconducting metal (dashed)
- Bean's critical-state model for Type-II superconductors (fine dashed): current either flows at the critical level $\boldsymbol{J}_c$ or not at all $\Rightarrow$ not fully applicable
- The power law by Rhyner for Type-II superconductors (continuous)

$$\boldsymbol{E} = |\boldsymbol{J}|^{n-1}\boldsymbol{J}, \qquad n \in (7, 1000)$$

- The full Maxwell equations ($\tilde{\delta} = 1$) and the quasi-static Maxwell equations ($\tilde{\delta} = 0$) for linear materials are considered

$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \tilde{\delta}\epsilon\partial_t \boldsymbol{E}$	Ampère's law	$\boldsymbol{H}$	magnetic field	$\epsilon > 0$	electric permittivity
$\nabla \times \boldsymbol{E} = -\mu\partial_t \boldsymbol{H}$	Faraday's law	$\boldsymbol{E}$	electric field	$\mu > 0$	magnetic permeability
$\nabla \cdot \boldsymbol{H}_0 = 0$		$\boldsymbol{J}$	current density		

- The formulation is in terms of electric field $\Rightarrow$ the power law has to be inverted:

$$\boldsymbol{J} = |\boldsymbol{E}|^{-\frac{1}{p}}\boldsymbol{E}, \qquad \text{for } p \in (1, 1.2) \text{ as } p = \frac{n}{n-1}$$

- Take the time derivative of Ampère's law and the curl of Faraday's law
$\Rightarrow$ nonlinear and degenerate partial differential equation for the electric field

$$\boxed{\tilde{\delta}\epsilon\partial_{tt}\boldsymbol{E} + \partial_t\left(|\boldsymbol{E}|^{-\frac{1}{p}}\boldsymbol{E}\right) + \frac{1}{\mu}\nabla \times \nabla \times \boldsymbol{E} = \boldsymbol{0}, \qquad \tilde{\delta} = 0 \vee 1}$$

- If $\tilde{\delta} = 0$:

$$\boxed{\mu\partial_t\boldsymbol{H} + \nabla \times \left(|\nabla \times \boldsymbol{H}|^{n-1}\nabla \times \boldsymbol{H}\right) = \boldsymbol{0}}$$

- Studied by: Barrett, Prigozhin, Sokolovsky, Yin, Li, Zou, Wei,...
- Is it possible to derive macroscopic models for type-I superconductors?

- $\Omega \subset \mathbb{R}^3$: bounded Lipschitz domain, $\boldsymbol{\nu}$ unit normal vector on $\partial\Omega$
- London and London (1935): a macroscopic description of type-I superconductors involves a two-fluid model

$$\boldsymbol{J} = \boldsymbol{J}_n + \boldsymbol{J}_s$$
$$\boldsymbol{J}_n = \sigma \boldsymbol{E} \qquad \text{Ohm's law}$$

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \boldsymbol{J}_s + \tilde{\delta}\epsilon\partial_t \boldsymbol{E}$$
$$\nabla \times \boldsymbol{E} = -\mu\partial_t \boldsymbol{H}$$
$$\nabla \cdot \boldsymbol{H}_0 = 0$$

$\boldsymbol{J}_n$	normal current density
$\boldsymbol{J}_s$	superconducting current density
σ	conductivity of normal electrons

- Below the critical temperature T_c, the current consists of superconducting electrons and normal electrons
- London equations (1935) $\Rightarrow$ local law for $\boldsymbol{J}_s$

$$\partial_t \boldsymbol{J}_s = \Lambda^{-1}\boldsymbol{E}$$
$$\nabla \times \boldsymbol{J}_s = -\Lambda^{-1}\boldsymbol{B}$$
$$\Lambda = \frac{m_e}{n_s e^2}$$

n_s	density of superelectrons
m_e	mass of an electron
$-e$	electric charge of an electron

$\Rightarrow$ Correct description of two basic properties of superconductors:

perfect conductivity and perfect diamagnetism (Meissner effect)

$$\nabla \cdot \boldsymbol{B} = 0 \Rightarrow \exists \boldsymbol{A} \in \mathbf{H}^1(\Omega) \text{ such that } \boldsymbol{B} = \nabla \times \boldsymbol{A} \text{ and } \nabla \cdot \boldsymbol{A} = 0$$

$$\nabla \times \boldsymbol{J}_s = -\Lambda^{-1}\boldsymbol{B} \quad \Rightarrow \quad \boldsymbol{J}_s(\boldsymbol{x},t) = -\Lambda^{-1}\boldsymbol{A}(\boldsymbol{x},t), \qquad (\boldsymbol{x},t) \in Q_T := \Omega \times (0,T)$$

Generalization of London and London: nonlocal laws

- Pippard (1953)

$$\boldsymbol{J}_{s,p}(\boldsymbol{x},t) = \int_\Omega Q(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{A}(\boldsymbol{x}',t)\,\mathrm{d}\boldsymbol{x}', \qquad (\boldsymbol{x},t)\in\Omega\times(0,T)$$

with

$$Q(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{A}(\boldsymbol{x}',t) = -\widetilde{C}\frac{\boldsymbol{x}-\boldsymbol{x}'}{\left|\boldsymbol{x}-\boldsymbol{x}'\right|^4}\left[\boldsymbol{A}(\boldsymbol{x}',t)\cdot(\boldsymbol{x}-\boldsymbol{x}')\right]\exp\left(-\frac{\left|\boldsymbol{x}-\boldsymbol{x}'\right|}{r_0}\right),$$

$$\widetilde{C} := \frac{3}{4\pi\xi_0\Lambda} > 0, \qquad r_0 = \frac{\xi_0 l}{\xi_0 + l}$$

ξ_0 the coherence length of the material, l is the mean free path

- Eringen (1984)

$$\boldsymbol{J}_{s,e}(\boldsymbol{x},t) = \int_\Omega \sigma_0\left(\left|\boldsymbol{x}-\boldsymbol{x}'\right|\right)(\boldsymbol{x}-\boldsymbol{x}')\times\boldsymbol{H}(\boldsymbol{x}',t)\,\mathrm{d}\boldsymbol{x}' =: -(\mathcal{K}_0 \star \boldsymbol{H})(\boldsymbol{x},t), \qquad (\boldsymbol{x},t)\in\Omega\times(0,T)$$

with

$$\sigma_0(s) = \begin{cases} \dfrac{\widetilde{C}}{2s^2}\exp\left(-\dfrac{s}{r_0}\right) & s < r_0; \\ 0 & s \geqslant r_0 \end{cases}$$

- Pippard's nonlocal law fails to explain the vanishing of electrical resistance
- It is possible to recover from Eringen's law the London equations and the form given by Pippard

$$\Rightarrow \boldsymbol{J}_s = \boldsymbol{J}_{s,e} = -\mathcal{K}_0 \star \boldsymbol{H} \quad \text{in} \quad \left\{ \begin{array}{ll} \nabla \times \boldsymbol{H} & = \sigma \boldsymbol{E} + \boldsymbol{J}_s + \tilde{\delta}\epsilon\partial_t \boldsymbol{E} \\ \nabla \times \boldsymbol{E} & = -\mu\partial_t \boldsymbol{H} \end{array} \right.$$

- Taking the curl of Ampère's law and the time derivative of Faraday's law result in

$$\boxed{\tilde{\delta}\epsilon\mu\partial_{tt}\boldsymbol{H} + \sigma\mu\partial_t\boldsymbol{H} + \nabla \times \nabla \times \boldsymbol{H} + \nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) = \boldsymbol{0}, \qquad \tilde{\delta} = 0 \vee 1}$$

- For ease of exposition, set $\mu = \sigma = \epsilon = 1$
- A possible source term $\boldsymbol{f}$ is added

A vectorial nonlocal linear parabolic and hyperbolic problem for type-I superconductors

- Two problems

$$\tilde{\delta} = 0 \quad \Rightarrow \quad \begin{cases} \partial_t \boldsymbol{H} + \nabla \times \nabla \times \boldsymbol{H} + \nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) & = \boldsymbol{f} & \text{in } Q_T; \\ \boldsymbol{H} \times \boldsymbol{\nu} & = \boldsymbol{0} & \text{on } \partial\Omega \times (0,T); \\ \boldsymbol{H}(\boldsymbol{x},0) & = \boldsymbol{H}_0 & \text{in } \Omega; \end{cases}$$

$$\tilde{\delta} = 1 \quad \Rightarrow \begin{cases} \partial_{tt} \boldsymbol{H} + \partial_t \boldsymbol{H} + \nabla \times \nabla \times \boldsymbol{H} + \nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) & = \boldsymbol{f} & \text{in } Q_T; \\ \boldsymbol{H} \times \boldsymbol{\nu} & = \boldsymbol{0} & \text{on } \partial\Omega \times (0,T); \\ \boldsymbol{H}(\boldsymbol{x},0) & = \boldsymbol{H}_0 & \text{in } \Omega; \\ \partial_t \boldsymbol{H}(\boldsymbol{x},0) & = \boldsymbol{H}_0' & \text{in } \Omega; \end{cases}$$

- Variational formulation ($\tilde{\delta} = 0 \vee 1$):

$$\tilde{\delta}\,(\partial_{tt}\boldsymbol{H},\varphi) + (\partial_t \boldsymbol{H},\varphi) + (\nabla \times \boldsymbol{H}, \nabla \times \varphi) + (\mathcal{K}_0 \star \boldsymbol{H}, \nabla \times \varphi) = (\boldsymbol{f},\varphi), \quad \forall \varphi \in \mathbf{H}_0(\mathbf{curl}\,,\Omega)$$

- Mathematical analysis: estimates on the kernels σ_0 and $\mathcal{K}_0$, time discretization
- The well-posedness of both problems is studied, two numerical schemes for computations are designed and error estimates for the time discretization are derived

Estimates on the singular kernels σ_0 and $\mathcal{K}_0$

Using spherical coordinates one can deduce that

- $\sigma_0(|\boldsymbol{x}|)\boldsymbol{x} \in \mathbf{L}^p(\Omega)$ for $p \in [1,3)$:

$$\int_\Omega |\sigma_0(|\boldsymbol{x}|)\,\boldsymbol{x}|^p \,\mathrm{d}\boldsymbol{x} \leqslant \int_{B(\boldsymbol{0},r_0)} \frac{C}{|\boldsymbol{x}|^{2p}} \left|\exp\left(-\frac{|\boldsymbol{x}|}{r_0}\right)\right|^p |\boldsymbol{x}|^p \,\mathrm{d}\boldsymbol{x}$$

$$\leqslant C \int_0^{2\pi} \mathrm{d}\varphi \int_0^\pi \sin(\theta)\mathrm{d}\theta \int_0^{r_0} r^{2-p}\mathrm{d}r \leqslant C\left[\frac{r^{3-p}}{3-p}\right]_0^{r_0} < \infty$$

- $|\boldsymbol{J}_s(\boldsymbol{x},t)| = |(\mathcal{K}_0 \star \boldsymbol{H})(\boldsymbol{x},t)| \leqslant C(q)\,\|\boldsymbol{H}(t)\|_q$ for $q > \frac{3}{2}$, $\quad \forall \boldsymbol{x} \in \Omega$:

$$|\boldsymbol{J}_s(\boldsymbol{x},t)| = \left|\int_\Omega \sigma_0\left(\left|\boldsymbol{x}-\boldsymbol{x}'\right|\right)(\boldsymbol{x}-\boldsymbol{x}') \times \boldsymbol{H}(\boldsymbol{x}',t)\,\mathrm{d}\boldsymbol{x}'\right| \leqslant \int_\Omega \left|\sigma_0\left(\left|\boldsymbol{x}-\boldsymbol{x}'\right|\right)(\boldsymbol{x}-\boldsymbol{x}')\right| \left|\boldsymbol{H}(\boldsymbol{x}',t)\right| \,\mathrm{d}\boldsymbol{x}'$$

$$\leqslant \sqrt[p]{\int_\Omega \left|\sigma_0\left(\left|\boldsymbol{x}-\boldsymbol{x}'\right|\right)(\boldsymbol{x}-\boldsymbol{x}')\right|^p \mathrm{d}\boldsymbol{x}'}\ \sqrt[q]{\int_\Omega |\boldsymbol{H}(\boldsymbol{x}',t)|^q \,\mathrm{d}\boldsymbol{x}'} \leqslant C\,\|\boldsymbol{H}(t)\|_q$$

- For instance, it holds that

$$\boxed{(\mathcal{K}_0 \star \boldsymbol{h}, \nabla \times \boldsymbol{h}) \leqslant C_\varepsilon \|\boldsymbol{h}\|^2 + \varepsilon \|\nabla \times \boldsymbol{h}\|^2, \quad \forall \boldsymbol{h} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega)}$$

Numerical schemes to approximate the solution ($\tilde{\delta} = 0 \vee 1$)

- Rothe's method: divide $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals (t_{i-1}, t_i) for $t_i = i\tau$, where $\tau = T/n$ and for any function u

$$u_i := u(t_i), \quad \partial_t u(t_i) \approx \delta u_i := \frac{u_i - u_{i-1}}{\tau}, \quad \partial_{tt} u(t_i) \approx \delta^2 u_i := \frac{\delta u_i - \delta u_{i-1}}{\tau}$$

- Convolution implicitly (from the actual time step):

$$\begin{cases} \tilde{\delta}\left(\delta^2 \boldsymbol{h}_i, \boldsymbol{\varphi}\right) + (\delta \boldsymbol{h}_i, \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) + (\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) & = (\boldsymbol{f}_i, \boldsymbol{\varphi}); \\ \boldsymbol{h}_0 & = \boldsymbol{H}_0 \end{cases}$$

- Lax-Milgram lemma: existence of a unique solution for any $i = 1, \ldots, n$ and any $\tau < \tau_0$
- Convolution explicitly (from the previous time step):

$$\begin{cases} \tilde{\delta}\left(\delta^2 \boldsymbol{h}_i, \boldsymbol{\varphi}\right) + (\delta \boldsymbol{h}_i, \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) & = (\boldsymbol{f}_i, \boldsymbol{\varphi}) - (\mathcal{K}_0 \star \boldsymbol{h}_{i-1}, \nabla \times \boldsymbol{\varphi}); \\ \boldsymbol{h}_0 & = \boldsymbol{H}_0 \end{cases}$$

- Lax-Milgram lemma: existence of a unique solution for any $i = 1, \ldots, n$ and any $\tau > 0$
- Now: look at both models separately

Convergence: a priori estimates as uniform bounds

Suppose that $\boldsymbol{f} \in L^2\left((0,T),\mathbf{L}^2(\Omega)\right)$

(i) Let $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$. Then, there exists a positive constant C such that for all $\tau < \tau_0$

$$\max_{1\leqslant i\leqslant n} \|\boldsymbol{h}_i\|^2 + \sum_{i=1}^{n} \left\|\boldsymbol{h}_i - \boldsymbol{h}_{i-1}\right\|^2 + \sum_{i=1}^{n} \|\nabla\times\boldsymbol{h}_i\|^2\,\tau \leqslant C$$

(ii) If $\nabla\cdot\boldsymbol{H}_0 = 0 = \nabla\cdot\boldsymbol{f}$ then $\nabla\cdot\boldsymbol{h}_i = 0$ for all $i = 1,\ldots,n$. Moreover, we have that

$$\sum_{i=1}^{n} \|\delta\boldsymbol{h}_i\|^2_{\mathbf{H}_0^{-1}(\mathbf{curl}\,,\Omega)}\,\tau \leqslant C$$

(iii) If $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,,\Omega)$ then for all $\tau < \tau_0$

$$\max_{1\leqslant i\leqslant n} \|\nabla\times\boldsymbol{h}_i\|^2 + \sum_{i=1}^{n} \left\|\nabla\times\boldsymbol{h}_i - \nabla\times\boldsymbol{h}_{i-1}\right\|^2 + \sum_{i=1}^{n} \|\delta\boldsymbol{h}_i\|^2\,\tau \leqslant C$$

(iv) If $\partial_t\boldsymbol{f} \in L^2\left((0,T),\mathbf{L}^2(\Omega)\right)$, $\nabla\times(\mathcal{K}_0\star\boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,,\Omega)$ and $\nabla\times\nabla\times\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ then for all $\tau < \tau_0$

$$\max_{1\leqslant i\leqslant n} \|\delta\boldsymbol{h}_i\|^2 + \sum_{i=1}^{n} \left\|\delta\boldsymbol{h}_i - \delta\boldsymbol{h}_{i-1}\right\|^2 + \sum_{i=1}^{n} \|\nabla\times\delta\boldsymbol{h}_i\|^2\,\tau \leqslant C$$

$\boldsymbol{h}_n$: piecewise linear in time spline of the solutions $\boldsymbol{h}_i, i = 1, \ldots, n$

Theorem (Existence solution and error estimate for par. problem)

- *Let* $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ *and* $\boldsymbol{f} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right)$. *Assume that* $\nabla \cdot \boldsymbol{H}_0 = 0 = \nabla \cdot \boldsymbol{f}(t)$ *for any time* $t \in [0,T]$. *Then there exists a solution* $\boldsymbol{H} \in C\left([0,T], \mathbf{L}^2(\Omega)\right) \cap L^2\left((0,T), \mathbf{H}^{\frac{1}{2}}(\Omega)\right)$ *with* $\partial_t \boldsymbol{H} \in L^2\left((0,T), \mathbf{H}_0^{-1}(\mathbf{curl}\,, \Omega)\right)$
- *Suppose that* $\boldsymbol{f} \in \mathrm{Lip}\left([0,T], \mathbf{L}^2(\Omega)\right)$
  - (i) *If* $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$ *then*
  $$\max_{t\in[0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla \times [\boldsymbol{h}_n - \boldsymbol{H}]\|^2 \leqslant C\tau$$
  - (ii) *If* $\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$ *and* $\nabla \times \nabla \times \boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ *then*
  $$\max_{t\in[0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla \times [\boldsymbol{h}_n - \boldsymbol{H}]\|^2 \leqslant C\tau^2$$

- Theorem holds for both numerical schemes!

Theorem (Existence solution and error estimate for hyp. problem)

- *Let* $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$, $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$ *and* $\boldsymbol{f} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right)$. *Assume that* $\nabla \cdot \boldsymbol{H}_0 = \nabla \cdot \boldsymbol{H}_0' = 0 = \nabla \cdot \boldsymbol{f}(t)$ *for any time* $t \in [0,T]$. *Then there exists a solution* $\boldsymbol{H}$ *such that* $\boldsymbol{H} \in C\left([0,T], \mathbf{H}^{\frac{1}{2}}(\Omega)\right)$, $\partial_t \boldsymbol{H} \in L^2\left((0,T), \mathbf{H}^{\frac{1}{2}}(\Omega)\right) \cap C\left([0,T], \mathbf{L}^2(\Omega)\right)$ *and* $\partial_{tt} \boldsymbol{H} \in L^2\left((0,T), \mathbf{H}_0^{-1}(\mathbf{curl}\,, \Omega)\right)$
- *Suppose that* $\boldsymbol{f} \in \mathrm{Lip}\left([0,T], \mathbf{L}^2(\Omega)\right)$.
  - (i) *If* $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$ *and* $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$ *then*
  $$\max_{t\in[0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \max_{t\in[0,T]} \left\|\nabla \times \int_0^t [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 \leqslant C\tau$$
  - (ii) *If* $\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$, $\boldsymbol{H}_0' \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$ *and* $\nabla \times \nabla \times \boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ *then*
  $$\max_{t\in[0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \max_{t\in[0,T]} \left\|\nabla \times \int_0^t [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 \leqslant C\tau^2$$

- Suboptimal convergence rates $\mathcal{O}(\tau)$ in the space $C\left([0,T],\mathbf{L}^2(\Omega)\right)$
- Grönwall lemma: $\mathcal{O}(\tau) = e^{CT}\tau$
- To get rid of the exponential character of this constant, the use of Grönwall's lemma should be avoided
- How? This can be tried by symmetrification of the problem, namely by incorporation of the curl operator $\nabla \times \boldsymbol{J}_s$ into a new convolution kernel

Lemma

$$(\boldsymbol{x},t) \in \Omega \times (0,T), \nabla \cdot \boldsymbol{H} = 0 \text{ and } \boldsymbol{H} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega$$

$$\Rightarrow \quad \nabla \times \boldsymbol{J}_s(\boldsymbol{x},t) = -\int_\Omega \mathcal{K}(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{H}(\boldsymbol{x}',t)\; d\boldsymbol{x}' =: -(\mathcal{K} \star \boldsymbol{H})(\boldsymbol{x},t),$$

$$\text{where } \mathcal{K} : \Omega \times \Omega \to \mathbb{R} : (\boldsymbol{x},\boldsymbol{x}') \mapsto \kappa(|\boldsymbol{x}-\boldsymbol{x}'|)$$

$$\text{with } \kappa : (0,\infty) \to \mathbb{R} : s \mapsto \begin{cases} \frac{\widetilde{C}}{2s^2}\left(1-\frac{s}{r_0}\right)\exp\left(-\frac{s}{r_0}\right) & s < r_0; \\ 0 & s \geqslant r_0 \end{cases}$$

- Is this approach successful for both problems?

Models in $\mathbf{H}^1(\Omega) \subset \mathbf{H}(\mathrm{div}, \Omega) \cap \mathbf{H}(\mathbf{curl}, \Omega)$ $\quad (\tilde{\delta} = 0 \vee 1)$

$$\nabla \times \boldsymbol{J}_s = -\mathcal{K} \star \boldsymbol{H} \quad \text{in} \quad \begin{cases} \nabla \times \nabla \times \boldsymbol{H} & = \sigma \nabla \times \boldsymbol{E} + \nabla \times \boldsymbol{J}_s + \tilde{\delta} \epsilon \nabla \times \partial_t \boldsymbol{E} \\ \nabla \times \boldsymbol{E} & = -\mu \partial_t \boldsymbol{H} \end{cases}$$

$$-\Delta \boldsymbol{H} = \nabla \times (\nabla \times \boldsymbol{H}) - \nabla(\nabla \cdot \boldsymbol{H})$$

$$\overset{\nabla \cdot \boldsymbol{H} = 0}{\Rightarrow} \quad \boxed{\tilde{\delta} \epsilon \mu \partial_{tt} \boldsymbol{H} + \sigma \mu \partial_t \boldsymbol{H} - \Delta \boldsymbol{H} + \mathcal{K} \star \boldsymbol{H} = \mathbf{0}}$$

Variational formulation:

$$\tilde{\delta}\left(\partial_{tt} \boldsymbol{H}, \varphi\right) + (\partial_t \boldsymbol{H}, \varphi) + (\nabla \boldsymbol{H}, \nabla \varphi) + (\mathcal{K} \star \boldsymbol{H}, \varphi) = (\boldsymbol{f}, \varphi), \qquad \forall \varphi \in \mathbf{H}_0^1(\Omega)$$

Again two numerical schemes (convolution implicitly $\Leftrightarrow$ convolution explicitly), $i = 1, \ldots, n$:

$$\begin{cases} \tilde{\delta}\left(\delta^2 \boldsymbol{h}_i, \varphi\right) + (\delta \boldsymbol{h}_i, \varphi) + (\nabla \boldsymbol{h}_i, \nabla \varphi) + (\mathcal{K} \star \boldsymbol{h}_i, \varphi) & = (\boldsymbol{f}_i, \varphi), \qquad \varphi \in \mathbf{H}_0^1(\Omega); \\ \boldsymbol{h}_0 & = \boldsymbol{H}_0 \end{cases}$$

$$\begin{cases} \tilde{\delta}\left(\delta^2 \boldsymbol{h}_i, \varphi\right) + (\delta \boldsymbol{h}_i, \varphi) + (\nabla \boldsymbol{h}_i, \nabla \varphi) & = (\boldsymbol{f}_i, \varphi) - (\mathcal{K} \star \boldsymbol{h}_{i-1}, \varphi), \qquad \varphi \in \mathbf{H}_0^1(\Omega); \\ \boldsymbol{h}_0 & = \boldsymbol{H}_0 \end{cases}$$

Properties of the kernel $\mathcal{K}$

- $\mathcal{K}(\boldsymbol{x},\cdot) \in L_p(\Omega)$ if $p \in \left[1, \frac{3}{2}\right), \quad \forall \boldsymbol{x} \in \Omega$

$$\int_\Omega |K(\boldsymbol{x},\boldsymbol{x}')|^p \, \mathrm{d}\boldsymbol{x}' \leqslant \int_{B(\boldsymbol{x},r_0)} \frac{C}{|\boldsymbol{x}-\boldsymbol{x}'|^{2p}} \left| \left(1 - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{r_0} \right) \right|^p \left| \exp\left(- \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{r_0} \right) \right|^p \mathrm{d}\boldsymbol{x}'$$

$$\leqslant \int_{B(\boldsymbol{x},r_0)} \frac{C}{|\boldsymbol{x}-\boldsymbol{x}'|^{2p}} \leqslant C \left[\frac{r^{3-2p}}{3-2p} \right]_0^{r_0} < \infty$$

- $|(\mathcal{K} \star \boldsymbol{H})(\boldsymbol{x},t)| = \left| \int_\Omega \mathcal{K}(\boldsymbol{x},\boldsymbol{x}') \boldsymbol{H}(\boldsymbol{x}',t) \, \mathrm{d}\boldsymbol{x}' \right| \leqslant C(q) \left\| \boldsymbol{H}(t) \right\|_q, \quad \forall q > 3, \quad \forall \boldsymbol{x} \in \Omega$

$$|(\mathcal{K} \star \boldsymbol{H})(\boldsymbol{x},t)| \leqslant \sqrt[p]{\int_\Omega |\mathcal{K}(\boldsymbol{x},\boldsymbol{x}')|^p \, \mathrm{d}\boldsymbol{x}'} \sqrt[q]{\int_\Omega |\boldsymbol{H}(\boldsymbol{x}',t)|^q \, \mathrm{d}\boldsymbol{x}'} \leqslant C(q) \left\| \boldsymbol{H}(t) \right\|_q$$

- Schoenberg interpolation theorem: $\mathcal{K}$ is positive definite
- For instance, it holds that (Sobolev embeddings theorem in $\mathbb{R}^3$: $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ + Friedrichs inequality)

$$\boxed{0 \leqslant (\mathcal{K} \star \boldsymbol{h}, \boldsymbol{h}) \leqslant C_\varepsilon \left\| \boldsymbol{h} \right\|^2_{\mathbf{H}^1(\Omega)} + \varepsilon \left\| \boldsymbol{h} \right\|^2 \leqslant C_\varepsilon \left\| \nabla \boldsymbol{h} \right\|^2 + \varepsilon \left\| \boldsymbol{h} \right\|^2, \quad \forall \boldsymbol{h} \in \mathbf{H}_0^1(\Omega)}$$

- This leads only to a better estimate if $\tilde{\delta} = 0$

Theorem (Error estimate for the par. problem in $\mathbf{H}^1(\Omega)$)

Assume that $\boldsymbol{f} \in \mathrm{Lip}\left([0, T], \mathbf{L}^2(\Omega)\right)$.

(i) *If* $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega)$ *then*

$$\max_{t\in[0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla[\boldsymbol{h}_n - \boldsymbol{H}]\|^2 \leqslant C\tau$$

(ii) *If* $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ *then*

$$\max_{t\in[0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla[\boldsymbol{h}_n - \boldsymbol{H}]\|^2 \leqslant C\tau^2$$

Proof:

$$\mathcal{K} \text{ positive definite} \Rightarrow \int_0^t (\mathcal{K} \star \boldsymbol{h}, \boldsymbol{h}) \geqslant 0 \Rightarrow \text{no Grönwall}$$

□

- Theorem holds for both numerical schemes!

- Backward Euler method for the time discretization: $\tau = 2^{-j}$, $2 \leqslant j \leqslant 7$
- The following scheme is followed:

$$\begin{cases} (\boldsymbol{h}_i, \varphi) + \tau(\nabla \boldsymbol{h}_i, \nabla \varphi) & = \tau(\boldsymbol{f}_i, \varphi) - \tau(\mathcal{K} \star \boldsymbol{h}_{i-1}, \varphi) + (\boldsymbol{h}_{i-1}, \varphi), \quad \varphi \in \mathbf{H}_0^1(\Omega), \\ \boldsymbol{h}_0 & = \boldsymbol{H}_0 \end{cases}$$

- First order Lagrange elements for the space discretization
- $\Omega =$ unitcube, $T = 1$, $r_0 = 0.1$
- $\mathcal{T}_h$: triangulation of Ω
- $\boldsymbol{x}_{m,T}$ and $\mathrm{Vol}(T)$: the midpoint and the volume of a tetrahedron $T \in \mathcal{T}_h$
- Define the set

$$\mathcal{T}_{\boldsymbol{x}} := \{T \in \mathcal{T}_h : |\boldsymbol{x}_{m,T} - \boldsymbol{x}| < r_0\} \subset \mathcal{T}_h$$

- The convolution integral arising in the numerical experiments is solved numerically:

$$\boxed{\mathcal{K}(\boldsymbol{x}, \cdot) \star \boldsymbol{h} \approx \sum_{T \in \mathcal{T}_{\boldsymbol{x}}} \mathrm{Vol}(T) \mathcal{K}(\boldsymbol{x} - \boldsymbol{x}_{m,T}) \boldsymbol{h}(\boldsymbol{x}_{m,T})}$$

- Implementation: in FEniCS

$$\boldsymbol{H}^{\text{ex}} = (1+t^2)\begin{pmatrix} y-z \\ z-x \\ x-y \end{pmatrix}, \qquad E = \max_{t\in[0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2$$

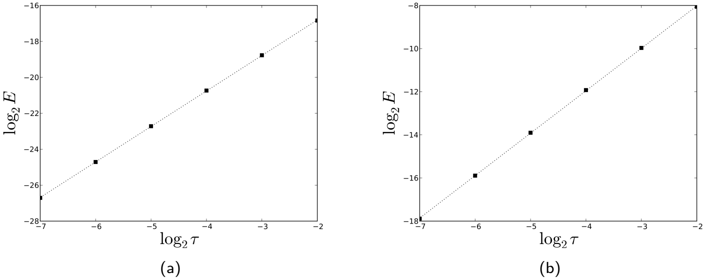


Figure: Convergence rate (a) $\widetilde{C} = 2$: regression line is $\log_2 E = 1.9753 \log_2 \tau - 12.858$
(b) $\widetilde{C} = 150$: regression line is $\log_2 E = 1.9678 \log_2 \tau - 4.0842$

Open questions

- Numerical experiment is time consuming: speed up the computations
- Implementation: convolution implicit
- Way out via Fourier transform?
- Full discretization of the models
- Modelling and analysis of a combined model for type-I and type-II superconductivity?

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