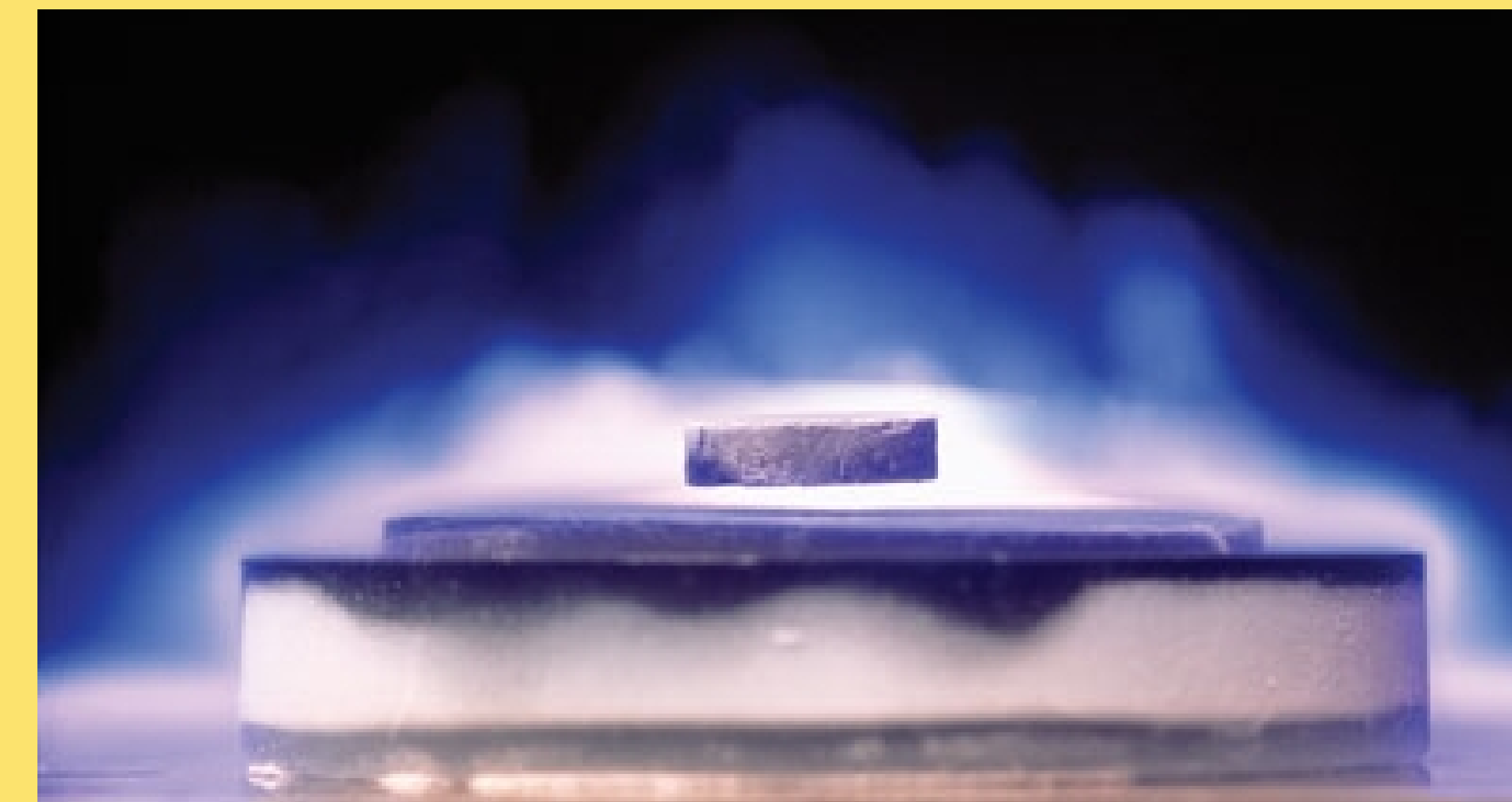


Domain

- ▷ Bounded domain Ω in \mathbb{R}^3 with a Lipschitz continuous boundary Γ
- ▷ ν denotes the outward unit normal vector on Γ
- ▷ Ω is occupied by a superconductive material
- ▷ This is a material, which loses all resistivity below a certain temperature T_c
- ▷ Figure: a magnet levitates above a ceramic superconductor cooled by liquid nitrogen



Full Maxwell's equations ($\delta = 1$) and quasi-static Maxwell's equations ($\delta = 0$) for linear materials

$$\begin{aligned}\nabla \times \mathbf{H} &= \mathbf{J} + \delta \varepsilon \partial_t \mathbf{E} \\ \nabla \times \mathbf{E} &= -\mu \partial_t \mathbf{H} \\ \nabla \cdot \mathbf{H} &= 0\end{aligned}$$

\mathbf{H} magnetic field
 \mathbf{E} electric field
 \mathbf{J} current density

$\varepsilon > 0$ electric permittivity
 $\mu > 0$ magnetic permeability

Two-fluid model

$$\begin{aligned}\mathbf{J} &= \mathbf{J}_n + \mathbf{J}_s \\ \mathbf{J}_n &= \sigma \mathbf{E}\end{aligned}$$

Ohm's law

$$\begin{aligned}\nabla \times \mathbf{H} &= \sigma \mathbf{E} + \mathbf{J}_s + \delta \varepsilon \partial_t \mathbf{E} \\ \nabla \times \mathbf{E} &= -\mu \partial_t \mathbf{H} \\ \nabla \cdot \mathbf{H} &= 0\end{aligned}$$

\mathbf{J}_n normal current density
 \mathbf{J}_s superconducting current density
 σ conductivity of normal electrons

Local law for \mathbf{J}_s : London equations (1935)

$$\begin{aligned}\partial_t \mathbf{J}_s &= \Lambda^{-1} \mathbf{E} & n_s & \text{density of superelectrons} \\ \nabla \times \mathbf{J}_s &= -\Lambda^{-1} \mathbf{B} & m & \text{mass of superelectrons} \\ \Lambda &= \frac{m}{n_s q^2} & q & \text{electric charge of superelectrons}\end{aligned}$$

⇒ Correct description of two basic properties of superconductors: perfect conductivity and perfect diamagnetism (Meissner effect)

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \exists \mathbf{A} \in \mathbf{H}^1(\Omega) \text{ such that } \mathbf{B} = \nabla \times \mathbf{A} \text{ and } \nabla \cdot \mathbf{A} = 0$$

$$\nabla \times \mathbf{J}_s = -\Lambda^{-1} \mathbf{B}$$

$$\Downarrow$$

$$\mathbf{J}_s(\mathbf{x}, t) = -\Lambda^{-1} \mathbf{A}(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times (0, T)$$

Generalization: nonlocal laws

▷ Pippard (1953)

$$\mathbf{J}_{s,p}(\mathbf{x}, t) = \int_{\Omega} Q(\mathbf{x} - \mathbf{x}') \mathbf{A}(\mathbf{x}', t) d\mathbf{x}', \quad (\mathbf{x}, t) \in \Omega \times (0, T)$$

with

$$Q(\mathbf{x} - \mathbf{x}') \mathbf{A}(\mathbf{x}', t) = -C \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^4} [\mathbf{A}(\mathbf{x}', t) \cdot (\mathbf{x} - \mathbf{x}')] \exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|}{r_0}\right)$$

▷ $C > 0$, $r_0 > 0$ is related to the mean free path in the material

▷ Eringen (1984)

$$\mathbf{J}_{s,e}(\mathbf{x}, t) = \int_{\Omega} \sigma_0(|\mathbf{x} - \mathbf{x}'|) (\mathbf{x} - \mathbf{x}') \times \mathbf{H}(\mathbf{x}', t) d\mathbf{x}', \quad (\mathbf{x}, t) \in \Omega \times (0, T)$$

with

$$\sigma_0(s) = \begin{cases} \frac{C}{2s^2} \exp\left(-\frac{s}{r_0}\right) & s < r_0; \\ 0 & s \geq r_0 \end{cases}$$

Lemma

$$(\mathbf{x}, t) \in \Omega \times (0, T), \mathbf{H} \cdot \nu = 0 \text{ on } \Gamma \Rightarrow \nabla \times \mathbf{J}_{s,e}(\mathbf{x}, t) = - \int_{\Omega} K(\mathbf{x} - \mathbf{x}') \mathbf{H}(\mathbf{x}', t) d\mathbf{x}' = - (K \star \mathbf{H})(\mathbf{x}, t) \text{ with } K : \Omega \rightarrow \mathbb{R} : \mathbf{y} \mapsto \begin{cases} \frac{C}{2|\mathbf{y}|^2} \left(1 - \frac{|\mathbf{y}|}{r_0}\right) \exp\left(-\frac{|\mathbf{y}|}{r_0}\right) & |\mathbf{y}| < r_0; \\ 0 & |\mathbf{y}| \geq r_0 \end{cases}$$

Consequence: two models ($\delta = 0, 1$)

$$\nabla \times \nabla \times \mathbf{H} = \sigma \nabla \times \mathbf{E} + \nabla \times \mathbf{J}_{s,e} + \delta \varepsilon \nabla \times \partial_t \mathbf{E} \Rightarrow \delta \varepsilon \mu \partial_{tt} \mathbf{H} + \sigma \mu \partial_t \mathbf{H} - \Delta \mathbf{H} + K \star \mathbf{H} = \mathbf{0}$$

Solution method

- ▷ Time discretization is based on Backward Euler's method
- ▷ Rothe's method ⇒ unique weak solution for both problems, numerical scheme

Numerical Experiment

- ▷ Penetration of magnetic field and induced current into the material (unit cube)
- ▷ Main difficulty: programming convolution term in Fénics (not finished)

Further research

- ▷ Comparison of the two models, practical applications
- ▷ Nonlinear model for \mathbf{H}

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