2009-06-01

Diffusion and Fractional Diffusion Based Image Processing

Jonathan Blackledge
Dublin Institute of Technology, jonathan.blackledge59@gmail.com

Recommended Citation

Diffusion and Fractional Diffusion Based Image Processing, Blackledge, Jonathan; EG UK Theory and Practice of Computer Graphics, pages: 233 - 240, Cardiff, 2009

EG UK Theory and Practice of Computer Graphics (2009), pp. 1–8
Wen Tang, John Collomosse (Editors)

Diffusion and Fractional Diffusion Based Image Processing

J. M. Blackledge[†]

School of Electrical Engineering Systems
Faculty of Engineering, Dublin Institute of Technology

Abstract

We consider the background to describing strong scattering in terms of diffusive processes based on the diffusion equation. Intermediate strength scattering is then considered in terms of a fractional diffusion equation which is studied using results from fractional calculus. This approach is justified in terms of the generalization of a random walk model with no statistical bias in the phase to a random walk that has a phase bias and is thus, only 'partially' or 'fractionally' diffusive. A Green's function solution to the fractional diffusion equation is studied and a result derived that provides a model for an incoherent image generated by light scattering from a tenuous random medium. Applications include image enhancement of star fields and other cosmological bodies imaged through interstellar dust clouds. An example of this application is given.

Categories and Subject Descriptors (according to ACM CCS): I.4.5 [Reconstruction]: Transform Methods

1. Introduction

A conventional approach to modelling light scattering in random media is to consider the scatterer to be a stochastic function with a characteristic Probability Density Function (PDF) under the weak scattering approximation [Bla06]. In the far field, the scattering amplitude is then given by the Fourier transform of the scattering function and the intensity of the scattered field (i.e. the measurable quantity, at optical frequencies and above) is determined by the Fourier transform of the autocorrelation of the scattering function. The inverse scattering problem is then reduced to estimating the correlation function by Fourier inversion and then solving the phase reconstruction problem to recover the scattering function from its autocorrelation function.

Multiple scattering processes are often modelled using a statistical approach [Mar06]. The aim is to develop a model of the PDF for the scattered field itself rather than for the scattering function. This involves concepts traditionally associated with the kinetic theory of gases in which the random motion of particles undergoing elastic collisions and following 'random walks' is 'replaced' with the random scattering of an electric field, for example, from multiple scattering sites. The total contribution of the multiple scattering process after N scattering interactions is given by [Fie09]

$$E=\sum_{j=1}^{N}a_j\exp(i\theta_j)$$

where the amplitude a_j, the phase θ_j and N are independent random variables. While this approach provides physically informative models for the PDF that can be used for the statistical characterisation of an image and image segmentation (using a moving 'window') to locate statistically significant features, it does not directly help in the development of algorithms for image restoration and reconstruction [BM86]. On the other hand, random walk models provide the physical basis for diffusion processes in general. This is the essential 'link' to modelling multiple scattering processes in terms of solutions to the diffusion equation for the intensity of light.

In certain circumstances, multiple scattering may only involve a small number of interactions. This occurs when light interacts with tenuous media, for example, and is considered to be one of the most difficult scenarios to model accurately. Diffusion processes are not applicable in such cases. In this paper, we study an approach to solving this problem using the fractional diffusion equation.

[†] SFI Stokes Professor of DSP

2. Optical Scattering

Analysis of scattering from a random medium ideally requires a model for the physical behaviour of the random variable(s) that is derived from basic principles. This involves modelling the scattered field in terms of its interaction with an ensemble of 'scattering sites' based on an assumed stochastic process. If the density of these scattering sites is low enough so that multiple scattering is minimal, then we can apply the weak scattering approximation to develop a model for the intensity of a wavefield interacting with a random (weak) scatterer.

In the far field, the (weak) scattered field (i.e. the scattering amplitude) is given by the Fourier transform of the scattering function. If this function is known *a priori*, then the scattering amplitude can be determined. This is an example of a deterministic model. If the scattering function is stochastic (i.e. a randomly distributed scatterer) such that it can only be quantified in terms of a PDF then we can simulate the scattered field by designing a random number generator that outputs deviates that conform to this distribution. The Fourier transform of this stochastic field then provides the scattering amplitude. Thus, given a three dimensional optical Helmholtz scattering function of compact support $\gamma(\mathbf{r}), \quad \mathbf{r}\in V, \quad \mathbf{r}=\hat{\mathbf{x}}x+\hat{\mathbf{y}}y+\hat{\mathbf{z}}z$ with $\Pr[\gamma(\mathbf{r})]$ known *a priori*, the scattering amplitude A is given by [Bla06]

$$A(\hat{\mathbf{N}},k)=k^2\int\limits_V \exp(-ik\hat{\mathbf{N}}\cdot\mathbf{r})\gamma(\mathbf{r})d^3\mathbf{r}$$

where $\hat{\mathbf{N}}=\hat{\mathbf{n}}_s-\hat{\mathbf{n}}_i$ and $\gamma(\mathbf{r})$ is a stochastic function whose deviates conform to the PDF $\Pr[\gamma(\mathbf{r})]$. Here, $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_s$ denote the direction of the incident and scattered fields respectively and $\gamma(\mathbf{r})=\epsilon_r(\mathbf{r})-1$ where $\epsilon_r\geq 1$ is the relative permittivity (a real function), a result that is based on application of a scalar electromagnetic scattering model for a non-conductive dielectric.

The intensity of the scattering amplitude is given by

$$I(\hat{\mathbf{N}},k)=\mid A(\hat{\mathbf{N}},k)\mid^2=A(\hat{\mathbf{N}},k)A^*(\hat{\mathbf{N}},k)$$

$$=k^4\int\limits_V \exp(-ik\hat{\mathbf{N}}\cdot\mathbf{r})\gamma(\mathbf{r})d^3\mathbf{r}\int\limits_V \exp(ik\hat{\mathbf{N}}\cdot\mathbf{r}')\gamma(\mathbf{r}')d^3\mathbf{r}'.$$

Using the autocorrelation theorem, we have

$$I(\hat{\mathbf{N}},k)=k^4\int\limits_V \exp(-ik\hat{\mathbf{N}}\cdot\mathbf{r})\Gamma(\mathbf{r})d^3\mathbf{r}$$

where Γ is the autocorrelation function given by

$$\Gamma(\mathbf{r})=\int\limits_V \gamma(\mathbf{r}')\gamma(\mathbf{r}'+\mathbf{r})d^3\mathbf{r}'.$$

This result allows us to evaluate the intensity of the scattered amplitude by computing the Fourier transform of the autocorrelation function of the scattering function which is taken to be composed of a number of scatterers distributed at random throughout V. This requires the autocorrelation function to be defined for a particular type of random medium. Thus, a random medium can be characterized via its autocorrelation function by measuring the scattered intensity and inverse Fourier transforming the result.

From the autocorrelation theorem, the characteristics of the autocorrelation function can be formulated by considering its expected spectral properties since

$$\Gamma(\mathbf{r})\Longleftrightarrow\mid\widetilde{\gamma}(\mathbf{k})\mid^2$$

where $\widetilde{\gamma}$ is the Fourier transform of γ, $\mathbf{k}$ is the spatial frequency vector and $\Longleftrightarrow$ denotes the transformation from real space $\mathbf{r}$ to Fourier space $\mathbf{k}$. Hence, in order to evaluate the most likely form of the autocorrelation function we can consider the properties of the power spectrum of the scattering function. If this function is 'white' noise, for example (i.e. its Power Spectral Density Function or PSDF is a constant), then the autocorrelation function is a delta function whose Fourier transform is a constant. However, in practice, we can expect that few scattering functions have a PSDF characterized by white noise, rather, the PSDF will tend to decay as the frequency increases. For example, we can consider a model for the PSDF based on the Gaussian function

$$\mid\widetilde{\gamma}(\mathbf{k})\mid^2=\widetilde{\gamma}_0^2\exp\left(-\frac{k^2}{k_0^2}\right),$$

where $\widetilde{\gamma}_0=\widetilde{\gamma}(0)$, $k=\mid\mathbf{k}\mid$ and k_0 is the standard deviation which is a measure of the correlation length. This form yields an autocorrelation function which is of the same type, i.e. a Gaussian function. If the geometry of the scattering function is self-affine, then we can model the scattering function as a random scattering fractal whose PSDF is characterized by (for a Topological Dimension of 3 and Fractal Dimension denoted by D_F) [TBA97]

$$\mid\widetilde{\gamma}(\mathbf{k})\mid^2\sim\frac{1}{k^{11-2D_F}}$$

where $3<D_F<4$, the autocorrelation function being characterized by [TBA97]

$$\Gamma(\mathbf{r})\sim\frac{1}{r^{D_F-2.5}}.$$

Other issues in determining the nature of the autocorrelation function are related to the physical conditions imposed on the stochastic characteristics of the scatterer.

The method discussed above can be used to model the weak scattered intensity from a random medium which requires an estimate of the autocorrelation of the scattering function to be known. However, this approach assumes that the density of scattering sites from which the scatterer is composed is low so that the weak scattering approximation is valid. When the density of scattering sites increases and multiple scattering is present, the problem become progressively intractable. One approach to overcoming this problem is to resort to a purely stochastic approach which involves developing a statistical model, not for the scattering function, but for the scattered field itself [Fie09]. Another approach is to model the problem in terms of the diffusion of light which is the approach considered here.

3. Optical Diffusion

When light is scattered by one localized centre, the single or 'weak' scattering approximation can be used, i.e. the Born approximation [Bla06]. However, when these centres are grouped together, multiple light scattering occurs. The randomness of multiple interactions tends to be averaged out by the large number of scattering events that occur leading to a deterministic distribution of intensity. This is exemplified by a light beam propagating through thick fog, for example. In this sense, multiple scattering is highly analogous to diffusion, and the terms multiple scattering and diffusion are interchangeable in many contexts. Optical elements designed to produce multiple scattering are thus known as diffusers. The diffusion equation can then be used to model such systems in the same way as it is used to model temperature distributions or particle concentrations, for example, and any system that is the result of a large ensemble of particles or waves undergoing random elastic collisions or scattering interactions respectively.

Suppose we consider the three-dimensional diffusion of light to be based on a three-dimensional random walk. Each scattering event is taken to be a point of the random walk in which a ray of light changes its direction randomly (any direction between 0 and 4π radians). The light field is taken to be composed of a complex of rays, each of which propagates through the diffuser in a way that is incoherent and uncorrelated in time. If this is the case, then the propagation of light can be considered to be analogous to a process of (classical) diffusion. Instead of modelling the process in terms of the three-dimensional inhomogeneous wave equation (for the a spatially variable wavespeed $c(\mathbf{r})$ with PDF $\Pr[c(\mathbf{r})]$)

$$\left(\nabla^2 - \frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2}\right)u(\mathbf{r},t) = 0$$

with light intensity given by $I = \mid u \mid^2$, we consider the intensity to be given by the solution of the homogeneous diffusion equation

$$\left(\nabla^2 - \frac{1}{D}\frac{\partial}{\partial t}\right)I(\mathbf{r},t) = 0$$

with initial condition $I(\mathbf{r},t) = I_0(\mathbf{r})$ at $t = 0$. This model assumes that the diffusivity D is constant throughout the diffuser which is equivalent to a random scattering model (based on a solution to the wave equation) in which $\Pr[c(\mathbf{r})]$ is constant throughout the diffuser, i.e. stationary statistics.

In multiple wave scattering theory, we consider a wavefront travelling through space and scattering from multiple interaction sites, each of which changes the direction of propagation in an entirely random way with no directional bias over 4π radian. The mean free path is taken to be the average number of wavelengths required for the wavefront to propagate from one interaction to another as described by the free space Green's function. After scattering from many sites, the wavefront can be considered to have diffused through the 'diffuser'. Here, the mean free path is a measure of the density of scattering sites, which in turn, is a measure of the diffusivity of the medium D. As $D \to \infty$, the medium becomes increasingly tenuous.

4. Optical Diffusion Equation

Consider the three-dimensional homogeneous time dependent wave equation

$$\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}u = 0$$

where c_0 is taken to be a constant (light speed). Let

$$u(x,y,z,t) = \phi(x,y,z,t)\exp(i\omega t)$$

where it is assumed that field ϕ varies significantly slowly in time compared with $\exp(i\omega t)$ and note that

$$u^*(x,y,z,t) = \phi^*(x,y,z,t)\exp(-i\omega t)$$

is also a solution to the wave equation. Differentiating

$$\nabla^2 u = \exp(i\omega t)\nabla^2\phi,$$

and

$$\frac{\partial^2}{\partial t^2}u = \exp(i\omega t)\left(\frac{\partial^2}{\partial t^2}\phi + 2i\omega\frac{\partial\phi}{\partial t} - \omega^2\phi\right)$$

$$\simeq \exp(i\omega t)\left(2i\omega\frac{\partial\phi}{\partial t} - \omega^2\phi\right)$$

when

$$\left|\frac{\partial^2\phi}{\partial t^2}\right| << 2\omega\left|\frac{\partial\phi}{\partial t}\right|.$$

Under this condition, the wave equation reduces to

$$(\nabla^2 + k^2)\phi = \frac{2ik}{c_0}\frac{\partial\phi}{\partial t}$$

where $k = \omega/c_0$. However, since u^* is also a solution,

$$(\nabla^2 + k^2)\phi^* = -\frac{2ik}{c_0}\frac{\partial\phi^*}{\partial t}$$

and thus,

$$\phi^*\nabla^2\phi - \phi\nabla^2\phi^* = \frac{2ik}{c_0}\left(\phi^*\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi^*}{\partial t}\right)$$

which can be written in the form

$$\nabla^2 I - 2\nabla\cdot(\phi\nabla\phi^*) = \frac{2ik}{c_0}\frac{\partial I}{\partial t}$$

where $I = \phi\phi^* = |\phi|^2$. Let ϕ be given by

$$\phi(\mathbf{r},t) = A(\mathbf{r},t)\exp(ik\hat{\mathbf{n}}\cdot\mathbf{r})$$

where $\hat{\mathbf{n}}$ is a unit vector and A is the amplitude function. Differentiating, and noting that $I = A^2$, we obtain

$$\hat{\mathbf{n}}\cdot\nabla A = \frac{2}{c_0}\frac{\partial A}{\partial t}$$

or

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)A(x,y,z,t) = \frac{2}{c_0}\frac{\partial}{\partial t}A(x,y,z,t)$$

which is the unconditional continuity equation for the amplitude A of a wavefield

$$u(\mathbf{r},t) = A(\mathbf{r},t)\exp[i(k\hat{\mathbf{n}}\cdot\mathbf{r}+\omega t)]$$

where A varies slowly with time.

The equation

$$\nabla^2 I - 2\nabla\cdot(\phi\nabla\phi^*) = \frac{2ik}{c_0}\frac{\partial I}{\partial t}$$

is valid for $k = k_0 - i\kappa$ (i.e. $\omega = \omega_0 - i\kappa c_0$) and so, by equating the real and imaginary parts, we have

$$D\nabla^2 I + 2\mathrm{R}e[\nabla\cdot(\phi\nabla\phi^*)] = \frac{\partial I}{\partial t}$$

and

$$\mathrm{I}m[\nabla\cdot(\phi\nabla\phi^*)] = -\frac{k_0}{c_0}\frac{\partial I}{\partial t}$$

respectively where $D = c_0/2\kappa$, so that under the condition

$$\mathrm{R}e[\nabla\cdot(\phi\nabla\phi^*)] = 0$$

we obtain

$$D\nabla^2 I = \frac{\partial I}{\partial t}.$$

This is the diffusion equation for the intensity of light I. The condition required to obtain this result can be justified by applying a boundary condition on the surface S of a volume V over which the equation is taken to conform. Using the divergence theorem

$$\mathrm{R}e\int_V \nabla\cdot(\phi\nabla\phi^*)d^3\mathbf{r} = \mathrm{R}e\oint_S \phi\nabla\phi^*\cdot\hat{\mathbf{n}}d^2\mathbf{r}$$

$$= \oint_S (\phi_r\nabla\phi_r + \phi_i\nabla\phi_i)\cdot\hat{\mathbf{n}}d^2\mathbf{r}$$

and if

$$\phi_r(\mathbf{r},t)\nabla\phi_r(\mathbf{r},t) = -\phi_i(\mathbf{r},t)\nabla\phi_i(\mathbf{r},t), \quad \mathbf{r}\in S$$

then the surface integral is zero and

$$D\nabla^2 I(\mathbf{r},t) = \frac{\partial}{\partial t}I(\mathbf{r},t), \quad \mathbf{r}\in V.$$

This boundary condition can be written as

$$\frac{\nabla\phi_r}{\nabla\phi_i} = -tan\theta$$

where θ is the phase of the field ϕ which implies that the amplitude A of ϕ is constant on the boundary (i.e. $A(\mathbf{r},t) = A_0, \quad \mathbf{r}\in S, \quad \forall t$), since

$$\frac{\nabla A_0\cos\theta(\mathbf{r},t)}{\nabla A_0\sin\theta(\mathbf{r},t)} = -\frac{A_0\sin\theta(\mathbf{r},t)\nabla\theta(\mathbf{r},t)}{A_0\cos\theta(\mathbf{r},t)\nabla\theta(\mathbf{r},t)}$$

$$= -tan\theta(\mathbf{r},t), \quad \mathbf{r}\in S.$$

4.1. Diffused Image Equation

Suppose we record the intensity I of a light field in the xy-plane for a fixed value of z. Then for $z = z_0$ say,

$$I(x,y,t) \equiv I(x,y,z_0,t)$$

so that

$$\frac{\partial}{\partial t}I(x,y,t) = D\nabla^2 I(x,y,t).$$

Let this two-dimensional diffusion equation be subject to the initial condition

$$I(x,y,0) = I_0(x,y).$$

Then, at any time $T > 0$, it can be assumed that light diffusion is responsible for generating image I and that as time increases, the image becomes progressively more diffused, the solution being given by, for the infinite domain and ignoring scaling [EBY00]

$$I(x,y,T) = \exp\left[-\left(\frac{x^2+y^2}{4DT}\right)\right]\otimes_2 I_0(x,y)$$

where $\otimes_2$ denotes the two-dimensional convolution integral.

4.2. Inverse Solution

If we record an image at a time $t = T$ then by Taylor expanding I at $t = 0$ we can write

$$I(x,y,0) = I(x,y,T) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}T^n\left[\frac{\partial^n}{\partial t^n}I(x,y,t)\right]_{t=T}.$$

From the diffusion equation

$$\frac{\partial^2 I}{\partial t^2} = D\nabla^2\frac{\partial I}{\partial t} = D^2\nabla^4 I$$

$$\frac{\partial^3 I}{\partial t^3} = D\nabla^2\frac{\partial^2 I}{\partial t^2} = D^3\nabla^6 I$$

so that, by induction, we can write

$$\left[\frac{\partial^n}{\partial t^n}I(x,y,t)\right]_{t=T} = D^n\nabla^{2n}I(x,y,T).$$

Substituting this result into the series for $I(x,y,0) \equiv I_0(x,y)$, we get

$$I_0(x,y) = I(x,y,T) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}(DT)^n \nabla^{2n} I(x,y,T)$$

The 'high emphasis filter' [BM86] is then obtained when $DT << 1$, i.e.

$$I_0(x,y) \sim I(x,y,T) - DT\nabla^2 I(x,y,T).$$

For $DT = 1$ the FIR filter corresponding to this result is given by

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Higher order FIR filters can be obtained on a term-by-term basis. For $I_0 = I - \nabla^2 I + \frac{1}{2}\nabla^4 I$ the FIR filter is

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -10 & 2 & 0 \\ 1 & -10 & 30 & -10 & 1 \\ 0 & 2 & -10 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and for $I_0 = I - \nabla^2 I + \frac{1}{2}\nabla^4 I - \frac{1}{6}\nabla^6 I$ the FIR filter is

$$\frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 15 & -3 & 0 & 0 \\ 0 & -3 & 24 & -87 & 24 & -3 & 0 \\ -1 & 15 & -87 & 202 & -87 & 15 & -1 \\ 0 & -3 & 24 & -87 & 24 & -3 & 0 \\ 0 & 0 & -3 & 15 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

5. Hurst Processes and Fractional Diffusion

The diffusion equation models a macroscopic field which is the result of an ensemble of incoherent random walks characterised by a $\sqrt{t}$ scaling law. Hurst processes, describe random walks that have a directional bias and are characterised by the scaling law t^H, $H \in (0.5, 1]$ [Hur44], [TBA97]. As the value of H approaches 1, the processes become increasingly persistent. In terms of the multiple scattering of light from a random medium, increasing persistence relates to multiple scattering from fewer sites so that the light path has a greater directional bias. We consider the characterisation of this by generalizing the diffusion operator

$$\nabla^2 - \sigma\frac{\partial}{\partial t}$$

to the fractional form [Hil95b], [Hil95a]

$$\nabla^2 - \sigma^q \frac{\partial^q}{\partial t^q}$$

where $q \in [1,2]$ and $D^q = 1/\sigma^q$ is the fractional diffusivity. Fractional diffusive processes can therefore be interpreted as intermediate between diffusive processes proper (random phase walks with $H = 0.5$; diffusive processes with $q = 1$) and 'propagative process' (coherent phase walks for $H = 1$; propagative processes with $q = 2$). It should be noted that the fractional diffusion operator given above is the result of a phenomenology. It is a generalisation of a well known differential operator to fractional form which follows from a physical analysis of a fully incoherent random process and its generalisation to fractional, just as the Hurst exponent H is a generalisation of the $\sqrt{t}$ scaling law. The solution to fractional partial differential equations of this type requires application of the fractional calculus (e.g. [OS74], [MR93], [DE75], [SKM93] and [Kir94]) which is considered in the following section.

6. Fractionally Diffused Imaging Equation

Consider the two-dimensional fractional diffusion equation for the intensity $I(x,y,t)$ of light in the image plane located at z given by

$$\nabla^2 I(\mathbf{r},t) - \sigma^q \frac{\partial^q}{\partial t^q} I(\mathbf{r},t) = I_0(\mathbf{r},t)$$

where $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$, $r \equiv | \mathbf{r} |$ and $I_0(\mathbf{r},t)$ is now a (two-dimensional) source function. Using the Fourier based operator for a fractional derivative, we can transform this equation into the form

$$(\nabla^2 + \Omega_q^2)\widetilde{I}(\mathbf{r},\omega) = \widetilde{I}_0(\mathbf{r},\omega)$$

where

$$\widetilde{I}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} I(\mathbf{r},t)\exp(-i\omega t)dt,$$

$$\widetilde{I}_0(\mathbf{r},\omega) = \int_{-\infty}^{\infty} I_0(\mathbf{r},t)\exp(-i\omega t)dt$$

and

$$\Omega_q^2 = -i\omega\sigma, \quad \Omega_q = \pm i(i\omega\sigma)^{q/2}.$$

The Green's function solution for this equation (in the infinite domain) is

$$\widetilde{I}(\mathbf{r},\omega) = g(r,\omega) \otimes_2 \widetilde{I}_0(\mathbf{r},\omega)$$

where g is the 'outgoing' Green function given by (for $| \Omega_q r | >> 1$ and ignoring scaling) [EBY00]

$$g(r,\omega) \simeq \frac{\exp(i\Omega_q r)}{\sqrt{\Omega_q r}}.$$

For $\Omega_q = i(i\omega\sigma)^{q/2}$, Fourier inversion, yields the time dependent Green's function (obtained by writing the exponential function in its series form).

$$G(r,t) = \frac{1}{\sqrt{r}}\frac{1}{\sigma^{q/4}t^{1-q/4}} - \sqrt{r}\sigma^{q/4}\delta^{q/4}(t)$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}r^{(2n+1)/2}\sigma^{3nq/4}\delta^{3nq/4}(t) \qquad (1)$$

the solution for $I(\mathbf{r},t)$ being given by

$$I(\mathbf{r},t)=G(r,t)\otimes_2\otimes_t I_0(\mathbf{r},t)$$

where $\otimes_t$ denotes the convolution integral over time t. Simplification of this infinite sum representation for G can be addressed be considering suitable asymptotics, the most significant of which (for arbitrary values of r) is the case when the (fractional) diffusivity D is large. In particular, we note that as $\sigma\to 0$,

$$G(r,t)=\frac{1}{\sqrt{r\sigma^{q/4}t^{1-q/4}}}.$$

Thus, for $I_0(\mathbf{r},t)=I_0(x,y)\delta(t)$ we can consider a solution to the two-dimensional fractional diffusion equation (for a tenuous medium when $D\to\infty$) of the form (ignoring scaling)

$$I(x,y)=\frac{1}{(x^2+y^2)^{\frac{1}{4}}}\otimes_2 I_0(x,y).$$

7. Deconvolution

In the presence of additive noise $n(x,y)$, the deconvolution problem is as follows: Given that

$$I(x,y)=p(x,y)\otimes_2 I_0(x,y)+n(x,y)$$

where $\Pr[n(x,y)]$ is known (ideally), find an estimate for I_0. This is a common problem in optics (digital image processing) known as the deconvolution problem whose solution is fundamental to image restoration and reconstruction [BM86], [BB98]. In terms of the material presented in this paper, there are two Point Spread Functions (PSF) $p(x,y)$ that have been considered: For full diffusion (strong scattering)

$$p(x,y)=\exp\left[-\left(\frac{x^2+y^2}{4DT}\right)\right]$$

and for fractional diffusion (intermediate scattering in a tenuous medium with large diffusivity)

$$p(x,y)=\frac{1}{(x^2+y^2)^{\frac{1}{4}}}.$$

We note that (ignoring scaling)

$$\exp\left[-\left(\frac{x^2+y^2}{4DT}\right)\right]\leftrightarrow\exp[-4DT(k_x^2+k_y^2)]$$

and [TBA97]

$$\frac{1}{(x^2+y^2)^{\frac{1}{4}}}\leftrightarrow\frac{1}{(k_x^2+k_y^2)^{\frac{3}{4}}}$$

where $\leftrightarrow$ denotes transformation from real space to Fourier space. In the latter case, the filter is a 'fractal filter' and thus, if I_0 is characterised by white noise, then the output I is a Mandelbrot surface with a fractal dimension of 2.5 [TBA97]. In the absence of noise, the inverse solution for I_0 can be written in the form

$$I_0(x,y)=\nabla^{\frac{3}{2}}I(x,y),$$

a result that is based on the application of the fractional Laplacian or Riesz operator [TBA97]

$$\nabla^q\leftrightarrow(k_x^2+k_y^2)^{\frac{q}{2}}.$$

There are a range of approaches to solving the one-dimensional and two-dimensional deconvolution problem in practice (i.e. with additive noise) leading to the classification of different 'inverse filters' (e.g. [BM86], [BB98]). If *a priori* information on the statistics of the noise function n and the object function I_0 is available, then Bayesian estimation methods are preferable in the design of filters whose performance then depends on statistical parameters such as the standard deviation. In some cases, an estimate of $\Pr[n(x,y)]$ can be obtained by taking an image (or a number of images to obtain a statistically significant result) with zero input, i.e. with $I_0=0$. This provides a method of validating an idealised PDF through data fitting and, thus, determination of the statistical parameters from which a theoretical PDF is composed. In cases when experimental determinism is not practically possible, statistical models must be utilized [Fie09]. However, with regard to incoherent imaging systems, the noise function tends to be Gaussian distributed - a result of the noise being a linear combination of many different independent noise sources which combine to produce Gaussian noise (a consequence of the Central Limit Theorem).

Using Bayes rule, the aim is to find an estimate for I_0 such that

$$\frac{\partial}{\partial I_0}\ln\Pr[n(x,y)]+\frac{\partial}{\partial I_0}\ln\Pr[I_0(x,y)]=0.$$

Consider the following models for the PDFs: (i) A Gaussian distribution for the noise (ignoring scaling and where σ_n^2 is the standard deviation of n)

$$\Pr[n(x,y)]=$$

$$\exp\left(-\frac{1}{\sigma_n^2}\int\int[(I(x,y)-p(x,y)\otimes_2 I_0(x,y)]^2dxdy\right).$$

(ii) A Gaussian distribution for the object function (ignoring scaling and where $\sigma_{I_0}^2$ is the standard deviation of I_0)

$$\Pr[I_0(x,y)]=\exp\left(-\frac{1}{\sigma_{I_0}^2}\int\int I_0^2(x,y)dxdy\right).$$

Differentiating, these statistical models yield the equation

$$I(x,y)\odot_2 p(x,y)$$

$$=\frac{\sigma_n^2}{\sigma_{I_0}^2}I_0(x,y)+[p(x,y)\otimes_2 I_0(x,y)]\odot_2 p(x,y)$$

where $\odot_2$ denotes the two-dimensional correlation integral. In Fourier space, this equation becomes

$$\widetilde{I}(k_x,k_y)P^*(k_x,k_y)$$

$$=\frac{\sigma_n^2}{\sigma_{I_0}^2}\widetilde{I}_0(k_x,k_y)+\mid P(k_x,k_y)\mid^2 I_0(k_x,k_y).$$

The Bayesian *a Posteriori* filter $F(k_x,k_y)$ (for Gaussian statistics) is then given by

$$F(k_x,k_y)=\frac{P^*(k_x,k_y)}{\mid P(k_x,k_y)\mid^2+\sigma_n^2/\sigma_{I_0}^2}$$

where σ_n/σ_{I_0} defines the Signal-to-Noise Ratio of $I(x,y)$ and $\widetilde{I}_0(k_x,k_y)=F(k_x,k_y)\widetilde{I}(k_x,k_y)$. The reconstruction for I_0 is then given by

$$I_0(x,y)=$$

$$\frac{1}{(2\pi)^2}\int\int F(k_x,k_y)\widetilde{I}(k_x,k_y)\exp(ik_xx)\exp(ik_yy)dk_xdk_y \tag{2}$$

Given $P(k_x,k_y)$, the performance of this filter depends on the value of $\Sigma=\sigma_n^2/\sigma_{I_0}^2$. In general, as $\Sigma\rightarrow 0$ the reconstruction sharpens but at the expense of 'ringing'. Thus, an optimum value of Σ is obtained by computing I_0 over a range of values of Σ and, for each reconstruction, computing the ratio of the number of zero crossings Z_c to the sum of the magnitude of a digital gradient $\sum\mid\mathcal{D}I_0[i,j]\mid$, i.e.

$$R=\frac{Z_c}{\sum\mid\mathcal{D}I_0[i,j]\mid}.$$

This ratio is based on the principle that an optimum reconstruction is one which provides a sharp image with minimal ringing, i.e. a reconstruction for which R is a minimum. This principle has been applied in the example results given in the following section. Note that the Fourier based approach to image restoration relies on the ability to implement the convolution and correlation theorems. This requires that the data has been recorded by an (optical) imaging system that is isoplanatic (i.e. the PSF is stationary).

8. Example Applications

We consider examples of image reconstruction based on equation (2) for fully diffusive and fractional diffusive models using the optimization procedure discussed above and the following 'digital Laplacian'

$$\mathcal{D}I_0[i,j]=\begin{pmatrix}0 & 1 & 0\\ 1 & -4 & 1\\ 0 & 1 & 0\end{pmatrix}.$$

8.1. Diffusion

Figure 1 shows the application of equation (2) where (ignoring scaling and with $\sigma=4DT$)

$$P(k_x,k_y)=\exp[-\sigma(k_x^2+k_y^2)]$$

In this example, the diffusion of the object has been generated by turbulence of the earths atmosphere through which light from the object has been fully diffused. In this case, the reconstruction depends on the value of both σ and Σ and an optimization scheme based on computing $I_0[i,j;\sigma,\Sigma]$ for min$R$.

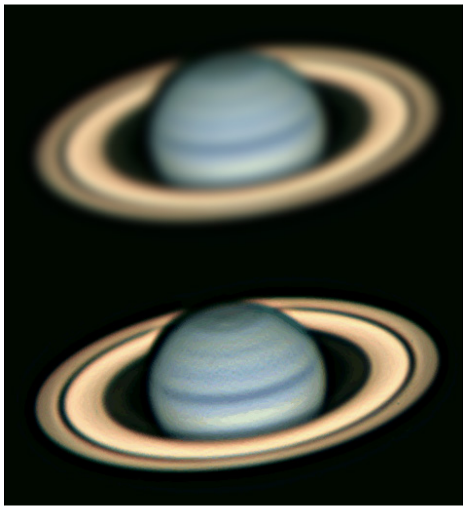

Figure 1: *Diffusion based deconvolution (below) of an image of Saturn observed by a ground based telescope with light diffused by the atmosphere (above).*

8.2. Fractional Diffusion

Fractional diffusion models apply to scattering processes that occur in a tenuous and extremely rarefied medium. In applied optics, one of the most common examples of this phenomenon occurs in astronomy and the processes associated with light scattering from cosmic dust which is composed of particles which are a few molecules to the order of 10^{-4} metres in size. Cosmic dust is defined in terms of its astronomical location including intergalactic dust, interstellar dust, interplanetary dust and circumplanetary dust (such as in a planetary ring). In our own Solar System, interplanetary dust is generated from sources such as comet dust, asteroidal dust, dust from the Kuiper belt and interstellar dust passing through our solar system. This dust is responsible for zodiacal light which is produced by sunlight reflecting off dust particles. Cosmic dust can be categorised in terms of different types of nebulae associated with different physical causes and processes. These include: diffuse nebulae, infrared reflection nebula, supernova remnants and molecular clouds, for example. However, in a more general sense, cosmic dust often characterises the interstellar medium which is the gas and dust that pervade interstellar space. This medium consists of an extremely dilute (by terrestrial standards) mixture of ions, atoms, molecules, and larger dust grains, consisting of about 99% gas and 1% dust by mass. Densities range from a few thousand to a few hundred million particles per cubic meter with an average value in the Milky Way Galaxy, for example, of a million particles per cubic meter. In comparison with the scattering of light from earth-based random media, for example, the interstellar medium is highly diffuse and therefore ideal for applying light scattering models based on fractional diffusion when $D \to \infty$.

Figure 2 shows the application of equation (2) with

$$P(k_x, k_y) = \frac{1}{(k_x^2 + k_y^2)^{0.75}}$$

for an optical image obtained by the Hubble Space Telescope (part of the constellation of Perseus observed through an interstellar dust cloud that covers nearly 4 degrees on the sky and observed approximately 1,000 light-years away).

Figure 2: *Fractional diffusion based deconvolution (right) of a dust clouded star field (left) in the constellation of Pegasus.*

9. Conclusions

The use of fully diffusive processes for modelling strong scattering provides a result that is applicable in solving the inverse (multiple) scattering problem. This requires the formulation of a deconvolution algorithm for a Gaussian PSF. We have extended this approach to model intermediate scattering by generalizing the diffusion equation to fractional form

$$\left(\nabla^2 - \frac{1}{D^q}\frac{\partial^q}{\partial t^q}\right) I(x,y,t) = I_0(x,y)\delta(t)$$

for a fractional diffusivity D. An asymptotic solution has been considered based on the condition $D \to \infty$ which yields a characteristic Optical Transfer Function of the form $(k_x^2 + k_y^2)^{-0.75}$. This filter is the transfer function associated with an optical system involving the intermediate strength scattering of light in a tenuous medium and is equivalent to a self-affine filter with fractal dimension 2.5. The inverse (deconvolution) problem has been considered in terms of Bayesian estimation which has been applied to example images associated with fully diffusive and partially diffusive processes. It is noted that unlike the Gaussian PSF associated with fully diffusive processes, the PSF derived for fractional diffusion is not dependent on time (other than scaling) and the diffusivity D. This is because the result is based on $D \to \infty$ and it is therefore of value to study non-asymptotic solutions based on including higher order terms in equation (1). On the other hand, the asymptotic solution considered, means that application of the optimization procedure used to compute the filter $F(k_x, k_y)$ is reduced to a single-parameter problem, i.e. the computation of $I_0[i, j; \Sigma]$.

References

[BB98] BERTERO M., BOCCACCI B.: *Introduction to Inverse Problems in Imaging*. Institute of Physics Publishing, 1998.

[Bla06] BLACKLEDGE J. M.: *Digital Image Processing*. Horwood Publishing, 2006.

[BM86] BATES R. H. T., MCDONNAL M. J.: *Image Restoration and Reconstruction*. Oxford Science Publications, 1986.

[DE75] DOLD A., ECKMANN B.: *Fractional Calculus and its Applications*. Springer, 1975.

[EBY00] EVANS G. A., BLACKLEDGE J. M., YARDLEY P.: *Analytical Solutions to Partial Differential Equations*. Springer-Verlag, 2000.

[Fie09] FIELD T. R.: *Electromagnetic Scattering from Random Media*. Oxford Science Publications, 2009.

[Hil95a] HILFER R.: Exact solutions for a class of fractal time random walks. *Fractals 3*, 1 (1995), 211–216.

[Hil95b] HILFER R.: Foundations of fractional dynamics. *Fractals 3*, 3 (1995), 549–556.

[Hur44] HURST E. H.: *A Short Account of the Nile Basin*. Cairo, Government Press, 1944.

[Kir94] KIRYAKOVA V.: *Generalized Fractional Calculus and Applications*. Longman, 1994.

[Mar06] MARTIN P. A.: *Multiple Scattering: Interaction of Time-Harmonic Waves with N Obstacles*. Cambridge University Press, 2006.

[MR93] MILLER K. S., ROSS B.: *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, 1993.

[OS74] OLDHAM K. B., SPANIER J.: *The Fractional Calculus*. Academic Press, 1974.

[SKM93] SAMKO S. G., KILBAS A., MARICHEV O. I.: *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, 1993.

[TBA97] TURNER M. J., BLACKLEDGE J. M., ANDREWS P.: *Fractal Geometry in Digital Imaging*. Academic Press, 1997.