Solutions of the compatibility conditions for a Wigner quantum oscillator

N.I. Stoilova[a] and J. Van der Jeugt[b]

Department of Applied Mathematics and Computer Science, University of Ghent, Krijgslaan 281-S9, B-9000 Gent, Belgium.

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Abstract

We consider the compatibility conditions for a N-particle D-dimensional Wigner quantum oscillator. These conditions can be rewritten as certain triple relations involving anticommutators, so it is natural to look for solutions in terms of Lie superalgebras. In the recent classification of "generalized quantum statistics" for the basic classical Lie superalgebras [1], each such statistics is characterized by a set of creation and annihilation operators plus a set of triple relations. In the present letter, we investigate which cases of this classification also lead to solutions of the compatibility conditions. Our analysis yields some known solutions and several classes of new solutions.

[a]E-mail: Neli.Stoilova@UGent.be; Permanent address: Institute for Nuclear Research and Nuclear Energy, Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria

[b]E-mail: Joris.VanderJeugt@UGent.be

In a previous paper [1] we made a classification of all generalized quantum statistics (GQS) associated with the basic classical Lie superalgebras $A(m|n)$, $B(m|n)$, $C(n)$ and $D(m|n)$. Each such statistics is determined by M creation operators x_i^+ $(i=1,\ldots,M)$ and M annihilation operators x_i^- $(i=1,\ldots,M)$, which generate the corresponding superalgebra G subject to certain triple relations $\mathcal{R}$. This leads to a $\mathbb{Z}$-grading of G of the form

$$G = G_{-2}\oplus G_{-1}\oplus G_0\oplus G_{+1}\oplus G_{+2}, \tag{1}$$

with $G_{\pm1} = \text{span}\{x_i^{\pm},\ i=1,\ldots,M\}$ and $G_{j+k} = [\![G_j, G_k]\!]$, where $[\![\cdot,\cdot]\!]$ is the Lie superalgebra bracket. The known cases, namely para-Bose and para-Fermi statistics [2], and A-(super)statistics [3]-[6] appear as simple examples in the classification.

In the present letter we are dealing with a different problem, namely finding solutions of the compatibility conditions (CCs) of a Wigner quantum oscillator system. These compatibility conditions take the form of certain triple relations for operators. So formally the CCs appear as special triple relations among operators which resemble the creation and annihilation operators of a generalized quantum statistics. One can thus investigate which formal GQSs also provide solutions of the CCs. It turns out that the classification presented in [1] yields new solutions of these compatibility conditions.

The concepts of Wigner quantization [7] and of a Wigner Quantum System (WQS) [8] were introduced by Palev, inspired by [9]. WQSs are noncanonical generalized quantum systems for which Hamilton's equations are identical to the Heisenberg equations and for which certain additional properties, valid for any quantum system, are also fulfilled. For more examples of WQSs and physical aspects, see [10]-[18].

Let us briefly describe a WQS consisting of N D-dimensional isotropic harmonic oscillators. The Hamiltonian of this N-particle D-dimensional ($D=1,2,3$) harmonic oscillator system is given by

$$\hat{H} = \sum_{\alpha=1}^{N}\Big(\frac{\hat{\mathbf{P}}_\alpha^2}{2m} + \frac{m\omega^2}{2}\hat{\mathbf{R}}_\alpha^2\Big), \tag{2}$$

with m the mass and ω the frequency of each oscillator. The Hamiltonian $\hat{H}$ depends on the $2DN$ variables $\hat{R}_{\alpha i}$ and $\hat{P}_{\alpha i}$, with $\alpha=1,\ldots,N$ and $i=1,\ldots,D$. In practice, the cases $D=1,2,3$ will be the most interesting, but we shall treat the general situation here.

In a Wigner quantum system, the operators $\hat{\mathbf{R}}_1,\ldots,\hat{\mathbf{R}}_N$ and $\hat{\mathbf{P}}_1,\ldots,\hat{\mathbf{P}}_N$ have to be defined in such a way that Hamilton's equations

$$\dot{\hat{\mathbf{P}}}_\alpha = -m\omega^2\hat{\mathbf{R}}_\alpha,\quad \dot{\hat{\mathbf{R}}}_\alpha = \frac{1}{m}\hat{\mathbf{P}}_\alpha \ \text{ for }\ \alpha=1,2,\ldots,N, \tag{3}$$

and the Heisenberg equations

$$\dot{\hat{\mathbf{P}}}_\alpha = \frac{i}{\hbar}[\hat{H},\hat{\mathbf{P}}_\alpha],\quad \dot{\hat{\mathbf{R}}}_\alpha = \frac{i}{\hbar}[\hat{H},\hat{\mathbf{R}}_\alpha] \ \text{ for }\ \alpha=1,2\ldots,N, \tag{4}$$

are identical as operator equations. These compatibility conditions (CCs) are as follows

$$[\hat{H}, \hat{\mathbf{P}}_\alpha] = i\hbar m\omega^2 \hat{\mathbf{R}}_\alpha, \quad [\hat{H}, \hat{\mathbf{R}}_\alpha] = -\frac{i\hbar}{m}\hat{\mathbf{P}}_\alpha \quad \text{for} \quad \alpha = 1, 2, \ldots, N. \tag{5}$$

To make the connection with basic classical Lie superalgebras we write the operators $\hat{\mathbf{P}}_\alpha$ and $\hat{\mathbf{R}}_\alpha$ ($\alpha = 1, 2, \ldots, N$) in terms of new operators (or vice versa):

$$a^\pm_{\alpha j} = \sqrt{\frac{cm\omega}{4\hbar}}\hat{R}_{\alpha j} \pm i\mu\sqrt{\frac{c}{4m\omega\hbar}}\hat{P}_{\alpha j}, \qquad (\alpha = 1, \ldots, N;\ j = 1, \ldots, D) \tag{6}$$

where $\mu = +1$ or -1 and c is an arbitrary positive constant (which can be chosen as an integer). The Hamiltonian $\hat{H}$ is then

$$\hat{H} = \frac{\omega\hbar}{c}\sum_{\alpha=1}^N\sum_{i=1}^D \{a^+_{\alpha i}, a^-_{\alpha i}\}, \tag{7}$$

with $\{\cdot,\cdot\}$ an anticommutator. The compatibility conditions (5) take the form:

$$\sum_{\alpha=1}^N\sum_{i=1}^D [\{a^+_{\alpha i}, a^-_{\alpha i}\}, a^\pm_{\beta j}] = \mp\mu c\, a^\pm_{\beta j}, \qquad (\beta = 1, \ldots, N;\ j = 1, \ldots, D). \tag{8}$$

In the present form, the compatibility conditions are expressed as certain triple relations for a set of odd operators $a^\pm_{\alpha i}$. Thus it is natural to look for solutions of (8) in the framework of Lie superalgebras. The classification of GQSs [1], also expressed by means of certain creation and annihilation operators (CAOs) $x^\pm_i$ ($i = 1, \ldots, M$) satisfying triple relations $\mathcal{R}$, can thus be used to investigate solutions of (8). In the classification list of [1], we should now restrict ourselves to cases where all CAO's of $\mathcal{R}$ consist of odd elements only. Therefore G_{-1} and G_{+1} are odd subspaces, and the grading (1) is *consistent* with the $\mathbb{Z}_2$-grading of the Lie superalgebra. Then, after identifying the $x^\pm_i$ with the operators $a^\pm_{\alpha j}$ (eventually up to an overall constant), it remains to verify whether (8) is satisfied. We shall now perform this investigation for the basic classical Lie superalgebras.

For the Lie superalgebra $sl(m|n) = A(m-1|n-1)$ there are two GQSs with all CAOs odd elements [1]. The first of these corresponds to a grading of length 3 (i.e. $G_{\pm 2} = 0$ in (1)). In this case, the CAOs are given by:

$$x^-_{rk} = e_{k,r+m+1}, \quad x^+_{rk} = e_{r+m+1,k}, \qquad r = 1, \ldots, n;\ k = 1, \ldots, m, \tag{9}$$

where e_{ij} is a $(m+n)\times(m+n)$ matrix with zeros everywhere except a 1 on the intersection of row i and column j (corresponding to the defining $sl(m|n)$ representation). These operators satisfy the triple relations (we write in this paper only the relations from $\mathcal{R}$ that are needed here; $r, s, t = 1, \ldots, n$; $i, j, k = 1, \ldots, m$)

$$\begin{aligned} [\{x^+_{ri}, x^-_{sj}\}, x^+_{tk}] &= \delta_{ij}\delta_{st}x^+_{rk} - \delta_{jk}\delta_{rs}x^+_{ti}, \\ [\{x^+_{ri}, x^-_{sj}\}, x^-_{tk}] &= -\delta_{ij}\delta_{rt}x^-_{sk} + \delta_{ik}\delta_{rs}x^-_{tj}, \end{aligned} \tag{10}$$

and thus

$$\sum_{r=1}^{n}\sum_{k=1}^{m}[\{x_{rk}^+,x_{rk}^-\},x_{sj}^\pm]=\pm(m-n)x_{sj}^\pm. \tag{11}$$

It is clear that such systems provide solutions for the CCs (as long as $m\neq n$). First of all, taking $m=D$ and $n=N$ yields the $sl(D|N)$ solution of the CCs (8) for the N-particle D-dimensional oscillator, by taking $a_{\alpha j}^\pm = x_{\alpha j}^\pm$ ($\alpha=1,\ldots,N$; $j=1,\ldots,D$). This is (at least for $D=3$) a known solution: see [19] for a discussion and some properties corresponding to this $sl(3|N)$ case.

Secondly, one can take $m=1$ and $n=DN$, yielding the $sl(1|DN)$ solution of the CCs. In this case, one takes $a_{\alpha j}^\pm = x_{j+(\alpha-1)D,1}^\pm$ ($\alpha=1,\ldots,N$; $j=1,\ldots,D$). This is again a known solution: see [7], [20]-[22] for an investigation of the physical properties of the $sl(1|3N)$ solution of the Wigner quantum oscillator.

Observe that one can always interchange the operators $a_{\alpha j}^+$ with $a_{\alpha j}^-$.

Note that the cases ($m=N,n=D$) or ($m=DN,n=1$) also provide solutions, but these are not considered because of the isomorphism of $sl(m|n)$ and $sl(n|m)$. More generally, it is clear that by repartitioning the mn operators x_{rk}^+ ($r=1,\ldots,n$; $k=1,\ldots,m$) into N sets of D operators (and analogously for the x_{rk}^-), (11) still yields a solution of (8). This means that all Lie superalgebras $sl(m|n)$ with $mn=DN$ provide a solution to the compatibility conditions for the N-particle D-dimensional Wigner quantum oscillator.

The second type of GQS for the Lie superalgebra $sl(m|n)$ with all CAOs odd elements corresponds to a grading of length 5 [1]. In this situation there are several inequivalent GQSs, all of them leading to solutions of the CCs. Since the description is somewhat more complicated than the other cases, we shall give it in the Appendix.

Next, we turn our attention to the Lie superalgebras $B(m|n)=osp(2m+1|2n)$. We know from [1] that there is one GQS with odd elements only. In terms of the defining $(2m+2n+1)$-dimensional representation of $B(m|n)$, the corresponding CAOs are given by:

$$\begin{aligned}
x_{ri}^+ &= e_{m+i,2m+1+r}-e_{2m+1+n+r,i}, & x_{ri}^- &= e_{i,2m+1+n+r}+e_{2m+1+r,m+i},\\
x_{r,-i}^+ &= e_{i,2m+1+r}-e_{2m+1+n+r,i+m}, & x_{r,-i}^- &= e_{m+i,2m+1+n+r}+e_{2m+1+r,i},\\
x_{r0}^+ &= e_{2m+1,2m+1+r}-e_{2m+1+n+r,2m+1}, & x_{r0}^- &= e_{2m+1,2m+1+n+r}+e_{2m+1+r,2m+1},
\end{aligned}$$

with $r=1,\ldots,n$ and $i=1,\ldots,m$. If we introduce the notation

$$\langle j\rangle = \begin{cases} 1 & \text{if } j=1,\ldots,m\\ -1 & \text{if } j=-m,\ldots,-1\\ 0 & \text{if } j=0\end{cases} \tag{12}$$

the triple relations needed can be written as follows:

$$[\{x_{rk}^+,x_{rk}^-\},x_{sj}^\pm]=\pm\langle k\rangle\langle j\rangle\delta_{|k||j|}\,x_{sj}^\pm\mp\delta_{rs}\,x_{sj}^\pm,\qquad (r,s=1,\ldots,n;\ k,j=-m,\ldots,m). \tag{13}$$

This implies

$$\sum_{r=1}^{n}\sum_{k=-m}^{m}[\{x_{rk}^{+},x_{rk}^{-}\},x_{sj}^{\pm}]=\mp(2m+1)x_{sj}^{\pm},\qquad (s=1,\ldots,n;\ j=-m,\ldots,m). \tag{14}$$

Again it is clear that this provides solutions for the CCs. For $D=2m+1$ and $N=n$, one obtains the $osp(D|2N)$ solution of the CCs (8) for the N-particle D-dimensional oscillator, by taking $a_{\alpha j}^{\pm}=x_{\alpha j}^{\pm}$ ($\alpha=1,\ldots,N;\ j=-m,\ldots,m$). This is a new class of solutions of WQSs. Note that even the simplest case ($D=3$ and $N=1$, or $osp(3|2)$) is different from the $osp(3|2)$ solution of [23], since in the current case the operators $a_{\alpha j}^{\pm}$ correspond to root vectors of $osp(3|2)$ (which was not the case in [23]).

Alternatively, one can also take $N=2m+1$ and $D=n$ in (14). This yields the $osp(N|2D)$ solution of the CCs for the N-particle D-dimensional oscillator, by taking $a_{\alpha j}^{\pm}=x_{j\alpha}^{\pm}$ ($\alpha=-m,\ldots,m$; $j=1,\ldots,D$). More generally, it is clear that by repartitioning the $(2m+1)n$ operators x_{rk}^{+} ($r=1,\ldots,n$; $k=-m,\ldots,m$) into N sets of D operators (and analogously for the x_{rk}^{-}), (14) still yields a solution of (8). This means that all Lie superalgebras $osp(2m+1|2n)$ with $(2m+1)n=DN$ provide a solution to the compatibility conditions.

Finally, one can have $m=0$ and $n=DN$, yielding the $B(0|DN)=osp(1|2DN)$ solution of the CCs. In this case, one obtains a solution for the N-particle D-dimensional oscillator, by taking $a_{\alpha j}^{\pm}=x_{j+(\alpha-1)D,0}^{\pm}$ ($\alpha=1,\ldots,N$; $j=1,\ldots,D$). This solution is not new; in fact it is (up to a constant) the known para-Bose solution [7], [24]. Indeed, let us put

$$b_r^{+}=\sqrt{2}\,x_{r0}^{+},\qquad b_r^{-}=-\sqrt{2}\,x_{r0}^{-}, \tag{15}$$

for $r=1,\ldots,DN$. Then these operators satisfy

$$\begin{gathered}[\{b_r^{\xi},b_s^{\eta}\},b_t^{\epsilon}]=(\epsilon-\xi)\delta_{rt}b_s^{\eta}+(\epsilon-\eta)\delta_{st}b_r^{\xi}, \\ \xi,\eta,\epsilon=\pm \text{ or } \pm 1;\quad r,s,t=1,\ldots,DN.\end{gathered} \tag{16}$$

These are the para-Bose operators of [2]. For $osp(1|6N)$, it was observed in [19] that this yields a solution of the CCs for the N-particle 3-dimensional Wigner quantum oscillator.

Let us now consider the Lie superalgebras $D(m|n)=osp(2m|2n)$. From [1] it follows that there are two GQSs with odd elements only. In terms of the defining $(2m+2n)$-dimensional representation of $D(m|n)$, the CAOs of the first system are given by:

$$\begin{aligned} x_{ri}^{+}&=e_{m+i,2m+r}-e_{2m+n+r,i}, & x_{ri}^{-}&=e_{i,2m+n+r}+e_{2m+r,m+i},\\ x_{r,-i}^{+}&=e_{i,2m+r}-e_{2m+n+r,i+m}, & x_{r,-i}^{-}&=e_{m+i,2m+n+r}+e_{2m+r,i}, \end{aligned} \tag{17}$$

with $r=1,\ldots,n$ and $i=1,\ldots,m$. It is easy to verify that these satisfy

$$[\{x_{rk}^{+},x_{rk}^{-}\},x_{sj}^{\pm}]=\pm\langle k\rangle\langle j\rangle\delta_{|k||j|}\,x_{sj}^{\pm}\mp\delta_{rs}\,x_{sj}^{\pm},\qquad (r,s=1,\ldots,n;\ k,j=\pm1,\ldots,\pm m). \tag{18}$$

Thus it follows that

$$\sum_{r=1}^{n} \sum_{0\neq k=-m}^{m} [\{x_{rk}^{+}, x_{rk}^{-}\}, x_{sj}^{\pm}] = \mp 2m\, x_{sj}^{\pm}, \qquad (s = 1, \ldots, n;\ j = \pm 1, \ldots, \pm m). \quad (19)$$

For $D = 2m$ and $N = n$, this yields the $osp(D|2N)$ solution of the CCs (8) for the N-particle D-dimensional oscillator, by taking $a_{\alpha j}^{\pm} = x_{\alpha j}^{\pm}$ ($\alpha = 1, \ldots, N$; $j = \pm 1, \ldots, \pm m$). This is a new class of solutions for the WQSs.

Alternatively, one can take $N = 2m$ and $D = n$ in (19). This yields the $osp(N|2D)$ solution of the CCs (8), by taking $a_{\alpha j}^{\pm} = x_{j\alpha}^{\pm}$ ($j = 1, \ldots, D$; $\alpha = \pm 1, \ldots, \pm m$). As before, one can more generally repartition the $2mn$ operators x_{rk}^{+} ($r = 1, \ldots, n$; $k = \pm 1, \ldots, \pm m$) into N sets of D operators (and analogously for the x_{rk}^{-}); then (19) still yields a solution of (8). This means that all Lie superalgebras $osp(2m|2n)$ with $2mn = DN$ provide a solution to the compatibility conditions for the N-particle D-dimensional Wigner quantum oscillator.

The Lie superalgebra $D(m|n) = osp(2m|2n)$ also admits a different GQS with odd elements only [1]. The CAOs of this second system are given by:

$$\begin{aligned} x_{ri}^{+} &= e_{2m+n+i,r} - e_{m+r,2m+i}, & x_{ri}^{-} &= e_{r,2m+n+i} + e_{2m+i,m+r}, \\ x_{r,-i}^{+} &= e_{m+r,2m+n+i} + e_{2m+i,r}, & x_{r,-i}^{-} &= e_{r,2m+i} - e_{2m+n+i,m+r}, \end{aligned} \quad (20)$$

with $r = 1, \ldots, m$ and $i = 1, \ldots, n$. Although this looks similar to the first system, observe that it is essentially different. In (17), the subalgebra $G_0 = [\![G_{-1}, G_{+1}]\!]$ is $sl(n) \oplus so(2m)$, whereas in (20), it is $sl(m) \oplus sp(2n)$ [1]. In this case the operators (20) satisfy

$$[\{x_{rk}^{+}, x_{rk}^{-}\}, x_{sj}^{\pm}] = \pm\langle k\rangle\langle j\rangle\delta_{|k||j|}\, x_{sj}^{\pm} \mp \delta_{rs}\, x_{sj}^{\pm}, \qquad (r, s = 1, \ldots, m;\ k, j = \pm 1, \ldots, \pm n). \quad (21)$$

Now we have

$$\sum_{r=1}^{m} \sum_{0\neq k=-n}^{n} [\{x_{rk}^{+}, x_{rk}^{-}\}, x_{sj}^{\pm}] = \mp 2n\, x_{sj}^{\pm}, \qquad (s = 1, \ldots, m;\ j = \pm 1, \ldots, \pm n). \quad (22)$$

For $N = 2m$ and $D = n$, this yields the second $osp(N|2D)$ solution of the CCs (8), by taking $a_{\alpha j}^{\pm} = x_{\alpha j}^{\pm}$ ($j = 1, \ldots, D$; $\alpha = \pm 1, \ldots, \pm m$). As for the other cases, one can more generally repartition the $2mn$ operators x_{rk}^{+} ($r = 1, \ldots, m$; $k = \pm 1, \ldots, \pm n$) into N sets of D operators (and analogously for the x_{rk}^{-}) and still obtain a solution of (8). Hence all Lie superalgebras $osp(2m|2n)$ with $2mn = DN$ provide a second type of solution to the compatibility conditions for the N-particle D-dimensional Wigner quantum oscillator.

The solutions presented here for $D(m|n)$ remain valid also when $m = 1$. In that case, the Lie superalgebra is usually denoted by $C(n+1)$: $C(n+1) = D(1|n) = osp(2|2n)$. In particular, $C(N+1)$ yields solutions for the N-particle 2-dimensional Wigner quantum oscillator.

To conclude, our analysis of the compatibility conditions (8) using the formal classification of GQS in [1] has given rise to several classes of new solutions for the N-particle D-dimensional Wigner quantum oscillator. The most interesting solutions are those with $D = 1, 2, 3$. For example, for $D = 1$ there are solutions in terms of the Lie superalgebras $sl(1|N)$ and $osp(1|2N)$; for $D = 2$ there are solutions in terms of $sl(1|2N)$, $sl(2|N)$, $osp(2|2N)$ and $osp(2N|2)$; for $D = 3$ there are solutions in terms of $sl(1|3N)$, $sl(3|N)$ and $osp(3|N)$ (apart from other types of partitioning).

In order to study physical properties of the new Wigner quantum systems (energy spectrum, position and momentum operators, etc.) one is lead to representation theory of the corresponding Lie superalgebra. The class of representations should be "unitary", in the sense that $(a^{\pm}_{\alpha j})^{\dagger} = a^{\mp}_{\alpha j}$ must hold (by the Hermiticity of the position and momentum operators, see (6)). For interesting examples with intriguing physical properties, see the $sl(1|3)$ [7],[21] (or $sl(1|3N)$) solution for the (N-particle) 3-dimensional Wigner quantum oscillator [22]. With the current list of new solutions obtained in this letter, we hope to investigate the physical properties of some of these in the future.

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Appendix

We describe here the remaining GQSs for the Lie superalgebra $sl(m|n)$ with odd CAOs only. According to [1], there are two classes. For the first class, l can be any index between 1 and $m-1$, so assume that l is fixed ($1 \leq l < m$). The CAOs are then described by the root vectors of [1, eq. (3.9)], but in order to deduce solutions for the CCs we need to multiply them by some overall constant. This gives, for $k = 1, \ldots, m$ and $r = 1, \ldots, n$:

$$x^{+}_{rk} = \begin{cases} \sqrt{|2m-n-2l|}\; e_{m+r,k} & \text{for} \quad k \leq l \\ \sqrt{|n-2l|}\; e_{k,m+r} & \text{for} \quad k > l \end{cases} \tag{23}$$

and

$$x^{-}_{rk} = \begin{cases} \sqrt{|2m-n-2l|}\; e_{k,m+r} & \text{for} \quad k \leq l \\ \epsilon\sqrt{|n-2l|}\; e_{m+r,k} & \text{for} \quad k > l \end{cases} \tag{24}$$

where $\epsilon = \mathrm{sgn}((n-2l)(2m-n-2l))$. Of course, we have to assume that l is such that these factors do not vanish, i.e. $(n-2l)(2m-n-2l) \neq 0$. Then, one can deduce that

$$\sum_{r=1}^{n}\sum_{k=1}^{m}[\{x^{+}_{rk}, x^{-}_{rk}\}, x^{\pm}_{sj}] = \mp\nu\; n(m-n)x^{\pm}_{sj} \tag{25}$$

where $\nu = \mathrm{sgn}(2m-n-2l)$. Clearly, for $m \neq n$ such systems provide solutions for the CCs for the N-particle D-dimensional oscillator whenever $mn = DN$.

For the second class, l can be any index between 1 and $n-1$. Now the CAOs are described by the root vectors of [1, eq. (3.8)], again multiplied by some appropriate constant. This gives, for $k=1,\ldots,m$ and $r=1,\ldots,n$:

$$x_{rk}^{+}=\begin{cases}\sqrt{|2n-m-2l|}\;e_{m+r,k} & \text{for}\quad r\leq l\\ \sqrt{|m-2l|}\;e_{k,m+r} & \text{for}\quad r>l\end{cases} \tag{26}$$

and

$$x_{rk}^{-}=\begin{cases}\sqrt{|2n-m-2l|}\;e_{k,m+r} & \text{for}\quad r\leq l\\ \epsilon\sqrt{|m-2l|}\;e_{m+r,k} & \text{for}\quad r>l\end{cases} \tag{27}$$

where $\epsilon=\mathrm{sgn}((m-2l)(2n-m-2l))$. Again we assume that l is such that these factors do not vanish, i.e. $(m-2l)(2n-m-2l)\neq 0$. Now one can deduce that

$$\sum_{r=1}^{n}\sum_{k=1}^{m}[\{x_{rk}^{+},x_{rk}^{-}\},x_{sj}^{\pm}]=\mp\nu\;m(n-m)x_{sj}^{\pm} \tag{28}$$

where $\nu=\mathrm{sgn}(2n-m-2l)$. For $m\neq n$ such systems provide another class of solutions for the CCs for the N-particle D-dimensional oscillator whenever $mn=DN$.

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