Invariance Groups of Three Term Transformations for Basic Hypergeometric Series

S. Lievens and J. Van der Jeugt

Department of Applied Mathematics and Computer Science,
Ghent University, Krijgslaan 281-S9, B-9000 Gent, Belgium
Stijn.Lievens@ugent.be, Joris.VanderJeugt@ugent.be

Abstract

The study of invariance groups associated with two term transformations between (basic) hypergeometric series has received its fair share of attention, and indeed, for most two term transformations between (basic) hypergeometric series, the underlying invariance group is explicitly known. In this article, we study the group structure underlying some three term transformation formulae, thereby giving an explicit and simple realization that is helpful in determining whether two of these transformation formulae are equivalent or not.

Keywords: Basic hypergeometric series, Three term transformation, Symmetry group
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1 Introduction

This article deals with transformations between basic hypergeometric series, and we use the (standard) notation of [5] when working with such series. The *q-shifted factorial* is

$$(a;q)_0 = 1, \quad \text{and } (a;q)_n = \prod_{k=0}^{n-1}(1-a\,q^k), \text{ for } n = 1,2,\ldots,\infty,$$

and $(a_1,\ldots,a_m;q)_n = (a_1;q)_n\cdots(a_m;q)_n$. The *basic hypergeometric series* we will be working with are all of the form

$${}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_r \end{matrix};q,z\right) = \sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_k}{(q,b_1,b_2,\ldots,b_r;q)_k}z^k.$$

The series appearing in this article are non-terminating, meaning that in general none of the numerator parameters a_i equals q^{-n}, with n a nonnegative integer. We will assume that the convergence conditions are always satisfied. A basic hypergeometric series is called of *type I* whenever $z=q$ and of *type II* when $z = b_1b_2\cdots b_r/a_1a_2\cdots a_{r+1}$. A basic hypergeometric series is called *very-well poised* if the following relations between its parameters hold:

$$qa_1 = a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r \quad \text{and} \quad a_2 = qa_1^{1/2},\ a_3 = -qa_1^{1/2},$$

and one uses the more compact notation ${}_{r+1}W_r(a_1;a_4,a_5,\ldots,a_{r+1};q,z)$ to denote such a series.

Each basic hypergeometric series has so called *trivial transformations*, since one can permute both the numerator parameters a_i and the denominator parameters b_i without changing the series. We immediately remark that this can destroy the very-well poised property of a series. If one wants to preserve this property, one has to restrict oneself to permutations of the parameters a_4 up to a_{r+1}, and the denominator parameters have to be permuted accordingly.

Besides these trivial transformations, for some basic hypergeometric series, there are also non-trivial transformations, and if such a transformation connects two series with the same number of numerator and denominator parameters, satisfying the same "boundary" conditions (e.g. being very-well poised), then the transformation can be iterated (and composed with the trivial transformations) to yield a so-called invariance group H for the series in question. As an example, the well known Heine transformation [5, Formula (1.4.1)]

$$ {}_2\phi_1(a,\ b;c;q,z) = \frac{(a,bz;q)_\infty}{(c,z;q)_\infty} {}_2\phi_1(c/a,\ z;bz;q,a), \tag{1} $$

together with permutations of a and b, yields a group H of order 12 which is isomorphic with the symmetry group of a regular hexagon [11]. The study of these invariance groups enables whole lists of known transformation formulae to be summarized as elegant one-line statements [11]. They can also be used for other purposes, e.g. to detect which expansion formulae are essentially different [8].

The following three term identity is an example of the sort of identities that we will be studying in this article, see [5, Appendix (III.31)]:

$$ \begin{aligned} &{}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) \\ &\quad = \frac{(b,c/a,az,q/az;q)_\infty}{(c,b/a,z,q/z;q)_\infty} {}_2\phi_1\left(\begin{matrix} a,aq/c \\ aq/b \end{matrix};q,\frac{cq}{abz}\right) + \frac{(a,c/b,bz,q/bz;q)_\infty}{(c,a/b,z,q/z;q)_\infty} {}_2\phi_1\left(\begin{matrix} b,bq/c \\ bq/a \end{matrix};q,\frac{cq}{abz}\right). \end{aligned} \tag{2} $$

Of course, the group G associated with such a three term identity will not be an invariance group in the same sense as it is for two term transformations. This is because, if one would apply this transformation to one of the series on the right hand side, the resulting identity would (in general) no longer be a three term identity (although it would be in this simple case), but would involve four series. Repeated application would cause the identity to expand to a multi term identity. In the cases we will consider however, as we will show, there will be an upper bound on the number of series that can occur, provided one groups series that are connected through a two term transformation.

What we will do however is interpret the arguments of the series on the right hand side as group element transforms of the arguments of the left hand side series. In this case, we thus have group elements mapping (a,b,c,q,z) on to $(a,aq/c,aq/b,q,cq/abz)$ and on to $(b,bq/c,bq/a,q,cq/abz)$.

To study the groups generated by these elements, we used the GAP [4] program, thereby presenting the group elements as matrices, acting on vectors from the left. Stated otherwise, we use group representations, although in this article the explicit calculations and representations are left out as they are quite trivial. The first of the previously mentioned group elements for instance would be represented by the matrix:

$$ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & -1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & -1 \end{pmatrix} \circ \begin{pmatrix} a \\ b \\ c \\ q \\ z \end{pmatrix} \equiv \begin{pmatrix} a \\ aq/c \\ aq/b \\ q \\ cq/abz \end{pmatrix}. $$

The action of this matrix on the five-dimensional vector is thus ordinary matrix multiplication with addition replaced by multiplication, and multiplication by exponentiation. Note that this matrix in principle acts on a column vector. To ease notation however, in the following we will not be too strict about the use of column or row vectors, as the context will make clear what is meant.

We will realize these groups as (signed) permutation groups acting on certain variables x_i. We will denote permutations by their cycle notation [6, Chapter 1]. For instance, the element $\tilde{h}$ of the symmetric group S_8 for which

$$\tilde{h}(0) = 4,\ \tilde{h}(1) = 6,\ \tilde{h}(2) = 7,\ \tilde{h}(3) = 5,\ \tilde{h}(4) = 0,\ \tilde{h}(5) = 2,\ \tilde{h}(6) = 3,\ \tilde{h}(7) = 1,$$

is denoted as $(0, 4)(1, 6, 3, 5, 2, 7)$. A signed permutation is equivalent to a permutation matrix (i.e. a matrix in which on each row and each column there appears exactly one "1", while the other elements of the matrix are "0") in which each non-zero element is "± 1"(instead of just "1").

Throughout this article $\mathbf{x}$ will be a shorthand for the vector $(x_0, x_1, \ldots, x_n)$, where n will always be clear from the context. The action of the permutation $\tilde{h}$ on $\mathbf{x}$ is simply to permute the indices as dictated by the permutation, e.g.:

$$\tilde{h} \circ \mathbf{x} = \tilde{h} \circ (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) \equiv (x_4, x_6, x_7, x_5, x_0, x_2, x_3, x_1).$$

The action of a signed permutation on $\mathbf{x}$ is the same as that of a permutation but for the fact that if the ith column of its matrix representation contains a -1, then x_i is replaced by $\frac{1}{x_i}$. Note that by our convention the application of a permutation $\tilde{h}$ followed by a permutation $\tilde{g}$ on the indices of x_i is realized by the matrix action $\tilde{h} \circ \tilde{g} \circ \mathbf{x}$.

In this article we will study three different cases of three term identities that appear in the literature, namely for ${}_2\phi_1$-series, for ${}_3\phi_2$-series of type II, and for very-well poised ${}_8\phi_7$-series. The main result can briefly be described as follows. For each term of a three term identity, there is a "local" invariance group H whose action leaves this term "invariant" (i.e. H is the invariance group of a two term transformation). Besides this, there exists the "invariance group" $G \supset H$ of the three term identity. Each term in the three term identity is characterized by a coset of H in G. Our analysis shows that conversely for each triple of cosets there is a three term identity. Using the group theoretical context, all three term identities between the series can thus be unified as a single statement.

For the ${}_2\phi_1$-series, we will find that the group G underlying the three term identities is the full group of symmetries of the cube; for ${}_3\phi_2$-series the group G is S_6; and for the very-well poised ${}_8\phi_7$-series G is the group of signed permutations on six elements that have an even number of -1 signs in their representation. Each of them will be treated in a different section. When appropriate, we will refer to known results concerning hypergeometric series, using a notation very similar to that in [10]. It should be mentioned that the first paper introducing this notion of "invariance group of a three term identity" is [1]. Some of the ideas are already present in the papers of Whipple [12, 13], however Whipple did not characterize the groups nor did he use any group theory arguments to unify the identities.

As a final remark for the introductory section, we stress that although in principal the order of the numerator and denominator parameters in basic hypergeometric series is immaterial, in this article we consider them to be fixed in the order given in the formulae, especially when considering permutations that are associated with the transformations dictated by these series.

2 Transformation formulae between basic ${}_2\phi_1$-series

Two term transformation formulae. Heine's two term transformation formula for ${}_2\phi_1$-series is well known, see e.g. [5]:

$$ {}_2\phi_1(a,\ b;c;q,z) = \frac{(a,bz;q)_\infty}{(c,z;q)_\infty} {}_2\phi_1(c/a,\ z;bz;q,a). \tag{3} $$

Using the following rescaling of the basic ${}_2\phi_1$ series

$$ \phi(a,\ b;c;q,z) \equiv (c,z;q)_\infty\, {}_2\phi_1(a,\ b;c;q,z), \tag{4} $$

the transformation (3) is rewritten as:

$$ \phi(a,\ b;c;q,z) = \phi(c/a,\ z;bz;q,a). $$

This transformation corresponds to the group element

$$ h \circ (a,b,c,q,z) = (c/a,z,bz,q,a). $$

Denote by p the group element corresponding to the trivial permutation of the numerator parameters:

$$ p \circ (a,b,c,q,z) = (b,a,c,q,z). $$

The invariance group H is generated by these elements: $H = \langle h,p\rangle$. This invariance group H (of the rescaled series) is of order 12 and is isomorphic to the dihedral group D_{12} [11].

Three term transformation formulae. Since it is the rescaled series that has the nice invariance property, we rewrite the three term identity (2), which is also formula T2162 from [7], using these rescaled series:

$$ \begin{aligned} &(b/a,a/b,qa/b,qb/a,q/z,qc/abz;q)_\infty\, \phi(a,\ b;c;q,z) \\ &\quad - (c/a,b,az,q/az,a/b,qb/a;q)_\infty\, \phi(a,\ qa/c;qa/b;q,qc/abz) \\ &\quad - (c/b,a,bz,q/bz,b/a,qa/b;q)_\infty\, \phi(b,\ qb/c;qb/a;q,qc/abz) = 0. \end{aligned} \tag{5} $$

In order to construct the group G, we need to consider also the group elements that map the first term of (5) to the other terms. Consider the element

$$ t \circ (a,b,c,q,z) = (b,qb/c,qb/a,q,qc/abz). $$

This element corresponds to the third and last series in (5). Now G is the group generated by h, p and t, i.e. $G = \langle h,p,t\rangle$. Note that we don't need the second series in (5) to generate the group, as it is already contained in G since $t \circ p \circ (a,b,c,q,z) = (a,qa/c,qa/b,q,qc/abz)$.

We will show that G is isomorphic with the full group of symmetries of the cube [2, Chapter 17]. In Figure 1 there is a drawing of a three dimensional cube together with a labelling of its vertices. This labelling is such that the sum of the labels on the corners of each of the six squares that make up the cube is 14 and such that the absolute value of the difference of the labels on opposite vertices equals 4. (Up to isomorphism there is only one way in which this can be done, a fact easily checked either by hand or with a simple computer program.)

Lemma 1 *The group G generated by the transformations h, p and t is isomorphic to the full group of symmetries of the cube.*

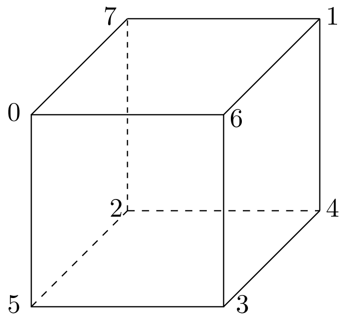

Figure 1: The cube with a specific labelling of its eight vertices.

Proof: In this proof we use the specific labelling of the cube as shown in Figure 1. Let x_0 up to x_7 be 8 variables, satisfying the condition that the product of each set of four variables that make up one of the faces of the cube equals 1. This gives 6 conditions, only 4 of which are independent from each other:

$$x_0x_1x_6x_7 = x_0x_2x_5x_7 = x_0x_3x_5x_6 = x_1x_2x_4x_7 = x_1x_3x_4x_6 = x_2x_3x_4x_5 = 1.$$

The 8 variables x_i now have 4 degrees of freedom left. Next, we let

$$a = q^{1/2}\frac{x_0x_3}{x_5x_6}, \quad b = q^{1/2}\frac{x_0x_1}{x_6x_7}, \quad c = q\frac{x_0x_4}{x_2x_6}, \quad \text{and} \quad z = q^{1/2}\frac{x_4x_6}{x_1x_3}. \tag{6}$$

These four equations can be inverted and solved for the variables x_i. Together with the previously mentioned constraints we get a unique solution, a fact easily verified by taking the logarithm of all equations involved to yield a system of linear equations in the variables $\log(x_i)$.

We now determine to which permutation of the cube vertices each of the group elements h, p and t corresponds. Since

$$c/a = q^{1/2}\frac{x_4x_5}{x_2x_3}, \quad z = q^{1/2}\frac{x_4x_6}{x_1x_3}, \quad bz = q\frac{x_0x_4}{x_3x_7}, \quad \text{and} \quad a = q^{1/2}\frac{x_0x_3}{x_5x_6},$$

the transformation h corresponds to the permutation $\tilde{h} \equiv (0,4)(1,6,3,5,2,7)$. This is an element of the group of full symmetries of the cube since it can be seen as a rotation over $2\pi/3$ radians around the 04 axis (permutation $(1,2,3)(5,6,7)$), followed by a reflection through the centre of the cube (permutation $(0,4)(1,5)(2,6)(3,7)$).

The transformation p (the swapping of the numerator parameters) corresponds to the permutation $\tilde{p} \equiv (1,3)(5,7)$, which can be realized as a reflection about the plane 0624.

It is easily verified that the transformation t corresponds to the permutation $\tilde{t} \equiv (0,1,2,3)(4,5,6,7)$ which can be seen as the permutation $(0,5,2,7)(1,6,3,4)$ (a rotation around the axis through the centre and perpendicular to the plane 0725 over $3\pi/2$ radians) followed again by a reflection through the centre of the cube. Since the transformations h, p and t generate a group of order 48, and since each of them corresponds to a symmetry of the cube, of which the full symmetry group has order 48 as well, the statement is proved. □

Transformation of the three term identity. We will now write the identity (5) in terms of the variables x_i. To this end, we first define the following function of eight variables, which is a translation of the rescaled basic hypergeometric series (4):

$$\tilde{\phi}(\mathbf{x}) \equiv \phi(q^{1/2}\frac{x_0x_3}{x_5x_6},\, q^{1/2}\frac{x_0x_1}{x_6x_7}; q\frac{x_0x_4}{x_2x_6}; q, q^{1/2}\frac{x_4x_6}{x_1x_3}). \tag{7}$$

For any element $\tilde{g}$ of the group $H = \langle \tilde{h}, \tilde{p} \rangle$ it holds that $\tilde{\phi}(\mathbf{x}) = \tilde{\phi}(\tilde{g} \circ \mathbf{x})$. Since $\tilde{h}$ only swaps the labels 0 and 4, and $\tilde{p}$ keeps them both in place, the numerator of the third argument of the series in $\tilde{\phi}(\tilde{g} \circ \mathbf{x})$ always equals qx_0x_4.

Since the order of $H = \langle \tilde{h}, \tilde{p} \rangle$ (the invariance group of $\tilde{\phi}(\mathbf{x})$) is 12, and since the full group G of symmetries of the cube has order 48, there are four cosets of H in G. The following four elements are simple representatives of these four cosets: $\tilde{t}^0 = ()$ (the identity), $\tilde{t}$, $\tilde{t}^2$ and $\tilde{t}^3$. It is thus the image of x_0 that determines the coset to which a particular group element belongs. For instance, the permutation $(0,3)(1,2)(4,7)(5,6)$ belongs to the coset of $\tilde{t}^3$, since the image of 0 is 3. (It is readily seen that this permutation corresponds to the group action dictated by the first series on the rhs of (2).) As a second example, if the permutation is $(0,6)(1,7)(2,4)(3,5)$ then it follows from the previous paragraph that it belongs to the coset $\tilde{t}^2$. Generally, if the image of 0 is k then the corresponding coset is $\tilde{t}^{[k]}$ with $[k] = k \bmod 4$.

We introduce the following abbreviation:

$$\tilde{\phi}_i(\mathbf{x}) \equiv \tilde{\phi}(\tilde{t}^i \circ \mathbf{x}).$$

The identity (5), when rewritten in terms of the eight variables x_0 up to x_7, thus involves the functions $\tilde{\phi}_0(\mathbf{x})$, $\tilde{\phi}_1(\mathbf{x})$ and $\tilde{\phi}_3(\mathbf{x})$. It is more elegant to have an identity involving $\phi_0(\mathbf{x})$, $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$, and this is why we apply the permutation $(2,3)(6,7)$ (a member of the full group of symmetries of the cube!) to this identity, resulting in the following:

$$\alpha(\mathbf{x})\,\tilde{\phi}_0(\mathbf{x}) + \beta(\mathbf{x})\,\tilde{\phi}_1(\mathbf{x}) + \gamma(\mathbf{x})\,\tilde{\phi}_2(\mathbf{x}) = 0, \tag{8}$$

with

$$\begin{aligned}
\alpha(\mathbf{x}) &= (\frac{x_1x_5}{x_2x_6}, \frac{x_2x_6}{x_1x_5}, q\frac{x_2x_6}{x_1x_5}, q\frac{x_5x_1}{x_2x_6}, q^{1/2}\frac{x_1x_2}{x_4x_7}, q^{1/2}\frac{x_5x_6}{x_0x_3}; q)_\infty \\
\beta(\mathbf{x}) &= -(\frac{x_6x_2}{x_0x_4}, \frac{x_1x_5}{x_2x_6}, q\frac{x_0x_4}{x_2x_6}, q\frac{x_2x_6}{x_1x_5}, q^{1/2}\frac{x_0x_2}{x_5x_7}, q^{1/2}\frac{x_4x_6}{x_1x_3}; q)_\infty \\
\gamma(\mathbf{x}) &= \beta(x_0, x_2, x_1, x_3, x_4, x_6, x_5, x_7).
\end{aligned} \tag{9}$$

Let $\tilde{g}$ be an element of the group $G = \langle \tilde{h}, \tilde{p}, \tilde{t} \rangle$. Applying $\tilde{g}$ to the identity (8) transforms it (trivially) in the following way:

$$\alpha(\tilde{g} \circ \mathbf{x})\,\tilde{\phi}_0(\tilde{g} \circ \mathbf{x}) + \beta(\tilde{g} \circ \mathbf{x})\,\tilde{\phi}_1(\tilde{g} \circ \mathbf{x}) + \gamma(\tilde{g} \circ \mathbf{x})\,\tilde{\phi}_2(\tilde{g} \circ \mathbf{x}) = 0.$$

Let k, l and m be three different elements of $\{0,1,2,3,4,5,6,7\}$, and let $G_{k,l,m}$ denote the subset of G containing elements that map x_0 to x_k, x_1 to x_l and x_2 to x_m. Note that the set $G_{k,l,m}$ either consists of just one element (if $\{k,l,m\}$ is a subset of $\{0,1,2,3\}$ or $\{4,5,6,7\}$) or is empty (otherwise). To see this is easy, remembering that the full group of symmetries of the cube is isomorphic to the group $S_4 \times C_2$ [2]. For k, there are thus $4 \times 2 = 8$ possibilities; when k is fixed, there are just 3 possibilities for l, and once k and l are fixed, there are only 2 possibilities for m. In this way, we have considered the 48 elements of the full group of symmetries of the cube.

We introduce the notation $\alpha_{k,l,m}(\mathbf{x}) \equiv \alpha(\tilde{g} \circ \mathbf{x})$ for each element $\tilde{g}$ of $G_{k,l,m}$, and analogously for β and γ. Considering the earlier remark about the size of $G_{k,l,m}$ this is rather trivial, but an analogous notation will prove its use in the later sections of this article. It is easy to see that $\alpha_{k,l,m}(\mathbf{x}) = \alpha_{[k],[l],[m]}(\mathbf{x})$, and analogously for β and γ. For any element $\tilde{g}$ of $G_{k,l,m}$, acting with $\tilde{g}$ on the identity (8) transforms it into:

$$\alpha_{k,l,m}(\mathbf{x})\tilde{\phi}_{[k]}(\mathbf{x}) + \beta_{k,l,m}(\mathbf{x})\tilde{\phi}_{[l]}(\mathbf{x}) + \gamma_{k,l,m}(\mathbf{x})\tilde{\phi}_{[m]}(\mathbf{x}) = 0,$$

so to describe the distinct relations it will be sufficient to consider indices from $\{0, 1, 2, 3\}$ only. We can now summarize the result as follows.

Theorem 1 *Let G be the group of symmetries of the cube (of order 48), and $H = \langle \tilde{h}, \tilde{p} \rangle$ its subgroup of order 12. The series $\tilde{\phi}(\mathbf{x})$ satisfies $\tilde{\phi}(\tilde{g} \circ \mathbf{x}) = \tilde{\phi}(\mathbf{x})$ for each $\tilde{g} \in H$. The 48 series $\tilde{\phi}(\tilde{g} \circ \mathbf{x})$, with $\tilde{g} \in G$, can be divided according to the four cosets of H in G, each coset being represented by $\tilde{\phi}_i(\mathbf{x})$, with $i \in \{0, 1, 2, 3\}$. For any three different elements k, l and m of $\{0, 1, 2, 3\}$ the following identity holds:*

$$\alpha_{k,l,m}(\mathbf{x})\tilde{\phi}_k(\mathbf{x}) + \beta_{k,l,m}(\mathbf{x})\tilde{\phi}_l(\mathbf{x}) + \gamma_{k,l,m}(\mathbf{x})\tilde{\phi}_m(\mathbf{x}) = 0.$$

We thus have found $\binom{4}{3} = 4$ three term identities between the four functions $\tilde{\phi}_i(\mathbf{x})$ (or among the four cosets). These identities are already known in the literature, and an instantiation of each of them is listed in [3, Eqs. (34),(37),(39),(41)]. Since, in fact we have already found all three term identities between the four functions $\tilde{\phi}_i(\mathbf{x})$, there is no point in trying to find new identities by an elimination procedure.

Remark. In [9] the group associated with the Kummer solutions is studied, i.e. a group associated with ordinary ${}_2F_1$-series. Is is shown that this group of order 48 is isomorphic with the full group of symmetries of the cube, which is a clear resemblance with what we have just seen. There are however some important differences:

- The invariance group of the ${}_2F_1(a,\, b; c; z)$ series generated by the Euler and Pfaff transformations is only of order 8, whereas the group generated by the Heine transformation is of order 12. In the former case there are thus 6 cosets, each of size 8, which is different from the 4 cosets of size 12 in the basic case.
- When the mirror symmetries are removed, one gets in both cases a group of order 24, which are however non-isomorphic. In the ordinary case, it is the group of direct symmetries of the cube, while in the basic case it contains elements of order 6 (and hence cannot be isomorphic with S_4).

3 Transformation formulae between basic ${}_3\phi_2$-series

Two term transformation formulae. Just as in the case of the basic ${}_2\phi_1$-series, we start off with a known two term transformation formula for which an interesting transformation group exists. This is the case for non-terminating ${}_3\phi_2$-series of type II, and the transformation group H involved follows from iterating

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right) = \frac{(b, de/ab, de/bc; q)_\infty}{(d, e, de/abc; q)_\infty}\, {}_3\phi_2\left(\begin{matrix} d/b, e/b, de/abc \\ de/ab, de/bc \end{matrix}; q, b\right), \tag{10}$$

together with permutations of the numerator and denominator parameters. Using the rescaling

$$\phi\,(a, b, c;\, d, e; q) \equiv (d, e, de/abc; q)_\infty\, {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right), \tag{11}$$

the transformation (10) is rewritten as:

$$\phi(a, b, c;\, d, e;\, q) = \phi(d/b, e/b, de/abc;\, de/ab, de/bc;\, q). \tag{12}$$

The invariance group H of $\phi(a, b, c;\, d, e)$ is isomorphic to the group S_5 [11] .

Three term transformation formulae. There exist three term transformation formulae connecting three ${}_3\phi_2$ series of type II, e.g. [5, III.33] (which is also [7, T3261] apart from a slight typo). Using the rescaling (11) this identity reads:

$$\begin{aligned}(cq/d, q/a, e/bc, d/q, bq/d, bcq/e, q^2/d; q)_\infty\, \phi(a, b, c;\, d, e;\, q)\\ -(e/b, e/c, q/d, d/q, d, de/abc, q^2/d; q)_\infty\, \phi(c, d/a, cq/e;\, cq/a, bcq/e;\, q)\\ +(q/d, b, d/a, de/bcq, bcq^2/de, c, d; q)_\infty\, \phi(aq/d, bq/d, cq/d;\, q^2/d, eq/d;\, q) = 0.\end{aligned} \tag{13}$$

It is then natural to consider the following group actions:

$$\begin{aligned} h\circ(a, b, c, d, e, q) &= (d/b, e/b, de/abc, de/ab, de/bc, q)\\ p_1\circ(a, b, c, d, e, q) &= (b, a, c, d, e, q)\\ p_2\circ(a, b, c, d, e, q) &= (b, c, a, d, e, q)\\ p_3\circ(a, b, c, d, e, q) &= (a, b, c, e, d, q)\\ t\circ(a, b, c, d, e, q) &= (c, d/a, cq/e, cq/a, bcq/e, q).\end{aligned}$$

The group $H = \langle h, p_1, p_2, p_3\rangle$ is isomorphic to S_5, and adding t as a generator to this group yields a group G of order 720, which we will show, by giving a simple realization, to be isomorphic to the group S_6. We note that the group element corresponding to the last series in (13) is not necessary as it is already contained in the group generated by the given elements.

Lemma 2 *The group G generated by h, p_1, p_2, p_3 and t is isomorphic to the group of permutations on six elements, denoted S_6.*

Proof: Let x_0 up to x_5 be six variables satisfying the constraint $x_0x_1x_2x_3x_4x_5 = 1$, and let

$$a = q^{1/2}\frac{x_0x_1x_2}{x_3x_4x_5},\quad b = q^{1/2}\frac{x_0x_2x_3}{x_1x_4x_5},\quad c = q^{1/2}\frac{x_0x_1x_3}{x_2x_4x_5},\quad d = q\frac{x_0^2}{x_4^2},\quad e = q\frac{x_0^2}{x_5^2}. \tag{14}$$

It is then easy to see that transformations h, p_1, p_2 and p_3 respectively correspond to the permutation $\tilde{h} = (5, 3, 4, 2, 1)$, $\tilde{p}_1 = (1, 3)$, $\tilde{p}_2 = (1, 2, 3)$, and $\tilde{p}_3 = (4, 5)$. Note that none of these permutations involve the index 0. The transformation t on the other hand corresponds to the permutation $\tilde{t} = (5, 4, 2, 0, 3)$. The given five permutations indeed generate the group S_6. □

We now proceed as before, and define the rescaled ${}_3\phi_2$-series in terms of the variables x_i using the realization (14):

$$\tilde{\phi}(\mathbf{x}) \equiv \phi(q^{1/2}\frac{x_0x_1x_2}{x_3x_4x_5}, q^{1/2}\frac{x_0x_2x_3}{x_1x_4x_5}, q^{1/2}\frac{x_0x_1x_3}{x_2x_4x_5};\, q\frac{x_0^2}{x_4^2},\, q\frac{x_0^2}{x_5^2};\, q). \tag{15}$$

It is known that for any element $\tilde{g}$ of the group $H = \langle \tilde{h}, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3\rangle$ it holds that $\tilde{\phi}(\mathbf{x}) = \tilde{\phi}(\tilde{g}\circ\mathbf{x})$. As already mentioned, all permutations of this group keep x_0 in place, and hence x_0 will be present in the numerators of each of the five arguments of the series appearing in $\tilde{\phi}$. The invariance group H has index 6 in the bigger group G that is associated with (13), and hence there are 6 cosets of H in G. Simple representatives of these six cosets are given by $\tilde{c}^0 = ()$, $\tilde{c} = (0, 1, 2, 3, 4, 5)$, up to $\tilde{c}^5 = (0, 5, 4, 3, 2, 1)$. As before we introduce the notation

$$\tilde{\phi}_i(\mathbf{x}) \equiv \tilde{\phi}\big(\tilde{c}^i\circ\mathbf{x}\big),$$

and we see that the identity (13), if rewritten in terms of the variables x_i, would involve the functions $\tilde{\phi}_0$, $\tilde{\phi}_3$ and $\tilde{\phi}_4$. We apply the permutation $(1,3)(2,4)$ to this identity and we get:

$$\alpha(\mathbf{x})\tilde{\phi}_0(\mathbf{x}) + \beta(\mathbf{x})\tilde{\phi}_1(\mathbf{x}) + \gamma(\mathbf{x})\tilde{\phi}_2(\mathbf{x}) = 0, \tag{16}$$

with

$$\begin{aligned}
\alpha(\mathbf{x}) &= (q^{1/2}\frac{x_3x_1x_2}{x_0x_4x_5}, q^{1/2}\frac{x_1x_2x_5}{x_0x_3x_4}, q^{1/2}\frac{x_1x_2x_4}{x_0x_3x_5}, \frac{x_2{}^2}{x_1{}^2}, \frac{x_0{}^2}{x_2{}^2}, \frac{qx_2{}^2}{x_0{}^2}, \frac{qx_1{}^2}{x_2{}^2}; q)_\infty \\
\beta(\mathbf{x}) &= -(q^{1/2}\frac{x_0x_2x_3}{x_1x_4x_5}, q^{1/2}\frac{x_0x_2x_4}{x_5x_1x_3}, q^{1/2}\frac{x_0x_2x_5}{x_3x_4x_1}, \frac{x_2{}^2}{x_0{}^2}, \frac{x_0{}^2}{x_2{}^2}, \frac{qx_0{}^2}{x_2{}^2}, \frac{qx_2{}^2}{x_0{}^2}; q)_\infty \\
\gamma(\mathbf{x}) &= \alpha(x_2, x_1, x_0, x_3, x_4, x_5).
\end{aligned}$$

As in the case of ${}_2\phi_1$-series, we introduce the notation $G_{k,l,m}$ to stand for the subset of S_6 containing the elements which map x_0 onto x_k, x_1 to x_l and x_2 to x_m, for any three different elements of $\{0,1,2,3,4,5\}$. In this case each set $G_{k,l,m}$ consists of six elements.

One sees immediately that each of the three coefficients α, β and γ is symmetric in the variables x_3, x_4 and x_5, allowing us to introduce the notations $\alpha_{k,l,m}(\mathbf{x})$, $\beta_{k,l,m}(\mathbf{x})$ and $\gamma_{k,l,m}(\mathbf{x})$ for three arbitrary, yet different, elements from $\{0,1,2,3,4,5\}$. Hereby $\alpha_{k,l,m}(\mathbf{x})$ stands for the resulting coefficient after acting on $\alpha(\mathbf{x})$ with an arbitrary element from $G_{k,l,m}$. In all, after acting with all 720 permutations, there will be $\binom{6}{3} = 20$ three term relations, each connecting three ${}_3\phi_2$-series of type II. This can be summarized as:

Theorem 2 *Let $G = S_6$ and $H = S_5$ as described. The series $\tilde{\phi}(\mathbf{x})$ satisfies $\tilde{\phi}(\tilde{g} \circ \mathbf{x}) = \tilde{\phi}(\mathbf{x})$ for each $\tilde{g} \in H$. The 720 elements $\tilde{\phi}(\tilde{g} \circ \mathbf{x})$, with $\tilde{g} \in G$, can be divided according to the six cosets of H in G, each coset being represented by $\tilde{\phi}_i(\mathbf{x})$, with $i \in \{0,1,2,3,4,5\}$. For any three different elements k, l and m from $\{0,1,2,3,4,5\}$ the following identity holds:*

$$\alpha_{k,l,m}(\mathbf{x})\phi_k(\mathbf{x}) + \beta_{k,l,m}(\mathbf{x})\phi_l(\mathbf{x}) + \gamma_{k,l,m}(\mathbf{x})\phi_m(\mathbf{x}) = 0.$$

For example taking $(k,l,m) = (0,5,2)$ gives [7, T3263].

Remark. In 1987, Beyer et al. [1], already studied the group underlying the three term transformations between ordinary non-terminating hypergeometric series of unit argument, i.e. ${}_3F_2(1)$-series. The group they obtain is of order 1440 and is isomorphic to $S_6 \times C_2$. The reason they obtain a larger group lies in the fact that both basic hypergeometric series of type I and II reduce to ordinary hypergeometric series of unit argument when taking the (formal) limit $q \uparrow 1$, and thus there are other three term relations to start with, which can be seen as limiting relations from three term relations connecting basic hypergeometric ${}_3\phi_2$-series of both type I and II, e.g. [5, III.34].

In the q-case, we cannot use this type of identity, as there is no known interesting invariance group for ${}_3\phi_2$-series of type I. One could argue that a series that is balanced is both of type I and II, but for such a series the relation [5, III.34] reduces to a sort of "summation" formula, since one of the series involved becomes summable.

On the other hand, the realization given in Lemma 2 is, with the usual modification of replacing q by 1 and multiplication resp. division, by addition resp. subtraction, directly applicable to the case of the ordinary hypergeometric ${}_3F_2(1)$-series. The first series on the right hand side of the resulting identity after taking the limit $q \uparrow 1$ in [5, III.34], namely

$${}_3F_2\left(\begin{matrix} d-a, b, c \\ d, 1+b+c-e \end{matrix}; 1\right)$$

corresponds to the signed permutation p that sends $\mathbf{x}$ to $(-x_4, -x_2, -x_1, -x_5, -x_0, -x_3)$, and thus this will yield (like in the following section, and as in [1]) the introduction of series with "starred" indices, giving twelve different functions in all.

4 Transformation formulae between very-well poised ${}_8\phi_7$-series

Two term transformation formulae. Again, we start with a well known two term transformation formula ([7, T8704] or [5, III.23]):

$$\begin{aligned} &{}_8W_7(a^2; ab, ac, ad, ae, af; q, \frac{q^2}{abcdef}) \\ &\quad = \frac{(a^2q, q/ef, q^2/bcde, q^2/bcdf; q)_\infty}{(aq/e, aq/f, aq^2/bcd, q^2/abcdef; q)_\infty}\, {}_8W_7(\frac{aq}{bcd}; \frac{q}{cd}, \frac{q}{bd}, \frac{q}{bc}, af, ae; q, \frac{q}{ef}). \end{aligned} \tag{17}$$

Using the following rescaling of the ${}_8W_7$-series:

$$\begin{aligned} &w(a; b, c, d, e, f) \\ &\quad \equiv \frac{(aq/b, aq/c, aq/d, aq/e, aq/f, a^2q^2/bcdef; q)_\infty}{(aq; q)_\infty}\, {}_8W_7(a; b, c, d, e, f; q, \frac{a^2q^2}{bcdef}) \end{aligned} \tag{18}$$

or more explicitly

$$\begin{aligned} &w(a^2; ab, ac, ad, ae, af) \\ &\quad = \frac{(aq/b, aq/c, aq/d, aq/e, aq/f, q^2/abcdef; q)_\infty}{(a^2q; q)_\infty}\, {}_8W_7(a^2; ab, ac, ad, ae, af; q, \frac{q^2}{abcdef}), \end{aligned} \tag{19}$$

equation (17) is written as:

$$w(a^2; ab, ac, ad, ae, af) = w(\frac{aq}{bcd}; \frac{q}{cd}, \frac{q}{bd}, \frac{q}{bc}, af, ae). \tag{20}$$

This is a two term transformation formula for very-well poised ${}_8\phi_7$-series, and it is known that transformation (20) together with permutations of (b, c, d, e, f) generates a group H of order 1920, and that this group is in fact the group WD_5 (the Weyl group of D_5, see [11, Section V]).

Three term transformation formulae. The first three term transformation formula between very-well poised ${}_8\phi_7$-series listed in [7] is transformation T8762. Using the rescaling (19) it reads:

$$\begin{aligned} &(adef, bd, be, bf, a/b, q/ac, q/adef, bq/a; q)_\infty\, w(a^2; ab, ac, ad, ae, af) \\ &+ (bdef, ad, ae, af, b/a, q/bc, q/bdef, aq/b; q)_\infty\, w(b^2; bc, bd, be, bf, ba) \\ &- (b/a, a/b, q/de, q/df, q/ef, aq/b, bq/a, q^2/abcdef; q)_\infty\, w(\frac{aef}{c}; \frac{q}{bc}, \frac{q}{cd}, ef, ae, af) = 0. \end{aligned} \tag{21}$$

One can apply (20) to the third series to write the relation in a "nicer" form, e.g. one can write:

$$w(\frac{aef}{c}; ef, ae, af, \frac{q}{bc}, \frac{q}{cd}) = w(\frac{q}{c^2}; \frac{q}{cd}, \frac{q}{ce}, \frac{q}{cf}, \frac{q}{ca}, \frac{q}{cb}).$$

Next, we consider the group elements associated with the two term transformation (17) or (20):

$$g \circ (a^2, ab, ac, ad, ae, af, q) = (\frac{aq}{bcd}, \frac{q}{cd}, \frac{q}{bd}, \frac{q}{bc}, af, ae, q),$$

and

$$p_1 \circ (a^2, ab, ac, ad, ae, af, q) = (a^2, ac, ad, ae, af, ab, q),$$
$$p_2 \circ (a^2, ab, ac, ad, ae, af, q) = (a^2, ac, ab, ad, ae, af, q).$$

One sees that p_1 and p_2 together allow any permutation of the parameters in the positions 2 up to 6; moreover, the elements g, p_1 and p_2 together generate a group H of order 1920, and this is exactly the invariance group of the transformation (20).

We also consider the group elements associated with the three term identity (21):

$$t_1 \circ (a^2, ab, ac, ad, ae, af, q) = (b^2, bc, bd, be, bf, ba, q),$$
$$t_2 \circ (a^2, ab, ac, ad, ae, af, q) = (q/c^2, q/cd, q/ce, q/cf, q/ca, q/cb, q).$$

The five elements g, p_1, p_2, t_1 and t_2 generate together a group $G = \langle g, p_1, p_2, t_1, t_2\rangle$ of order 23040, a fact easily verified using GAP [4]. We remark that the same group is already generated by g, p_1 and t_1, since both p_2 and t_2 can be expressed in terms of g, p_1 and t_1. One possibility (which is by no means guaranteed to be the simplest one) is the following:

$$p_2 \circ (a^2, ab, ac, ad, ae, af, q) = p_1^4 \circ t_1 \circ p_1^3 \circ t_1^2 \circ (a^2, ab, ac, ad, ae, af, q)$$

and

$$\begin{aligned} &t_2 \circ (a^2, ab, ac, ad, ae, af, q) \\ &\quad = g \circ p_1^2 \circ g \circ p_1^3 \circ t_1 \circ p_1^3 \circ g \circ p_1^2 \circ t_1 \circ (a^2, ab, ac, ad, ae, af, q). \end{aligned}$$

In [11] it was already shown that the invariance group H of (20) is isomorphic to the group of signed permutations on five elements with an even number of -1 signs in their representation. Since $|\langle g, p_1, t_1\rangle| = 23040 = 6! \times 2^5 = 6! \times \left(\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6}\right)$, we may suspect that this group is isomorphic to the group of signed permutations on six elements, with an even number of -1 signs in their representation. This is in fact the subject of the next lemma.

Lemma 3 *The group G generated by the elements g, p_1 and t_1 is isomorphic to the group of signed permutations on six elements with an even number of -1 signs in their representation.*

Proof: Let x_0, x_1, x_2, x_3, x_4 and x_5 be six arbitrary variables, and let:

$$\begin{aligned} a^2 &= q^{1/2}\frac{x_1x_2x_3x_4x_5}{x_0^3} & ab &= q^{1/2}\frac{x_2x_3x_4x_5}{x_0x_1} \\ ac &= q^{1/2}\frac{x_1x_3x_4x_5}{x_0x_2} & ad &= q^{1/2}\frac{x_1x_2x_4x_5}{x_0x_3} \\ ae &= q^{1/2}\frac{x_1x_2x_3x_5}{x_0x_4} & af &= q^{1/2}\frac{x_1x_2x_3x_4}{x_0x_5}, \end{aligned} \tag{22}$$

this is, we write the six arguments of the series on the left hand side of (20) in terms of the variables x_i, with $i \in \{0, 1, 2, 3, 4, 5\}$. We note that, as before, this rewriting is invertible.

It is easily verified that the arguments of the series on the right hand side of (20) become

$$\begin{aligned} aq/bcd &= q^{1/2}\frac{x_1x_2x_3}{x_0^3x_4x_5} & q/cd &= q^{1/2}\frac{x_2x_3}{x_0x_1x_4x_5} \\ q/bd &= q^{1/2}\frac{x_1x_3}{x_0x_2x_4x_5} & q/bc &= q^{1/2}\frac{x_1x_2}{x_0x_3x_4x_5} \\ af &= q^{1/2}\frac{x_1x_2x_3x_4}{x_0x_5} & ae &= q^{1/2}\frac{x_1x_2x_3x_5}{x_0x_4}. \end{aligned}$$

This means that the action of g on $(a^2, ab, ac, ad, ae, af, q)$ translates into the signed permutation

$$\tilde{g} \circ (x_0, x_1, x_2, x_3, x_4, x_5) = (x_0, x_1, x_2, x_3, \frac{1}{x_4}, \frac{1}{x_5}).$$

In the same way one verifies that the actions of p_1 and t_1 translate into:

$$\tilde{p}_1 \circ (x_0, x_1, x_2, x_3, x_4, x_5) = (x_0, x_2, x_3, x_4, x_5, x_1),$$
$$\tilde{t}_1 \circ (x_0, x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4, x_5, x_0).$$

Since the signed permutations $\tilde{g}$, $\tilde{p}_1$ and $\tilde{t}_1$ together generate the already mentioned subgroup of the group of signed permutations on six elements, this identifies the group G involved in the three term transformation formula for ${}_8W_7$-series. □

Since the index of the subgroup $H = \langle g, p_1, p_2 \rangle$ in the group $G = \langle g, p_1, t_1 \rangle$ is $23040/1920 = 12$, there are 12 cosets of the invariance group H in G, each of size 1920. The signed permutations $\tilde{g}$, $\tilde{p}_1$ and $\tilde{p}_2 = (1, 2)$ all leave the first element of their argument vector intact, while the other five arguments form a signed permutation with an even number of -1 signs. Let

$$\tilde{c} \circ \mathbf{x} \equiv (x_1, x_2, x_3, x_4, x_5, x_0) \quad \text{and} \quad \tilde{\imath} \circ \mathbf{x} \equiv (\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_1}, \frac{1}{x_3}, \frac{1}{x_4}, \frac{1}{x_5}).$$

It is then easy to see that the following twelve signed permutations are representatives of the 12 cosets: $\tilde{c}^j$ and $\tilde{c}_*^j \equiv \tilde{\imath} \circ \tilde{c}^j$, with $j \in \{0, 1, 2, 3, 4, 5\}$.

We next define the following function of six arguments:

$$\begin{aligned} &\tilde{w}(x_0, x_1, x_2, x_3, x_4, x_5) \\ &\quad \equiv w(q^{1/2}\frac{x_1x_2x_3x_4x_5}{x_0^3}; q^{1/2}\frac{x_2x_3x_4x_5}{x_0x_1}, q^{1/2}\frac{x_1x_3x_4x_5}{x_0x_2}, q^{1/2}\frac{x_1x_2x_4x_5}{x_0x_3}, q^{1/2}\frac{x_1x_2x_3x_5}{x_0x_4}, q^{1/2}\frac{x_1x_2x_3x_4}{x_0x_5}), \end{aligned} \tag{23}$$

which is (19), but rewritten in terms of the variables x_i. For each element i from $\{0, 1, 2, 3, 4, 5\}$, we define

$$\tilde{w_i}(\mathbf{x}) \equiv \tilde{w}(c^i \circ \mathbf{x}), \quad \text{and} \quad \tilde{w}_{i^*}(\mathbf{x}) \equiv \tilde{w}(c_*^i \circ \mathbf{x}).$$

Using the realization (22), one then rewrites the identity (21) as:

$$\alpha(\mathbf{x})\tilde{w}_0(\mathbf{x}) + \beta(\mathbf{x})\tilde{w}_1(\mathbf{x}) + \gamma(\mathbf{x})\tilde{w}_{2^*}(\mathbf{x}) = 0, \tag{24}$$

with

$$\begin{aligned} &\alpha(x_0, x_1, x_2, x_3, x_4, x_5) \\ &\quad = (qx_1^2x_2^2, \frac{1}{x_1^2x_2^2}, \frac{qx_0^2}{x_1^2}, \frac{x_1^2}{x_0^2}, \frac{q^{1/2}x_0x_2x_4x_5}{x_1x_3}, \frac{q^{1/2}x_0x_2x_3x_5}{x_1x_4}, \frac{q^{1/2}x_0x_2x_3x_4}{x_1x_5}, \frac{q^{1/2}x_0x_2}{x_1x_3x_4x_5}; q)_\infty, \\ &\beta(x_0, x_1, x_2, x_3, x_4, x_5) \\ &\quad = \alpha(x_1, x_0, x_2, x_3, x_4, x_5), \\ &\gamma(x_0, x_1, x_2, x_3, x_4, x_5) \\ &\quad = -(\frac{x_0^2}{x_1^2}, \frac{qx_1^2}{x_0^2}, \frac{x_1^2}{x_0^2}, \frac{qx_0^2}{x_1^2}, \frac{q^{1/2}x_3x_4}{x_0x_1x_2x_5}, \frac{q^{1/2}x_4x_5}{x_0x_1x_2x_3}, \frac{q^{1/2}x_3x_5}{x_0x_1x_2x_4}, \frac{q^{1/2}}{x_0x_1x_2x_3x_4x_5}; q)_\infty. \end{aligned} \tag{25}$$

Transformation of the three term identity. As usual, let $H = \langle \tilde{g}, \tilde{p}_1, \tilde{p}_2 \rangle$ denote the invariance group of $\tilde{w}$, and let $G = \langle \tilde{g}, \tilde{t}_1, \tilde{p}_1 \rangle$ denote the group associated with the three term transformation. If $\tilde{g}_1$ is an arbitrary element of G, then is trivial from (24) that

$$\alpha(\tilde{g}_1 \circ \mathbf{x})\tilde{w}_0(\tilde{g}_1 \circ \mathbf{x}) + \beta(\tilde{g}_1 \circ \mathbf{x})\tilde{w}_1(\tilde{g}_1 \circ \mathbf{x}) + \gamma(\tilde{g}_1 \circ \mathbf{x})\tilde{w}_{2^*}(\tilde{g}_1 \circ \mathbf{x}) = 0. \tag{26}$$

For any $\tilde{h} \in H$ it holds that $\tilde{w}(\mathbf{x}) = \tilde{w}(\tilde{h} \circ \mathbf{x})$, this is in fact the result of Section V of [11]. In all, there are 23040 possible arguments to $\tilde{w}$, corresponding to the 23040 signed permutation matrices on six elements with an even number of -1 signs. These 23040 series are classified into 12 cosets of 1920 elements each, all elements of a coset being equal to each other. It is the image of x_0 that determines the coset to which a group element belongs. Hence for $\tilde{w}_i(\tilde{g}_1 \circ \mathbf{x})$, it is the image of x_i that determines the resulting coset, and likewise for $\tilde{w}_{i^*}(\tilde{g}_1 \circ \mathbf{x})$.

For (24) the series that appear in the transformed equation (26) are determined by the images of x_0, x_1 and x_2. Let k, l and m be three different elements of $\{0, 1, \ldots, 5, 0^*, \ldots, 5^*\}$ and denote by $G_{k,l,m}$ the subset of the elements of G that map x_0 to x_k, x_1 to x_l and x_2 to x_m. Here we use the convention that $x_{k^*} = 1/x_k$, whenever $k \in \{0, 1, 2, 3, 4, 5\}$, and thus $x_{k^{**}} = x_k$. The element

$$\tilde{g}_1 \circ \mathbf{x} = (x_0, x_1, \frac{1}{x_2}, \frac{1}{x_3}, x_4, x_5) \tag{27}$$

for instance is an element of $G_{0,1,2^*}$. Each set $G_{k,l,m}$ contains 24 elements (if it is non-empty, i.e. if $\{k, l, m\} \cap \{k^*, l^*, m\} = \emptyset$), and there are $12 \times 10 \times 8 = 960$ such sets in all. We next look at how the coefficients α, β and γ transform under elements of $G_{k,l,m}$.

Lemma 4 *Let k, l, and m be three different elements from $\{0, \ldots, 5^*\}$, then it holds that:*

$$\alpha(\tilde{g}_1 \circ \mathbf{x}) = \alpha(\tilde{g}_2 \circ \mathbf{x}), \quad \beta(\tilde{g}_1 \circ \mathbf{x}) = \beta(\tilde{g}_2 \circ \mathbf{x}), \quad \text{and} \quad \gamma(\tilde{g}_1 \circ \mathbf{x}) = \gamma(\tilde{g}_2 \circ \mathbf{x}), \tag{28}$$

for any two elements $\tilde{g}_1$ and $\tilde{g}_2$ of $G_{k,l,m}$.

Proof: We prove the lemma for α, immediately implying the result for β. For γ, the lemma is proved analogously. Since the first four arguments of the infinite product in α do not involve x_3, x_4 or x_5, we only have to concentrate on the last four arguments:

$$(\frac{q^{1/2}x_0x_2x_4x_5}{x_1x_3}, \frac{q^{1/2}x_0x_2x_3x_5}{x_1x_4}, \frac{q^{1/2}x_0x_2x_3x_4}{x_1x_5}, \frac{q^{1/2}x_0x_2}{x_1x_3x_4x_5}; q)_\infty. \tag{29}$$

If there are zero or two "starred" elements under k, l and m, then there are also an even number of "starred" elements under the images of x_3, x_4 and x_5. It is clear that the product (29) is invariant under any permutation of x_3, x_4 and x_5. Moreover, the signed permutation $(x_3, x_4, x_5) \to (\frac{1}{x_3}, \frac{1}{x_4}, x_5)$ leaves the same product invariant, since it swaps the first with the second and the third with the fourth argument. This proves the statement in this case.

If there are one or three "starred" elements under k, l and m, then one has an odd number of "starred" elements under the images of x_3, x_4 and x_5. Since the image of the product (29) under any permutation of x_3, x_4 and x_5 with an odd number of -1 signs in their representation equals,

$$(\frac{q^{1/2}x_0x_2x_3}{x_1x_4x_5}, \frac{q^{1/2}x_0x_2x_4}{x_1x_3x_5}, \frac{q^{1/2}x_0x_2x_5}{x_1x_3x_4}, \frac{q^{1/2}x_0x_2x_3x_4x_5}{x_1}; q)_\infty,$$

the lemma is proved. □

This means that one can again introduce the notation $\alpha_{k,l,m}(\mathbf{x})$, to stand for $\alpha(\tilde{g}_1 \circ \mathbf{x})$ with $\tilde{g}_1$ an arbitrary element of $G_{k,l,m}$, and likewise for β and γ. For any element $\tilde{g}_1$ of $G_{k,l,m}$, the identity (24) transforms into

$$\alpha_{k,l,m}(\mathbf{x})\tilde{w}_k(\mathbf{x}) + \beta_{k,l,m}(\mathbf{x})\tilde{w}_l(\mathbf{x}) + \gamma_{k,l,m}(\mathbf{x})\tilde{w}_{m^*}(\mathbf{x}) = 0.$$

For instance, using the transformation (27), which is an element of $G_{0,1,2^*}$, transforms the original three term identity into:

$$\alpha_{0,1,2^*}(\mathbf{x})\tilde{w}_0(\mathbf{x}) + \beta_{0,1,2^*}(\mathbf{x})\tilde{w}_1(\mathbf{x}) + \gamma_{0,1,2^*}(\mathbf{x})\tilde{w}_2(\mathbf{x}) = 0.$$

Translating this back into variables a, b, c, d, e and f, using the inverse relations of (22) for the factors in front of the various series, and using an easy representative for each of the series gives:

$$\begin{aligned} &(bc, a/b, b/c, qb/a, qc/b, q/ad, q/ae, q/af; q)_\infty \, w(a^2; ab, ac, ad, ae, af) \\ &+ (ac, b/a, a/c, qa/b, qc/a, q/bd, q/be, q/bf; q)_\infty \, w(b^2; bc, bd, be, bf, ba) \\ &- (ab, b/a, a/b, qa/b, qb/a, q/cd, q/ce, q/cf; q)_\infty \, w(c^2; cd, ce, cf, ca, cb) = 0. \end{aligned} \tag{30}$$

This identity contains the same three series as transformation T8764 in Krattenthaler's list [7], and the factors in front can be rewritten to match those in [7].

Identities obtainable by transformation and elimination. Let k, l and m be three different elements from $\{0, 1, 2, 3, 4, 5\}$. From (24) we can, using transformations with elements from G, deduce three term identities of the following forms:

$$\begin{cases} \alpha_{k,l,m}(\mathbf{x})\tilde{w}_k(\mathbf{x}) + \beta_{k,l,m}(\mathbf{x})\tilde{w}_l(\mathbf{x}) + \gamma_{k,l,m}(\mathbf{x})\tilde{w}_{m^*}(\mathbf{x}) = 0, & \text{for } \tilde{g} \in G_{k,l,m} \\ \alpha_{k^*,l^*,m^*}(\mathbf{x})\tilde{w}_{k^*}(\mathbf{x}) + \beta_{k^*,l^*,m^*}(\mathbf{x})\tilde{w}_{l^*}(\mathbf{x}) + \gamma_{k^*,l^*,m^*}(\mathbf{x})\tilde{w}_m(\mathbf{x}) = 0, & \text{for } \tilde{g} \in G_{k^*,l^*,m^*} \\ \alpha_{k,l,m^*}(\mathbf{x})\tilde{w}_k(\mathbf{x}) + \beta_{k,l,m^*}(\mathbf{x})\tilde{w}_l(\mathbf{x}) + \gamma_{k,l,m^*}(\mathbf{x})\tilde{w}_m(\mathbf{x}) = 0, & \text{for } \tilde{g} \in G_{k,l,m^*} \\ \alpha_{k^*,l^*,m}(\mathbf{x})\tilde{w}_{k^*}(\mathbf{x}) + \beta_{k^*,l^*,m}(\mathbf{x})\tilde{w}_{l^*}(\mathbf{x}) + \gamma_{k^*,l^*,m}(\mathbf{x})\tilde{w}_{m^*}(\mathbf{x}) = 0, & \text{for } \tilde{g} \in G_{k^*,l^*,m}. \end{cases}$$

This gives $\binom{6}{3} + \binom{6}{3} + \binom{6}{2}\binom{4}{1} + \binom{6}{2}\binom{4}{1} = 160$ three term identities between the various coset elements. However, one would expect that there exist $\binom{12}{3} = 220$ such relations, and indeed this is the case, as the remaining 60 relations can be obtained by a simple elimination process.

Let k, l and m be three different elements from $\{0, 1, \ldots, 5\}$, and eliminate $\tilde{w}_{m^*}(\mathbf{x})$ from the following two relations:

$$\begin{cases} \alpha_{k,l,m}(\mathbf{x})\tilde{w}_k(\mathbf{x}) + \beta_{k,l,m}(\mathbf{x})\tilde{w}_l(\mathbf{x}) + \gamma_{k,l,m}(\mathbf{x})\tilde{w}_{m^*}(\mathbf{x}) = 0, \\ \alpha_{k,l^*,m}(\mathbf{x})\tilde{w}_k(\mathbf{x}) + \beta_{k,l^*,m}(\mathbf{x})\tilde{w}_{l^*}(\mathbf{x}) + \gamma_{k,l^*,m}(\mathbf{x})\tilde{w}_{m^*}(\mathbf{x}) = 0, \end{cases}$$

and one gets:

$$\begin{aligned} &\big(\alpha_{k,l,m}(\mathbf{x})\gamma_{k,l^*,m}(\mathbf{x}) - \alpha_{k,l^*,m}(\mathbf{x})\gamma_{k,l,m}(\mathbf{x})\big)\tilde{w}_k(\mathbf{x}) \\ &\qquad + \beta_{k,l,m}(\mathbf{x})\gamma_{k,l^*,m}(\mathbf{x})\tilde{w}_l(\mathbf{x}) - \beta_{k,l^*,m}(\mathbf{x})\gamma_{k,l,m}(\mathbf{x})\tilde{w}_{l^*}(\mathbf{x}) = 0. \end{aligned} \tag{31}$$

This yields another $6 \times 5 = 30$ relations. Replacing k by k^* in (31) yields the identity:

$$\begin{aligned} &\big(\alpha_{k^*,l,m}(\mathbf{x})\gamma_{k^*,l^*,m}(\mathbf{x}) - \alpha_{k^*,l^*,m}(\mathbf{x})\gamma_{k^*,l,m}(\mathbf{x})\big)\tilde{w}_{k^*}(\mathbf{x}) \\ &\qquad + \beta_{k^*,l,m}(\mathbf{x})\gamma_{k^*,l^*,m}(\mathbf{x})\tilde{w}_l(\mathbf{x}) - \beta_{k^*,l^*,m}(\mathbf{x})\gamma_{k^*,l,m}(\mathbf{x})\tilde{w}_{l^*}(\mathbf{x}) = 0, \end{aligned} \tag{32}$$

yielding another 30 relations. In all we thus have 220 relations.

Theorem 3 *Let G (resp. H) be the group of signed permutations on six (resp. five) elements with an even number of -1 signs, as described. The series $\tilde{w}(\mathbf{x})$ satisfies $\tilde{w}(\tilde{g} \circ \mathbf{x}) = \tilde{w}(\mathbf{x})$, for each $\tilde{g} \in H$. The 23040 series $\tilde{w}(\tilde{g} \circ \mathbf{x})$, with $\tilde{g} \in G$, can be divided according to the twelve cosets of H in G, each coset represented by $\tilde{w}_i(\mathbf{x})$, with $i \in \{0, \ldots, 5^*\}$. For any three different elements k, l and m from $\{0, \ldots, 5^*\}$ there is a three term identity between $\tilde{w}_k(\mathbf{x})$, $\tilde{w}_l(\mathbf{x})$ and $\tilde{w}_m(\mathbf{x})$. If $\{k, l, m\} \cap \{k^*, l^*, m^*\} = \emptyset$, the identity is given by*

$$\alpha_{k,l,m^*}(\mathbf{x})\tilde{w}_k(\mathbf{x}) + \beta_{k,l,m^*}(\mathbf{x})\tilde{w}_l(\mathbf{x}) + \gamma_{k,l,m^*}(\mathbf{x})\tilde{w}_m(\mathbf{x}) = 0;$$

otherwise it is given by (31) or (32).

Remark. The three term identities between very-well poised ${}_8\phi_7$-series are q-analogues of three term identities between well-poised ${}_7F_6$-series of unit argument. One of them is e.g. mentioned in [10, (4.3.7.8)], and with the realization given in Lemma 3 one sees that this identity connects the ordinary equivalents of the series $\tilde{w}_0(\mathbf{x})$, $\tilde{w}_{2^*}(\mathbf{x})$ and $\tilde{w}_{1^*}(\mathbf{x})$. Although the realization [10, (4.3.7.9)] is very similar to the one given in Lemma 3 this is not mentioned explicitly. Also, the underlying group is not mentioned nor identified explicitly.

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