A CHARACTER FORMULA FOR SINGLY ATYPICAL MODULES OF THE LIE SUPERALGEBRA sl(m/n)

by
J. Van der Jeugt [a,b] *, *J.W.B. Hughes* [c], *R.C. King* [a]
and J. Thierry-Mieg [d]

a) Faculty of Mathematical Studies, University of Southampton, Southampton SO9 5NH, U.K.
b) Laboratorium voor Numerieke Wiskunde en Informatica, Rijksuniversiteit Gent, Krijgslaan 281-S9, 9000 Gent, BELGIUM
c) School of Mathematical Sciences, Queen Mary College, Mile End Road, London E1 4NS, U.K.
d) Groupe d'Astrophysique Relativiste, CNRS, Observatoire de Paris Meudon, F-92195 Meudon, FRANCE

1. Introduction

Lie superalgebras, originating from physics [3], are $\mathbb{Z}_2$-graded algebras ($\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$) with a bracket operation which is "supersymmetric" (equation 2.1b in this paper) and which satisfies the "super Jacobi identity" (equation 2.1c). A classification of the finite dimensional simple Lie superalgebras over $\mathbb{C}$ was given over a decade ago by Kac [8, 9, 10, 13]. A subclass of these, closely

* Aangesteld Navorser N.F.W.O. (National Fund for Scientific Research of Belgium)

analogous to the finite dimensional Lie algebras over $\mathbb{C}$, is the class of so-called basic classical Lie superalgebras [9].

The problem of classifying the finite dimensional simple modules of the basic classical Lie superalgebras has also been considered by Kac [9, 11]. He showed that, as in the case of finite dimensional simple modules of the semi-simple Lie algebras, they are characterised up to equivalence by a highest weight Λ. The weight structure of a simple module $V(\Lambda)$ with highest weight Λ of such a Lie superalgebra G is determined by its character $\mathrm{ch}V(\Lambda)$. For a subclass of these simple modules, known as "typical" modules, Kac was able to derive a character formula closely analogous to the Weyl character formula for simple modules of simple Lie algebras. The problem of obtaining character formulae for the remaining "atypical" modules has been the subject of intense investigation but is still not solved other than in various special cases. In this paper, we solve this problem for the singly atypical modules of the Lie superalgebra $G = sl(m/n)$, where $sl(m/n)$ $(m, n \in \mathbb{N})$ is the special linear Lie superalgebra analogous to the special linear Lie algebra $sl(m)$.

We consider the indecomposable G modules $\overline{V}(\Lambda)$, introduced by Kac [11], which we refer to as Kac-modules. $\overline{V}(\Lambda)$ is well-defined for every integral dominant weight Λ and has the important property that every finite dimensional simple G module $V(\Lambda)$ is isomorphic to a quotient module of the form $\overline{V}(\Lambda)/M(\Lambda)$, where $M(\Lambda)$ is the unique maximal submodule of $\overline{V}(\Lambda)$. The character of $\overline{V}(\Lambda)$ is easy to determine, and has been given by Kac (equation 3.17 in this paper):

$$\mathrm{ch}\overline{V}(\Lambda) = \frac{\prod\limits_{\beta\in\Delta_1^+} (e^{\beta/2} + e^{-\beta/2})}{\prod\limits_{\alpha\in\Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2})} \sum_{w\in W} \varepsilon(w) e^{w(\Lambda+\rho)}, \tag{1.1}$$

where Δ_0^+ (Δ_1^+) is the set of even (resp. odd) positive roots of G, W is the Weyl group (defined to be the Weyl group of the even subalgebra of G), $\varepsilon(w)$ is the signature of $w \in W$, and $\rho = \rho_0 - \rho_1$ where ρ_0 (resp. ρ_1) is half the sum of all even (resp. odd) positive roots of G. The integral dominant weight Λ and the module $\overline{V}(\Lambda)$ are called typical if $\langle \Lambda + \rho \mid \beta \rangle \neq 0$ for all $\beta \in \Delta_1^+$, where $\langle\ \mid\ \rangle$ is a non-degenerate bilinear form [11, 12]. In this case, Kac showed that $M(\Lambda) = \{0\}$, and so (1.1) gives the character of the *simple* G module $V(\Lambda) = \overline{V}(\Lambda)$ [11].

If $\langle \Lambda+\rho \mid \beta \rangle = 0$ for some $\beta \in \Delta_1^+$, then $M(\Lambda) \neq \{0\}$ and so $\overline{V}(\Lambda) \neq V(\Lambda)$. In this case Λ, $\overline{V}(\Lambda)$ and $V(\Lambda)$ are called *atypical*; in particular if there is a unique $\gamma \in \Delta_1^+$ such that $\langle \Lambda + \rho \mid \gamma \rangle = 0$, then Λ, $\overline{V}(\Lambda)$ and $V(\Lambda)$ are said to be *singly atypical* of type γ, and γ is called the corresponding atypical root. In this paper we give a unique characterisation of $M(\Lambda)$ for the singly atypical case (Theorem 4.3): we show that $M(\Lambda)$ is itself a simple (singly atypical) G module. Using this theorem, we are then able to derive a character formula for $V(\Lambda)$, first for the case where the atypical root is the unique odd simple root α_m of G (Theorem 5.3). We then proceed to prove various properties relating a weight Λ, singly atypical of type γ, to a weight that is singly atypical of type α_m. Using these properties, we then derive a character formula for all singly atypical simple modules of $sl(m/n)$ in Section 7 (Theorem 7.2). Finally, we make some comments relating to *multiply* atypical modules.

We conclude this introduction by mentioning that characters of some atypical $sl(m/n)$ modules have been obtained by Berele and Regev [1] and Serge'ev [15]. Using Schur's method, they show that the tensor product $V^{\otimes N}$, where V is the natural $(m+n)$-dimensional module of $sl(m/n)$, is completely reducible. The irreducible components are the (simple) *covariant tensor modules*, the characters of which can be expressed in terms of Schur functions [1, 15]. These covariant tensor modules can be typical, singly atypical or even multiply atypical, but they do not by any means exhaust any of these categories of modules. Various formulae and conjectures have been published in order to accommodate the characters of all atypical simple $sl(m/n)$ modules [2, 5], but counterexamples to all formulae proposed so far have been found [17]. Realizing the failure of all these proposals, a new conjecture has been given in Ref. 17, to which no counterexamples are known. This is described briefly in Section 8.

2. The Lie superalgebra sl(m/n)

A complex Lie superalgebra G is a $\mathbb{Z}_2$-graded linear vector space, $G = G_{\bar{0}} \oplus G_{\bar{1}}$ over $\mathbb{C}$ with a bracket $[\ ,\ ]$ such that $\forall a \in G_\alpha$, $\forall b \in G_\beta$ and $\forall \alpha, \beta \in \mathbb{Z}_2$ [9, 13]

$$[a, b] \in G_{\alpha+\beta}, \tag{2.1a}$$

$$[a, b] = -(-1)^{\alpha\beta}[b, a], \tag{2.1b}$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]]. \tag{2.1c}$$

Note that the *even* part $G_{\bar{0}}$ is a complex Lie algebra, and that the *odd* part $G_{\bar{1}}$ is a $G_{\bar{0}}$ module under the adjoint action. The simplest example of a Lie superalgebra is given by $gl(m/n)$ with $m, n \in \mathbb{N}$. Its natural matrix realisation takes the form:

$$gl(m/n) = \Big\{x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} |A \in M_{m\times m}, B \in M_{m\times n}, \\ C \in M_{n\times m}, D \in M_{n\times n}\Big\}, \tag{2.2}$$

where $M_{p\times q}$ is the space of all $p \times q$ complex matrices. The "even" subspace $gl(m/n)_{\bar{0}}$ has $B = 0$ and $C = 0$; the "odd" subspace $gl(m/n)_{\bar{1}}$ has $A = 0$ and $D = 0$. In the case of $G = gl(m/n)$, the bracket is determined in the natural matrix representation by

$$[a, b] = ab - (-1)^{\alpha\beta} ba, \qquad \forall a \in G_\alpha \text{ and } \forall b \in G_\beta. \tag{2.3}$$

We denote by $gl(m/n)_{+1}$ the space of matrices $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ and by $gl(m/n)_{-1}$ the space of matrices $\left(\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right)$. Then $G = gl(m/n)$ has a $\mathbb{Z}$-grading which is consistent with the $\mathbb{Z}_2$-grading [13]:

$$G = G_{-1} \oplus G_0 \oplus G_{+1}, \qquad G_{\bar{0}} = G_0 \text{ and } G_{\bar{1}} = G_{-1} \oplus G_{+1}. \tag{2.4}$$

Note that $gl(m/n)_{\bar{0}} = gl(m) \oplus gl(n)$. With the definition of *supertrace* [9] as $\mathrm{str}(x) = \mathrm{tr}(A) - \mathrm{tr}(D)$ one can define the subalgebra $sl(m/n)$:

$$sl(m/n) = \{x \in gl(m/n) \mid \mathrm{str}(x) = 0\}. \tag{2.5}$$

If $m \neq n$ then $sl(m/n)$ is a *simple* Lie superalgebra [9, 13]. If $m = n$ it contains a one-dimensional ideal $\mathbb{C}I_{2m}$ and then $sl(m/m)/\mathbb{C}I_{2m}$ is simple. In what follows we put $G = sl(m/n)$. Note that $sl(m/n)_{\bar{0}} = sl(m) \oplus \mathbb{C} \oplus sl(n)$ is a reductive Lie algebra, the simple modules of which are well known.

A Cartan subalgebra H of G has dimension $m + n - 1$ and is spanned by

$$h_i = E_{ii} - E_{i+1,i+1} \ \ (1 \leq i \leq m-1 \text{ or } m+1 \leq i \leq m+n-1), \\ h_m = E_{mm} + E_{m+1,m+1}, \tag{2.6}$$

where E_{ij} is the matrix with entry 1 at position (i, j) and 0 elsewhere. The dual space H^* is described in the basis of forms ϵ_i ($i = 1, 2, \ldots, m$) and δ_j ($j = 1, 2, \ldots, n$), where $\epsilon_i: x \to A_{ii}$ and $\delta_j: x \to D_{jj}$ for $x = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, and $\sum_{i=1}^{m} \epsilon_i + \sum_{j=1}^{n} \delta_j = 0$. The roots and corresponding root vectors of $sl(m/n)$ are given by [9]

$$\begin{aligned} &\epsilon_i - \epsilon_j \ \leftrightarrow \ E_{ij} && (1 \leq i, j \leq m) \text{ (even)}, \\ &\delta_i - \delta_j \ \leftrightarrow \ E_{m+i,m+j} && (1 \leq i, j \leq n) \text{ (even)}, \\ &\epsilon_i - \delta_j \ \leftrightarrow \ E_{i,m+j} && (1 \leq i \leq m,\ 1 \leq j \leq n) \text{ (odd)}, \\ &\delta_i - \epsilon_j \ \leftrightarrow \ E_{m+i,j} && (1 \leq i \leq n,\ 1 \leq j \leq m) \text{ (odd)}. \end{aligned} \tag{2.7}$$

Denote by Δ the set of all roots, by Δ_0 the set of even roots, by Δ_1 the set of odd roots, and by $e(\alpha)$ the root vector (2.7) corresponding to the root $\alpha \in \Delta$. G has the root space decomposition

$$G = H \oplus \left(\bigoplus_{\alpha \in \Delta} \mathbb{C} e(\alpha) \right). \tag{2.8}$$

A set of simple roots of Δ may be chosen as follows:

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1}\ (1 \leq i \leq m-1),\ \alpha_m = \epsilon_m + \delta_1, \\ \alpha_{m+j} &= \delta_j - \delta_{j+1}\ (1 \leq j \leq n-1); \end{aligned} \tag{2.9}$$

this choice is often referred to as the "distinguished basis", for which there is only one odd simple root α_m [11]. With this distinguished choice, the elements of H^* are partially ordered by

$$\lambda, \mu \in H^* : \quad \lambda \geq \mu \quad \Leftrightarrow \quad \lambda - \mu = \sum_{i=1}^{m+n-1} k_i \alpha_i \text{ with } k_i \geq 0. \tag{2.10}$$

This partial ordering $\geq$ can be extended to a total ordering $\succeq$ compatible with $\geq$, i.e.

$$\lambda \geq \mu \quad \Rightarrow \quad \lambda \succeq \mu; \tag{2.11}$$

the most natural example of such a total ordering is lexicographical ordering with respect to the simple roots. The even and odd positive roots of $sl(m/n)$ are given by

$$\begin{aligned} \Delta_0^+ &= \{\epsilon_i - \epsilon_j\ (i < j);\ \delta_i - \delta_j\ (i < j)\}, \\ \Delta_1^+ &= \{\epsilon_i - \delta_j\}. \end{aligned} \tag{2.12}$$

It will be convenient to denote the mn odd positive roots by

$$\beta_{ij} = \epsilon_i - \delta_j \qquad 1 \leq i \leq m,\ 1 \leq j \leq n. \tag{2.13}$$

The invariant non-degenerate inner product on G is given by $\langle x|y \rangle = \text{str}(xy)$. The restriction of this to H is also non-degenerate and the pairing of H and H^* then defines a non-degenerate inner product $\langle\ |\ \rangle$ on H^*, explicitly determined by

$$\langle \epsilon_i | \epsilon_j \rangle = \delta_{ij},\ \langle \epsilon_i | \delta_j \rangle = 0,\ \langle \delta_i | \delta_j \rangle = -\delta_{ij}, \tag{2.14}$$

where δ_{ij} is the Kronecker-δ. An element $\Lambda \in H^*$ with $\Lambda = \sum_i \lambda_i \epsilon_i + \sum_j \mu_j \delta_j$ can be written in terms of its *components* in the $\epsilon\delta$-basis as $\Lambda =$

$(\lambda_1\lambda_2\ldots\lambda_m|\mu_1\mu_2\ldots\mu_n)$ with $\sum_i \lambda_i + \sum_j \mu_j = 0$, or in terms of its *Dynkin labels* $\Lambda = [a_1, \ldots, a_{m-1}; a_m; a_{m+1}, \ldots, a_{m+n-1}]$ where $a_i = \Lambda(h_i)$ and h_i is given in (2.6). We call a_i with $i \neq m$ an *even* Dynkin label and a_m the *odd* Dynkin label.

The *Weyl group* W of G is defined to be the Weyl group of $G_{\bar{0}}$ [9]. Hence $W = S_m \times S_n$, the direct product of the Weyl groups of $sl(m)$ and $sl(n)$. For $w = \sigma \times \tau \in W = S_m \times S_n$, the signature $\varepsilon(w)$ is the product of the signatures of σ and τ. We denote by w_0 the Coxeter element of W, i.e. $w_0 = \omega_m \times \omega_n$, where ω_m (resp. ω_n) is the element of maximal length in S_m (resp. S_n). The *dot action* is defined as usual:

$$w \cdot \Lambda = w(\Lambda + \rho) - \rho, \text{ where } \rho = \rho_0 - \rho_1 \tag{2.15}$$

with

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha \quad \text{and} \quad \rho_1 = \frac{1}{2} \sum_{\beta \in \Delta_1^+} \beta. \tag{2.16}$$

Explicitly,

$$\begin{aligned} \rho_0 &= \frac{1}{2} \sum_{i=1}^{m} (m - 2i + 1)\epsilon_i + \frac{1}{2} \sum_{j=1}^{n} (n - 2j + 1)\delta_j \\ \rho_1 &= \frac{n}{2} \sum_{i=1}^{m} \epsilon_i - \frac{m}{2} \sum_{j=1}^{n} \delta_j. \end{aligned} \tag{2.17}$$

Note that Δ_1^+, given in the distinguished basis by (2.12), is W-invariant. It follows from (2.16) that $w\rho_1 = \rho_1$ for all $w \in W$ (a property which can also be seen from the explicit form (2.17) for ρ_1), and hence

$$w \cdot \Lambda = w(\Lambda + \rho) - \rho = w(\Lambda + \rho_0) - \rho_0. \tag{2.18}$$

We set

$$\begin{aligned} N_0^{\pm} &= \text{span}\{e(\alpha) | \alpha \in \Delta_0^{\pm}\}, \\ N_1^{\pm} &= \text{span}\{e(\beta) | \beta \in \Delta_1^{\pm}\}, \\ N^{\pm} &= N_0^{\pm} \oplus N_1^{\pm}. \end{aligned} \tag{2.19}$$

Note that $N_1^{\pm} = G_{\pm 1}$ and, besides the decomposition (2.4), one has

$$\begin{aligned} G_{\bar{0}} &= N_0^- \oplus H \oplus N_0^+, \\ G &= N^- \oplus H \oplus N^+. \end{aligned} \tag{2.20}$$

Let $U(G)$ be the universal enveloping algebra of G, and $U(G')$ the enveloping algebra of any one of the subalgebras $G' = H, G_0, N^{\pm}, N_0^{\pm}, N_1^{\pm}$. The Poincaré-Birkhoff-Witt theorem for Lie algebras can be extended to the case of Lie superalgebras [9, 13]:

Theorem 2.1. *Let* $x_1, \ldots, x_M$ *be a basis of* $G_{\bar{0}}$ *and* $y_1, \ldots, y_N$ *be a basis of* $G_{\bar{1}}$. *Then the elements of the form*

$$(x_1)^{k_1} \ldots (x_M)^{k_M} y_{i_1} \ldots y_{i_s}, \text{ where } k_i \geq 0 \text{ and } 1 \leq i_1 < \ldots < i_s \leq N, \quad (2.21)$$

form a basis of $U(G)$.

A similar theorem is true for each $U(G')$ with G' one of the subalgebras given previously. Therefore $U(G')$ is H-diagonalisable and we can denote by $U(G')_\eta$ the subspace of all elements of $U(G')$ of weight η with respect to H.

Denote by σ the involutive antiautomorphism of G defined by the relations [11]

$$\begin{aligned} &\sigma(h) = h, \qquad \forall h \in H, \\ &\sigma(e(\alpha)) = e(-\alpha), \qquad \forall \alpha \in \Delta, \end{aligned} \quad (2.22)$$

where $e(\alpha)$ is the root vector corresponding to α. This antiautomorphism can be extended to $U(G)$ by $\sigma(xy) = \sigma(y)\sigma(x)$, for $x, y \in U(G)$.

3. The Kac-module

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded linear vector space over $\mathbb{C}$, and denote by $gl(V)$ the space of endomorphisms of V. Then $gl(V)$ is naturally $\mathbb{Z}_2$-graded: $gl(V) = gl(V)_{\bar{0}} \oplus gl(V)_{\bar{1}}$. A representation ϕ is a linear mapping from G to $gl(V)$ such that $\forall \alpha, \beta \in \{\bar{0}, \bar{1}\}$:

$$\begin{aligned} &\phi : x \to \phi(x) \text{ with } \phi(x) \in gl(V)_\alpha \text{ for } x \in G_\alpha, \\ &\phi([x, y]) = \phi(x)\phi(y) - (-1)^{\alpha\beta}\phi(y)\phi(x) \qquad \forall x \in G_\alpha \text{ and } \forall y \in G_\beta. \end{aligned} \quad (3.1)$$

Then V is a G module with $xv = \phi(x)v$ for $x \in G$ and $v \in V$.

Definition 3.1. V *is called a highest weight module for* G *(resp. for* $G_{\bar{0}}$*) with highest weight* $\Lambda \in H^*$ *if there exists a non-zero vector* $v_\Lambda \in V$ *such that*

$$\begin{aligned} &N^+ v_\Lambda = 0 \quad (\textit{resp. } N_0^+ v_\Lambda = 0), \\ &h v_\Lambda = \Lambda(h) v_\Lambda \qquad \forall h \in H, \\ &U(G) v_\Lambda = V \quad (\textit{resp. } U(G_{\bar{0}}) v_\Lambda = V). \end{aligned} \quad (3.2)$$

Then v_Λ *is called a* G *(resp.* $G_{\bar{0}}$*) highest weight vector.*

Highest weight modules are H-diagonalizable,

$$V = \bigoplus_{\lambda \leq \Lambda} V_\lambda, \text{ with } V_\lambda = \{v \in V \mid hv = \lambda(h)v, \quad \forall h \in H\}, \quad (3.3)$$

and so are all submodules or subquotients of highest weight modules.

Definition 3.2. *Let V be a G highest weight module with highest weight vector v_Λ. We call $v \in V$ a generating vector if and only if $V = U(G)v$ or, equivalently, if and only if $v_\Lambda \in U(G)v$.*

Definition 3.3. *Let V be a G module. A vector $v \in V$ is called a weakly primitive vector if there exists a G module $U \subset V$ such that $v \notin U$ and $N^+v \subseteq U$.*

If $U = \{0\}$ in Definition 3.3, the vector v is primitive :

Definition 3.4. *Let V be a G module (resp. a $G_{\bar{0}}$ module). A vector $v \in V$ is called a G primitive vector (resp. a $G_{\bar{0}}$ primitive vector) if $N^+v = 0$ (resp. $N_0^+v = 0$).*

A weight $\Lambda \in H^*$ is called *dominant* if $a_i = \Lambda(h_i) \geq 0$ for all $i \neq m$, *integral* if $a_i \in \mathbb{Z}$ for all $i \neq m$, and *integral dominant* if $a_i \in \mathbb{N}$ for all $i \neq m$. From the theory of reductive Lie algebras it follows that for every integral dominant weight Λ there exists a unique (up to isomorphism) finite dimensional simple $G_{\bar{0}}$ module $V_0(\Lambda)$ with highest weight Λ. Let v_Λ be a highest weight vector for $V_0(\Lambda)$. The G_0 module $V_0(\Lambda)$ can be extended to a $G_0 \oplus G_{+1}$ module by putting $G_{+1}V_0(\Lambda) = 0$. In this paper we shall make extensive use of the following G module, first defined by Kac [11]:

Definition 3.5. *For an integral dominant $\Lambda \in H^*$, the Kac-module $\overline{V}(\Lambda)$ is the induced module*

$$\overline{V}(\Lambda) = \mathrm{Ind}_{G_0 \oplus G_{+1}}^{G} V_0(\Lambda) = U(G) \otimes_{G_0 \oplus G_{+1}} V_0(\Lambda).$$

From Theorem 2.1 we see that $U(G) = U(G_{-1}) \otimes U(G_0) \otimes U(G_{+1})$. Therefore Definition 3.5 implies that

$$\overline{V}(\Lambda) \cong U(G_{-1}) \otimes V_0(\Lambda). \tag{3.4}$$

Since $[G_{-1}, G_{-1}] = 0$, $U(G_{-1})$ is isomorphic to $\wedge(G_{-1})$, the exterior algebra over G_{-1}. The dimension of G_{-1} is mn, thus $\dim(\wedge(G_{-1})) = 2^{mn}$, and hence $\overline{V}(\Lambda)$ is a finite dimensional G-module of dimension $2^{mn}\dim(V_0(\Lambda))$. It follows from the definition that $\overline{V}(\Lambda)$ is a G highest weight module. Unfortunately $\overline{V}(\Lambda)$ is not always a simple G module. Since $\overline{V}(\Lambda)$ is a G highest weight module, it contains a unique maximal submodule $M(\Lambda)$:

$$M(\Lambda) = \{v \in \overline{V}(\Lambda) \mid v_\Lambda \notin U(G)v\}, \tag{3.5}$$

such that the quotient module

$$V(\Lambda) = \overline{V}(\Lambda)/M(\Lambda) \tag{3.6}$$

is a finite dimensional simple G module with highest weight Λ. Kac proved the following theorem [11]:

Theorem 3.6 [Kac]. *Every finite dimensional simple G module is isomorphic to a module of type (3.6), where Λ is integral dominant. Moreover, every finite dimensional simple G module is uniquely characterized by its integral dominant highest weight Λ.*

Let T_+ and T_- be the following elements in $U(G)$:

$$T_\pm = \prod_{\beta \in \Delta_1^+} e(\pm\beta), \tag{3.7}$$

where the β's in the product (3.7) (and in all subsequent products of $e(\beta)$'s) appear in the chosen lexicographical ordering (note that a different ordering can only lead to a sign change). One can verify that

$$[e(\alpha), T_\pm] = 0, \qquad \forall \alpha \in \Delta_0. \tag{3.8}$$

In $\overline{V}(\Lambda)$, let

$$v_{\Lambda_-} = T_- v_\Lambda, \text{ where } \Lambda_- = \Lambda - 2\rho_1. \tag{3.9}$$

Note that (2.17) implies that Λ_- is also integral dominant; in fact if a_i are the Dynkin labels of Λ, then $[a_1, \ldots, a_{m-1}; a_m + m - n; a_{m+1}, \ldots, a_{m+n-1}]$ are the Dynkin labels of Λ_-. Since $G_{\bar{0}} \subset G$, $\overline{V}(\Lambda)$ is also a $G_{\bar{0}}$ module. It follows from (3.8) that the $G_{\bar{0}}$ module $\overline{V}(\Lambda)$ contains $T_- V_0(\Lambda)$ as a simple $G_{\bar{0}}$ submodule, with highest weight vector v_{Λ_-}. This submodule contains a unique (up to scalar multiplication) lowest weight vector v_- of weight $w_0\Lambda_-$. From (3.4) it follows that v_- is the unique (again, up to scalar multiplication) vector of $\overline{V}(\Lambda)$ annihilated by N^-.

Lemma 3.7. *$\overline{V}(\Lambda)$ is an indecomposable G module, indeed every non-zero G submodule Y of the G module $\overline{V}(\Lambda)$ contains the $G_{\bar{0}}$ module $T_- V_0(\Lambda)$ as a subspace.*

This follows from the fact that every submodule of $\overline{V}(\Lambda)$ contains the vector v_- that is annihilated by N^-.

The following lemma appears in the work of Gould [4]:

Lemma 3.8. *Let $X(\Lambda) = U(G)v_{\Lambda_-}$. Then $X(\Lambda)$ is a simple G submodule of $\overline{V}(\Lambda)$, and every non-zero submodule of $\overline{V}(\Lambda)$ contains $X(\Lambda)$.*

Indeed, $X(\Lambda)$ is by definition a submodule of $\overline{V}(\Lambda)$. Using Lemma 3.7, every non-zero submodule Y of $\overline{V}(\Lambda)$ contains v_{Λ_-}, and hence contains $X(\Lambda)$. This also implies that $X(\Lambda)$ has no proper submodules, so $X(\Lambda)$ is simple.

Lemma 3.9. *$\overline{V}(\Lambda)$ is a simple G module if and only if $T_+T_-v_\Lambda \neq 0$.*

Proof. The elements in $U(G)v_{\Lambda_-}$ of weight Λ must be multiples of $T_+v_{\Lambda_-}$. If $T_+T_-v_\Lambda = 0$, then it follows that $v_\Lambda \notin X(\Lambda)$, so $X(\Lambda)$ is then a proper non-zero submodule of $\overline{V}(\Lambda)$, so $\overline{V}(\Lambda)$ is not simple. Conversely, if $\overline{V}(\Lambda)$ were not simple, then $v_\Lambda \notin M(\Lambda)$. But according to Lemma 3.7 $T_-v_\Lambda \in M(\Lambda)$, hence $T_+T_-v_\Lambda \in M(\Lambda)$, and since $T_+T_-v_\Lambda$ is of weight Λ we conclude $T_+T_-v_\Lambda \propto v_\Lambda$. Thus $T_+T_-v_\Lambda = 0$. ∎

Lemma 3.10. *Let $Q(\Lambda)$ be the expression*

$$Q(\Lambda) = \prod_{\beta\in\Delta_1^+} \langle \Lambda + \rho \mid \beta \rangle. \tag{3.10}$$

Then

$$T_+T_-v_\Lambda = \pm Q(\Lambda) v_\Lambda. \tag{3.11}$$

For a proof, see Kac [11, 12]; whether the sign in (3.11) is $+$ or $-$ depends upon the ordering of the $e(\beta)$'s in (3.7) and is unimportant here.

Definition 3.11. *Let Λ be an integral dominant weight. We call Λ (resp. $\overline{V}(\Lambda)$, resp. $V(\Lambda)$) a typical weight (resp. a typical Kac-module, resp. a typical simple module) if and only if $\langle \Lambda + \rho \mid \beta \rangle \neq 0$ for all $\beta \in \Delta_1^+$. If there exists a $\beta \in \Delta_1^+$ such that $\langle \Lambda + \rho \mid \beta \rangle = 0$ then Λ, $\overline{V}(\Lambda)$ and $V(\Lambda)$ are called atypical, and β is called an atypical root for Λ. If there exists just one atypical root β for Λ, we call Λ, $\overline{V}(\Lambda)$ and $V(\Lambda)$ singly atypical of type β.*

The following theorem now follows from Lemmas 3.9 and 3.10 [11, 12]:

Theorem 3.12. *The Kac-module $\overline{V}(\Lambda)$ is a simple G module if and only if Λ is typical.*

The character ch$V$ of a G module V with weight space decomposition (3.3) is defined as

$$\mathrm{ch}V = \sum_{\lambda\in H^*} \dim(V_\lambda) e^\lambda, \tag{3.12}$$

where e^{λ} is the formal exponential. The action of the Weyl group W on such formal exponentials is defined by $w(e^{\lambda}) = e^{w\lambda}$. Let

$$L_0 = \prod_{\alpha\in\Delta_0^+}\left(e^{\alpha/2} - e^{-\alpha/2}\right) \text{ and } L_1 = \prod_{\beta\in\Delta_1^+}\left(e^{\beta/2} + e^{-\beta/2}\right). \tag{3.13}$$

From (3.4) it follows that the Kac-module has character

$$\mathrm{ch}\overline{V}(\Lambda) = \prod_{\beta\in\Delta_1^+}\left(1 + e^{-\beta}\right)\mathrm{ch}V_0(\Lambda), \tag{3.14}$$

where $\mathrm{ch}V_0(\Lambda)$ is given by Weyl's character formula [18]:

$$\mathrm{ch}V_0(\Lambda) = L_0^{-1}\sum_{w\in W}\varepsilon(w)e^{w(\Lambda+\rho_0)}. \tag{3.15}$$

Using the Weyl invariance of ρ_1, we have

$$\prod_{\beta\in\Delta_1^+}\left(1 + e^{-\beta}\right) = L_1 e^{-\rho_1} = L_1 e^{-w\rho_1}, \quad \forall w \in W, \tag{3.16}$$

and hence we obtain Kac's character formula [11, 12]:

$$\mathrm{ch}\overline{V}(\Lambda) = \frac{L_1}{L_0}\sum_{w\in W}\varepsilon(w)e^{w(\Lambda+\rho)}. \tag{3.17}$$

Due to the Weyl invariance of Δ_1^+ and of L_1, (3.17) can be rewritten in the form

$$\mathrm{ch}\overline{V}(\Lambda) = L_0^{-1}\sum_{w\in W}\varepsilon(w)w\left\{e^{\Lambda+\rho_0}\prod_{\beta\in\Delta_1^+}\left(1 + e^{-\beta}\right)\right\}. \tag{3.18}$$

Using Theorem 3.12, (3.18) gives the character of all typical simple modules of G. The problem of finding the characters of atypical simple G modules is unsolved so far. In this paper we shall deduce a character formula for singly atypical simple G modules.

Finally, let $\lambda \in H^*$ be integral. We define the "formal characters":

$$\chi_K(\lambda) = L_0^{-1}\sum_{w\in W}\varepsilon(w)w\left\{e^{\lambda+\rho_0}\prod_{\beta\in\Delta_1^+}\left(1 + e^{-\beta}\right)\right\}; \tag{3.19}$$

$$\chi_W(\lambda) = L_0^{-1}\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho_0)}. \tag{3.20}$$

If λ is integral dominant the expressions (3.19) and (3.20) coincide with Kac's character $\mathrm{ch}\overline{V}(\lambda)$ and Weyl's character $\mathrm{ch}V_0(\lambda)$ respectively. It is easy to verify that the formal characters satisfy the following properties:

$$\chi_K(\lambda) = e^{-\rho_1}L_1\,\chi_W(\lambda); \tag{3.21}$$

$$\chi_W(w\cdot\lambda) = \varepsilon(w)\chi_W(\lambda), \quad \text{and} \quad \chi_K(w\cdot\lambda) = \varepsilon(w)\chi_K(\lambda), \quad \forall w \in W. \tag{3.22}$$

4. The maximal submodule of the Kac-module

Let $\Lambda \in H^*$ be an integral dominant weight. In this section, we shall consider the even Dynkin labels $a_i = \Lambda(h_i)$ $(i \neq m)$ of Λ as fixed integers, and the odd Dynkin label a_m as a complex variable. Let $V_0(\Lambda)$ be the (finite dimensional) simple $G_{\bar{0}}$ module with highest weight Λ and highest weight vector v_Λ. $V_0(\Lambda)$ has the following weight space decomposition:

$$V_0(\Lambda) = \bigoplus_{\lambda} V_0(\Lambda)_\lambda. \tag{4.1}$$

Let $P_0(\Lambda)$ be the set of weights λ for which $V_0(\Lambda)_\lambda \neq \{0\}$, and denote by $m_0(\Lambda,\lambda)$ the dimension of $V_0(\Lambda)_\lambda$. The following lemma is a well known property of simple modules of semi-simple Lie algebras, and it is applicable here in the case of $G_{\bar{0}} = sl(m) \oplus \mathbb{C} \oplus sl(n)$.

Lemma 4.1. *For $\lambda \in P_0(\Lambda)$ there exists a set of elements $g_i(\lambda) \in U(N_0^-)_{\lambda-\Lambda}$, $(i = 1, 2, \ldots, m_0(\Lambda,\lambda))$, such that $\{g_i(\lambda)v_\Lambda\}$ forms a basis for $V_0(\Lambda)_\lambda$, and moreover, such that*

$$\sigma(g_i(\lambda))g_j(\lambda)v_\Lambda = \delta_{ij}v_\Lambda, \tag{4.2}$$

where σ is the antiautomorphism (2.22), and δ_{ij} is the usual Kronecker symbol.

Proof. From the results concerning the symmetric bilinear contravariant form associated with σ (see [7]), it follows that there exists a set of monomials $z_i(\lambda) \in U(N_0^-)_{\lambda-\Lambda}$ $(i = 1, \ldots, m_0(\Lambda,\lambda))$ (i.e. every $z_i(\lambda)$ is of the form $\prod_{\alpha\in\Delta_0^+}(e(-\alpha))^{k_\alpha}$ with $\sum_\alpha k_\alpha\alpha = \Lambda - \lambda$) such that $z_i(\lambda)v_\Lambda$ forms a basis for $V_0(\Lambda)_\lambda$ and such that the matrix Z of elements Z_{ij} in $\sigma(z_i(\lambda))z_j(\lambda)v_\Lambda = Z_{ij}v_\Lambda$ is non-singular [the elements Z_{ij} depend only upon the even Dynkin labels, and hence are numbers independent of a_m]. From the properties of σ and the real basis (2.7), it follows that Z is a real symmetric matrix. Diagonalising Z then also gives rise to a new basis $g_i'(\lambda)v_\Lambda$ with every $g_i'(\lambda)$ a linear combination of the $z_j(\lambda)$, and such that the matrix Z' of coefficients Z'_{ij} in $\sigma(g_i'(\lambda))g_j'(\lambda)v_\Lambda = Z'_{ij}v_\Lambda$ is diagonal with real non-zero entries. Rescaling the $g_i'(\lambda)$ gives $g_i(\lambda)$. ∎

Consider the weight space decomposition (3.3) for the Kac-module $\overline{V}(\Lambda)$,

$$\overline{V}(\Lambda) = \bigoplus_{\mu} \overline{V}(\Lambda)_\mu, \tag{4.3}$$

and let $P(\Lambda)$ be the set of all weights μ such that $\overline{V}(\Lambda)_\mu \neq \{0\}$. Let $\mathbf{k}$ be a sequence of numbers k_β $(\beta \in \Delta_1^+)$ such that every $k_\beta \in \{0,1\}$. For $\mu \in P(\Lambda)$, consider all partitions $(\mathbf{k}, \lambda)$ of μ of the form

$$\mu = \lambda - \sum_{\beta\in\Delta_1^+} k_\beta \beta \tag{4.4}$$

with $\lambda \in P_0(\Lambda)$. Then it follows from (3.4) that the dimension of $\overline{V}(\Lambda)_\mu$, $m(\Lambda, \mu)$, is given by

$$m(\Lambda,\mu) = \sum_{(\mathbf{k},\lambda)} m_0(\Lambda,\lambda), \tag{4.5}$$

where the summation in (4.5) is over all partitions $(\mathbf{k}, \lambda)$ of μ of the form (4.4). Moreover, it is easy to give a basis for $\overline{V}(\Lambda)_\mu$, namely

$$\prod_{\beta\in\Delta_1^+} e(-\beta)^{k_\beta} g_i(\lambda) v_\Lambda, \quad \text{with } (\mathbf{k},\lambda) \text{ a partition of type (4.4)} \\ \text{and } i = 1, 2, \ldots, m_0(\Lambda,\lambda). \tag{4.6}$$

We let

$$x_{\mathbf{k},i} = \prod_{\beta\in\Delta_1^+} e(-\beta)^{k_\beta} g_i(\lambda) \quad \in \quad U(N_1^-)U(N_0^-). \tag{4.7}$$

For convenience, we have dropped the dependence of $x_{\mathbf{k},i}$ upon Λ and λ in the notation. Denote by $\tilde{\mathbf{k}}$ the sequence complementary to $\mathbf{k}$, consisting of numbers $\tilde{k}_\beta = 1 - k_\beta$. Associated with $x_{\mathbf{k},i}$, we define

$$\tilde{x}_{\mathbf{k},i} = \sigma(g_i(\lambda)) \prod_{\beta\in\Delta_1^+} e(-\beta)^{\tilde{k}_\beta}. \tag{4.8}$$

Using $e(\beta)^2 = 0$ for $\beta \in \Delta_1$, (3.8), and (4.2), one obtains the following properties:

$$\tilde{x}_{\mathbf{k}',i'} x_{\mathbf{k},i} v_\Lambda = \delta_{\mathbf{k}'\mathbf{k}} \delta_{i'i} v_{\Lambda_-}, \tag{4.9a}$$

$$\sigma(x_{\mathbf{k},i})\sigma(\tilde{x}_{\mathbf{k}',i'}) v_{\Lambda_-} = \pm \delta_{\mathbf{k}\mathbf{k}'} \delta_{ii'} Q(\Lambda) v_\Lambda, \tag{4.9b}$$

where a $\pm$-sign appears because in general a reordering of the $e(+\beta)$'s is necessary to recover T_+ in (4.9b). Note that $Q(\Lambda)$ is considered as a polynomial of degree mn in the odd Dynkin label a_m.

Finally, let A be the matrix of size $m(\Lambda, \mu) \times m(\Lambda, \mu)$ defined by

$$\sigma(x_{\mathbf{k},i}) x_{\mathbf{k}',i'} v_\Lambda = A_{\mathbf{k}i,\mathbf{k}'i'} v_\Lambda. \tag{4.10}$$

This matrix is the Kac-module analogue of the Shapovalov matrix [14] for Verma modules of complex semi-simple Lie algebras [14, 7]. From Definition

3.2 and (4.10) it follows that the rank of A is equal to the number of linearly independent generating vectors v_μ in $\overline{V}(\Lambda)_\mu$. Hence, using (3.5) and (3.6):

$$\text{rank}(A) = \dim(V(\Lambda)_\mu). \tag{4.11}$$

Similarly, let B be the $m(\Lambda,\mu) \times m(\Lambda,\mu)$ matrix defined by

$$\tilde{x}_{\mathbf{k},i}\sigma(\tilde{x}_{\mathbf{k}',i'})v_{\Lambda_-} = B_{\mathbf{k}i,\mathbf{k}'i'}v_{\Lambda_-}. \tag{4.12}$$

Since $U(H)v_{\Lambda_-} = \mathbb{C}v_{\Lambda_-}$, $N_0^+ v_{\Lambda_-} = 0$ and $N_1^- v_{\Lambda_-} = 0$ (see equations (3.8)–(3.9)), $X(\Lambda) = U(G)v_{\Lambda_-} = U(N_1^+)U(N_0^-)v_{\Lambda_-}$. But $U(N_0^-)v_{\Lambda_-} = V_0(\Lambda_-)$ is isomorphic to $V_0(\Lambda)$ as an $sl(m)\oplus sl(n)$ module. Therefore $X(\Lambda)_\mu$ is spanned by the vectors of type $\sigma(\tilde{x}_{\mathbf{k}',i'})v_{\Lambda_-}$. Hence any maximal subset of linearly independent vectors of the set $\{\sigma(\tilde{x}_{\mathbf{k}',i'})v_{\Lambda_-}\}$ of $m(\Lambda,\mu)$ elements forms a basis for $X(\Lambda)_\mu{=}\big(U(G)v_{\Lambda_-}\big)_\mu$. It follows from (4.12) and the structure of the Kac-module that the rank of B is equal to the maximal number of linearly independent vectors of weight μ in $\overline{V}(\Lambda)_\mu$ that belong to $U(G)v_{\Lambda_-}{=}X(\Lambda)$. Thus

$$\text{rank}(B) = \dim(X(\Lambda)_\mu). \tag{4.13}$$

Now we can prove the main result of this section:

Lemma 4.2. *Let A and B be defined as in (4.10) and (4.12). Then*

$$det(A)det(B) = \pm(Q(\Lambda))^{m(\Lambda,\mu)}, \tag{4.14}$$

where $Q(\Lambda)$ is given in (3.10).

Proof. The vector $\sigma(\tilde{x}_{\mathbf{k}',i'})v_{\Lambda_-}$ is of weight μ in $\overline{V}(\Lambda)$, so it can be expressed as a linear combination of the basis vectors (4.6) of $\overline{V}(\Lambda)_\mu$. Thus

$$\sigma(\tilde{x}_{\mathbf{k}',i'})v_{\Lambda_-} = \sum_{\mathbf{k}''i''} C_{\mathbf{k}''i'',\mathbf{k}'i'}x_{\mathbf{k}'',i''}v_\Lambda, \tag{4.15}$$

where $C_{\mathbf{k}''i'',\mathbf{k}'i'}$ is the matrix of coefficients of the linear combinations. Acting on (4.15) with $\sigma(x_{\mathbf{k},i})$ and using (4.9b) yields:

$$\pm\delta_{\mathbf{k}\mathbf{k}'}\delta_{ii'}Q(\Lambda)v_\Lambda = \sum_{\mathbf{k}''i''} A_{\mathbf{k}i,\mathbf{k}''i''}C_{\mathbf{k}''i'',\mathbf{k}'i'}v_\Lambda. \tag{4.16}$$

Thus AC is a diagonal matrix, and in particular,

$$\det(A)\det(C) = \pm(Q(\Lambda))^{m(\Lambda,\mu)}. \tag{4.17}$$

Acting on (4.15) with $\tilde{x}_{\mathbf{k},i}$, and using (4.9a), yields

$$B_{\mathbf{k}i,\mathbf{k}'i'} v_{\Lambda_-} = \sum_{\mathbf{k}''i''} C_{\mathbf{k}''i'',\mathbf{k}'i'} \delta_{\mathbf{k}\mathbf{k}''} \delta_{ii''} v_{\Lambda_-}, \tag{4.18}$$

hence $B = C$, and in particular

$$\det(B) = \det(C). \tag{4.19}$$

The lemma now follows from (4.17) and (4.19). ∎

Theorem 4.3. *If Λ is singly atypical then $M(\Lambda) = X(\Lambda)$.*

Proof. Let Λ be singly atypical of type β. Then the polynomial $Q(\Lambda)$ in (4.14) has a zero of multiplicity $m(\Lambda,\mu)$ for $a_m = \Lambda(h_m)$. Now we use the following property: let $M(t)$ be a $N \times N$-matrix over $\mathbb{C}[t]$ (i.e. the entries of $M(t)$ are polynomials in the variable t); if $t = t_0$ is a zero of multiplicity k of $\det(M(t))$, then $\text{rank}(M(t_0)) \geq N - k$ (this property can be proved using elementary matrix operations). Applying this to A and B in (4.14), for $a_m = \Lambda(h_m)$, leads to

$$\text{rank}(A) + \text{rank}(B) \geq 2m(\Lambda,\mu) - m(\Lambda,\mu) = m(\Lambda,\mu), \tag{4.20}$$

or, using (4.11) and (4.13),

$$\text{dim}V(\Lambda)_\mu + \text{dim}X(\Lambda)_\mu \geq m(\Lambda,\mu). \tag{4.21}$$

But since $V(\Lambda) \cong \overline{V}(\Lambda)/M(\Lambda)$ and $X(\Lambda) \subseteq M(\Lambda)$,

$$\text{dim}V(\Lambda)_\mu + \text{dim}X(\Lambda)_\mu \leq \text{dim}\overline{V}(\Lambda)_\mu = m(\Lambda,\mu). \tag{4.22}$$

Hence

$$\text{dim}V(\Lambda)_\mu + \text{dim}X(\Lambda)_\mu = \text{dim}\overline{V}(\Lambda)_\mu, \qquad \forall \mu \in P(\Lambda). \tag{4.23}$$

This shows that $X(\Lambda)$ is the maximal submodule of $\overline{V}(\Lambda)$. ∎

5. Singly atypical modules of type α_m

In this section we shall consider the special case of a singly atypical Λ of type α_m, where α_m is the unique odd simple root given in (2.9). In this case it turns out to be rather easy to determine the highest weight of $X(\Lambda)$.

Lemma 5.1. *Let Λ be atypical of type α_m. Then $v = e(-\alpha_m)v_\Lambda$ is a G primitive vector in $\overline{V}(\Lambda)$.*

Proof. For $\alpha \in \Delta_0^+$, we have $e(\alpha)v = [e(\alpha), e(-\alpha_m)]v_\Lambda + e(-\alpha_m)e(\alpha)v_\Lambda$. But $[e(\alpha), e(-\alpha_m)] = 0$ for all $\alpha \in \Delta_0^+$, and $N^+ v_\Lambda = 0$, hence

$$e(\alpha)v = 0, \qquad \forall \alpha \in \Delta_0^+. \tag{5.1}$$

Then, using $e(+\alpha_m)v_\Lambda = 0$, one finds

$$\begin{aligned} e(+\alpha_m)v &= [e(+\alpha_m), e(-\alpha_m)]v_\Lambda = h_m v_\Lambda = \Lambda(h_m)v_\lambda \\ &= \langle \Lambda \mid \alpha_m \rangle v_\Lambda = \langle \Lambda + \rho \mid \alpha_m \rangle v_\Lambda = 0, \end{aligned} \tag{5.2}$$

since $\langle \rho \mid \alpha_m \rangle = 0$ and Λ is atypical of type α_m. Then (5.1) and (5.2) imply

$$e(\alpha_i)v = 0, \qquad i = 1, 2, \ldots, m+n-1, \tag{5.3}$$

where α_i are the simple roots introduced in (2.9). Since N^+ is generated by the $m+n-1$ elements $e(\alpha_i)$, it follows that $N^+ v = 0$. ∎

In the case of Lemma 5.1, $U(G)v$ is a proper submodule of $\overline{V}(\Lambda)$, hence Lemma 3.8 implies $X(\Lambda) \subseteq U(G)v \subseteq M(\Lambda)$. But Λ is singly atypical of type α_m, so by Theorem 4.3:

$$M(\Lambda) = X(\Lambda) = U(G)v, \tag{5.4}$$

where $v = e(-\alpha_m)v_\Lambda$ is a vector of weight $\Lambda - \alpha_m$. Since $U(G)v$ is a highest weight module with highest weight vector v, and since $X(\Lambda)$ is simple (see Lemma 3.8), we have the following

Corollary 5.2. *Let Λ be singly atypical of type α_m. Then $X(\Lambda) = U(G)v_{\Lambda_-}$ is the maximal proper submodule of $\overline{V}(\Lambda)$, and $X(\Lambda)$ is isomorphic to the simple G module $V(\Lambda - \alpha_m)$. Consequently,*

$$ch\overline{V}(\Lambda) = chV(\Lambda) + chV(\Lambda - \alpha_m). \tag{5.5}$$

Note that if Λ is dominant and singly atypical of type α_m, then $\Lambda - \alpha_m$ is also dominant and singly atypical of type α_m. Now we can prove a character formula for this particular case.

Theorem 5.3. *Let Λ be singly atypical of type α_m. Then*

$$chV(\Lambda) = L_0^{-1} \sum_{w \in W} \varepsilon(w) w \Big\{ e^{\Lambda + \rho_0} \prod_{\substack{\beta \in \Delta_1^+ \\ \beta \neq \alpha_m}} (1 + e^{-\beta}) \Big\}. \tag{5.6}$$

Proof. Using (5.5) as a recursion relation, we find

$$\begin{aligned}
&\mathrm{ch}V(\Lambda) = \mathrm{ch}\overline{V}(\Lambda) - \mathrm{ch}V(\Lambda-\alpha_m)\\
&= \mathrm{ch}\overline{V}(\Lambda) - \Big(\mathrm{ch}\overline{V}(\Lambda-\alpha_m) - \mathrm{ch}V(\Lambda-2\alpha_m)\Big)\\
&= \mathrm{ch}\overline{V}(\Lambda) - \mathrm{ch}\overline{V}(\Lambda-\alpha_m) + \Big(\mathrm{ch}\overline{V}(\Lambda-2\alpha_m) - \mathrm{ch}V(\Lambda-3\alpha_m)\Big) = \ldots\\
&= \mathrm{ch}\overline{V}(\Lambda) - \mathrm{ch}\overline{V}(\Lambda-\alpha_m) + \mathrm{ch}\overline{V}(\Lambda-2\alpha_m) - \mathrm{ch}\overline{V}(\Lambda-3\alpha_m) + \ldots \qquad (5.7)
\end{aligned}$$

which becomes a formal infinite series expression since (5.5) can be applied for every $\Lambda - k\alpha_m$ ($k \in \mathbb{N}$). Then we can substitute (3.18) for the characters of the Kac-modules appearing in (5.7), and sum over the formal series:

$$\begin{aligned}
\mathrm{ch}V(\Lambda) &= L_0^{-1} \sum_{w\in W} \varepsilon(w) w \Big\{ e^{\rho_0} \Big(e^{\Lambda} - e^{\Lambda-\alpha_m} + e^{\Lambda-2\alpha_m} - e^{\Lambda-3\alpha_m} + \ldots \Big)\\
&\qquad\qquad \times \prod_{\beta\in\Delta_1^+} (1+e^{-\beta}) \Big\}\\
&= L_0^{-1} \sum_{w\in W} \varepsilon(w) w \Big\{ e^{\Lambda+\rho_0} \Big(1+e^{-\alpha_m}\Big)^{-1} \prod_{\beta\in\Delta_1^+} (1+e^{-\beta}) \Big\}. \qquad (5.8)
\end{aligned}$$

This proves the theorem. ∎

Let λ be an integral weight, and $\gamma \in \Delta_1^+$. We define the formal character

$$\chi_\gamma(\lambda) = L_0^{-1} \sum_{w\in W} \varepsilon(w) w \left\{ e^{\lambda+\rho_0} \prod_{\substack{\beta\in\Delta_1^+\\ \beta\neq\gamma}} (1+e^{-\beta}) \right\}. \qquad (5.9)$$

Theorem 5.3 shows that if Λ is singly atypical of type α_m, then $\mathrm{ch}V(\Lambda) = \chi_{\alpha_m}(\Lambda)$. We shall show in Section 7 that if Λ is singly atypical of type γ, then $\mathrm{ch}V(\Lambda) = \chi_\gamma(\Lambda)$. Note that the formal character (5.9) satisfies the property:

$$\chi_{w(\gamma)}(w\cdot\lambda) = \varepsilon(w)\chi_\gamma(\lambda), \qquad \forall w \in W. \qquad (5.10)$$

Finally, one sees from (3.19) that

$$\chi_K(\lambda) = \chi_\gamma(\lambda) + \chi_\gamma(\lambda-\gamma), \qquad \forall\gamma \in \Delta_1^+. \qquad (5.11)$$

Remark 5.4. Let Λ be integral dominant. In Section 3 we have seen that $\overline{V}(\Lambda)$ has a unique (up to scalar multiplication) vector of weight $w_0(\Lambda_-) = w_0(\Lambda - 2\rho_1)$ that is annihilated by N^-; $w_0(\Lambda_-)$ is the lowest weight of $\overline{V}(\Lambda)$, and it also characterises the Kac-module uniquely. Then $\chi_K(\Lambda) = \mathrm{ch}\overline{V}(\Lambda)$ contains a unique lowest term $e^{w_0(\Lambda_-)}$, where the terms e^λ are partially ordered according to $e^\lambda \geq e^\mu \Leftrightarrow \lambda \geq \mu$. It follows from (5.9) that $e^{w_0(\Lambda_-)}$ is a term of $\chi_\gamma(\Lambda-\gamma)$ and not of $\chi_\gamma(\Lambda)$; in particular it is the unique lowest term appearing in $\chi_\gamma(\Lambda-\gamma)$.

6. The atypicality matrix

The atypicality of an integral dominant weight Λ is determined by the value of the mn numbers $\langle \Lambda + \rho \mid \beta \rangle$ with $\beta \in \Delta_1^+$. In this section we shall study some of the properties of a matrix consisting of these mn numbers [17], and in particular we prove some crucial lemmas concerning a singly atypical Λ.

Definition 6.1. *Let $\Lambda \in H^*$. The atypicality matrix $A(\Lambda)$ is the $m \times n$ complex matrix with entries $A(\Lambda)_{ij} = \langle \Lambda + \rho \mid \beta_{ij} \rangle$, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$, and β_{ij} is defined in (2.13).*

In terms of the $\epsilon\delta$-components of Λ, one has:

$$A(\Lambda)_{ij} = \lambda_i + \mu_j + m - i - j + 1. \tag{6.1}$$

The properties of this matrix have been studied in another paper [17], and can be summarized as follows:

a) Let $w = \sigma \times \tau \in W = S_m \times S_n$, then

$$A(w \cdot \Lambda)_{ij} = A(\Lambda)_{\sigma^{-1}(i),\tau^{-1}(j)}, \tag{6.2a}$$

where $w \cdot \Lambda$ is determined by (2.15) or (2.18).

b) Let a_i be the Dynkin labels of Λ, then

$$\begin{aligned} &A(\Lambda)_{ij} - A(\Lambda)_{i+1,j} = a_i + 1, \qquad (1 \le i < m) \\ &A(\Lambda)_{m1} = a_m, \\ &A(\Lambda)_{ij} - A(\Lambda)_{i,j+1} = a_{m+j} + 1. \qquad (1 \le j < n) \end{aligned} \tag{6.2b}$$

c) Any atypicality matrix $A(\Lambda)$ satisfies:

$$A(\Lambda)_{ij} + A(\Lambda)_{kl} = A(\Lambda)_{il} + A(\Lambda)_{kj} \; ; \tag{6.2c}$$

vice versa, any $m \times n$ matrix satisfying (6.2c) for all pairs (i, j) and (k, l) with $1 \le i, k \le m$ and $1 \le j, l \le n$ is the atypicality matrix of a unique element $\Lambda \in H^*$.

d) Λ is dominant if and only if

$$\begin{aligned} &A(\Lambda)_{ij} - A(\Lambda)_{i+1,j} - 1 \ge 0 \qquad (1 \le i < m,\ 1 \le j \le n) \quad \text{and} \\ &A(\Lambda)_{ij} - A(\Lambda)_{i,j+1} - 1 \ge 0 \qquad (1 \le i \le m,\ 1 \le j < n). \end{aligned} \tag{6.2d}$$

Moreover, Λ is integral dominant if the expressions on the l.h.s. of (6.2d) are all integers.

Lemma 6.2. *Let λ be any integral element of H^*. Then the following statements are equivalent:*

(1) $\chi_W(\lambda) = 0$;

(2) $\chi_K(\lambda) = 0$;

(3) $\exists w \in W$ *with* $\varepsilon(w) = -1$ *such that* $w \cdot \lambda = \lambda$;

(4) $\forall w \in W, w \cdot \lambda$ *is not dominant*;

(5) $A(\lambda)$ *has two equal columns or two equal rows.*

Proof. The equivalence of (1), (3) and (4) is a classical property of the Weyl group of a semi-simple Lie algebra [6]. From (3.21) it follows that (2) is equivalent to (1). Finally, if $A(\lambda)$ has two equal rows or columns, then (6.2a) implies that there exists a $w \in W$ with $\varepsilon(w) = -1$ such that $A(w \cdot \lambda) = A(\lambda)$ and hence $w \cdot \lambda = \lambda$, so that (5)$\Rightarrow$(3). Conversely, if $A(\lambda)$ has no equal rows or columns, (6.2a) together with (6.2b) implies that there exists a $w \in W$ such that in the matrix $A(w \cdot \lambda)$ the elements in every row are strictly decreasing from left to right and the elements in every column are strictly decreasing from top to bottom; then (6.2d) is satisfied for $A(w \cdot \Lambda)$ and implies that $w \cdot \lambda$ is dominant, contradicting (4). ∎

Definition 6.3. *An integral element $\lambda \in H^*$ is said to be vanishing if one of the statements (1)–(5) of Lemma 6.2 are satisfied. Otherwise, λ is said to be non-vanishing.*

In the rest of this section, Λ is an integral dominant weight. Note that if Λ is integral and atypical, then (6.2b) implies that all entries in the atypicality matrix $A(\Lambda)$ are integers.

Lemma 6.4. *Let Λ be singly atypical of type $\gamma = \beta_{ij}$. Then*

$$\{-A(\Lambda)_{il} \mid 1 \leq l \leq n\} \cap \{A(\Lambda)_{kj} \mid 1 \leq k \leq m\} = \{0\}. \tag{6.3}$$

Proof. Since $A(\Lambda)_{ij} = 0$, (6.2c) implies

$$A(\Lambda)_{kl} = A(\Lambda)_{il} + A(\Lambda)_{kj}.$$

But Λ is singly atypical, so $A(\Lambda)_{kl} \neq 0$ for $(k, l) \neq (i, j)$. This implies (6.3). ∎

Definition 6.5. *Let Λ be singly atypical of type $\gamma = \beta_{ij}$. Let*

$$r(\Lambda) = \{-A(\Lambda)_{il} \mid 1 \leq l \leq n\} \cup \{A(\Lambda)_{kj} \mid 1 \leq k \leq m\}. \tag{6.4}$$

Let $s(\Lambda)$ be the maximal subset of $r(\Lambda)$ consisting of consecutive integers $\{-q, \dots, p\}$ with $q, p \in \mathbb{N}$ and such that $0 \in s(\Lambda)$. Let $\{(i_t, j_t),\ -q \leq t \leq p\}$ be the sequence of matrix-positions defined by $(i_0, j_0) = (i, j)$ and

(a) for $t \geq 0$ $(i_{t+1}, j_{t+1}) = (i_t, j_t + 1)$ if $-(t+1)$ belongs to the ith row of $A(\Lambda)$, and $(i_{t+1}, j_{t+1}) = (i_t - 1, j_t)$ if $t+1$ belongs to the jth column of $A(\Lambda)$;

(b) for $t \leq 0$, $(i_{t-1}, j_{t-1}) = (i_t, j_t - 1)$ if $-(t-1)$ belongs to the ith row of $A(\Lambda)$, and $(i_{t-1}, j_{t-1}) = (i_t + 1, j_t)$ if $t-1$ belongs to the jth column of $A(\Lambda)$.

This sequence of matrix-positions is well defined thanks to Lemma 6.4. It is useful to introduce a notation for the subsequences:

$$\begin{aligned} S_+(\Lambda) &= \{(i_0, j_0), \dots, (i_p, j_p)\} \\ S_-(\Lambda) &= \{(i_{-q}, j_{-q}), \dots, (i_0, j_0)\} \end{aligned} \tag{6.5}$$

Also, we let

$$\tilde{S}_\pm(\Lambda) = \{\beta_{kl} \mid (k, l) \in S_\pm(\Lambda)\}. \tag{6.6}$$

Example. Let $G = sl(6/8)$ and

$$\Lambda = (7, 7, 5, 4, 4, 1 \mid 0, -2, -2, -4, -4, -4, -4, -7)$$

in the $\epsilon\delta$-basis or $\Lambda = [02103; 1; 2020003]$ in Dynkin labels. Then Λ is singly atypical of type $\beta_{3,5}$ and the atypicality matrix $A(\Lambda)$ is given in (6.7), where it is bordered at the top with the negatives of the third row and at the left with the fifth column. The numbers actually belonging to $s(\Lambda)$ are in *italic*, and these determine the sequences $S_+(\Lambda)$ and $S_-(\Lambda)$, also represented in (6.7) by $+$ and $-$ signs, respectively, in the table of matrix-positions.

$$A(\Lambda) = \begin{array}{r} \\ \mathit{4} \\ \mathit{3} \\ \mathit{0} \\ \mathit{-2} \\ \mathit{-3} \\ -7 \end{array} \begin{array}{c} \begin{array}{rrrrrrrr} -8 & \mathit{-5} & \mathit{-4} & \mathit{-1} & \mathit{0} & \mathit{1} & \mathit{2} & 6 \end{array} \\ \begin{pmatrix} 12 & 9 & 8 & 5 & 4 & 3 & 2 & -2 \\ 11 & 8 & 7 & 4 & 3 & 2 & 1 & -3 \\ 8 & 5 & 4 & 1 & 0 & -1 & -2 & -6 \\ 6 & 3 & 2 & -1 & -2 & -3 & -4 & -8 \\ 5 & 2 & 1 & -2 & -3 & -4 & -5 & -9 \\ 1 & -2 & -3 & -6 & -7 & -8 & -9 & -13 \end{pmatrix} \end{array} \tag{6.7}$$

						+	
						+	
			−	±	+	+	
			−				
	−	−	−				

Explicitly,

$$S_-(\Lambda) = \{(5, 2), (5, 3), (5, 4), (4, 4), (3, 4), (3, 5)\}$$

and

$$S_+(\Lambda) = \{(3, 5), (3, 6), (3, 7), (2, 7), (1, 7)\}.$$

Lemma 6.6. *Let Λ be singly atypical of type $\gamma = \beta_{ij}$. Then there exists a unique sequence of distinct elements $\beta_{-q} < \beta_{-q+1} < \cdots < \beta_0 = \gamma$ from Δ_1^+ such that the sequence of weights $\nu_{-q-1}, \nu_{-q}, \ldots, \nu_0 = \Lambda$, where $\nu_{t-1} = \nu_t - \beta_t$, satisfies*

$$\langle \nu_t + \rho \mid \beta_t \rangle = 0, \qquad -q \le t \le 0; \tag{6.8a}$$

$$\nu_t \text{ is vanishing for } -q \le t < 0; \tag{6.8b}$$

$$q(\Lambda) = \nu_{-q-1} \text{ is integral dominant and singly atypical of type } \beta_{-q}; \tag{6.8c}$$

$$\exists w \in W \text{ such that } \nu_t = w \cdot (\Lambda + t\gamma) \text{ with } \beta_{t+1} = w(\gamma) \text{ and } \varepsilon(w) = (-1)^{t+1},\ -q-1 \le t < 0; \tag{6.8d}$$

$$\beta_t = \beta_{i_t, j_t}, \text{ where } (i_t, j_t) \text{ is given in Definition 6.5.} \tag{6.8e}$$

Proof. From the inner product (2.14) one deduces

$$\langle \beta_{ab} \mid \beta_{kl} \rangle = \delta_{ak} - \delta_{bl}. \tag{6.9}$$

Using Definition 6.1 this implies that $A(\Lambda - \beta_{ab})$ is obtained from $A(\Lambda)$ by decreasing the elements in row a by one unit and simultaneously increasing the elements in column b by one unit. Hence the matrices $A_0 = A(\Lambda)$, $A_{-1} = A(\Lambda - \beta_{i_0, j_0})$, $A_{-2} = A(\Lambda - \beta_{i_0, j_0} - \beta_{i_{-1}, j_{-1}})$, …, where (i_t, j_t) is the sequence of Definition 6.5, satisfy

$$A_t \text{ has two zeroes, at positions } (i_{t+1}, j_{t+1}) \text{ and } (i_t, j_t) \text{ for } -q \le t \le -1; \tag{6.10a}$$

$$A_{-q-1} \text{ has one zero at position } (i_{-q}, j_{-q}); \tag{6.10b}$$

$$A_t \text{ is obtained from } A(\Lambda + t\gamma) \text{ by } -t-1 \text{ transpositions of rows and columns.} \tag{6.10c}$$

Thus the existence and the uniqueness of the sequence $\beta_0, \beta_{-1}, \ldots, \beta_{-q}$, and (6.8e), follow from the properties of the sequence $S_-(\Lambda)$, which also implies that $\beta_0 > \cdots > \beta_{-q}$. Then (6.8a) is a consequence of (6.10a). Moreover, from (6.10a) it follows that $A_t = A(\nu_t)$ $(-q \le t \le -1)$ has two equal rows or two equal columns, and then Lemma 6.2 implies (6.8b). The matrix A_{-q-1} has one zero at position (i_{-q}, j_{-q}), and by construction the elements in every row are strictly decreasing from left to right and the elements in every column are strictly decreasing from top to bottom; thus (6.2d) implies that $q(\Lambda) = \nu_{-q-1}$ is dominant, proving (6.8c). Finally, (6.10c) and (6.2a) imply (6.8d). ∎

Lemma 6.7. *Let Λ be singly atypical of type $\gamma = \beta_{ij}$. Then there exists a unique sequence of distinct elements $\beta_0 = \gamma < \beta_1 < \cdots < \beta_p$ from Δ_1^+ such that the sequence of weights $\nu_0 = \Lambda, \nu_1, \ldots, \nu_{p+1}$, where $\nu_{t+1} = \nu_t + \beta_t$, satisfies*

$$\langle \nu_t + \rho \mid \beta_t \rangle = 0, \qquad 0 \leq t \leq p; \tag{6.11a}$$

$$\nu_t \text{ is vanishing for } 0 < t \leq p; \tag{6.11b}$$

$$p(\Lambda) = \nu_{p+1} \text{ is integral dominant and singly atypical of type } \beta_p; \tag{6.11c}$$

$$\exists w \in W \text{ such that } \nu_t = w \cdot (\Lambda + t\gamma) \text{ with } \beta_{t-1} = w(\gamma) \text{ and } \varepsilon(w) = (-1)^{t-1},\ 0 < t \leq p+1; \tag{6.11d}$$

$$\beta_t = \beta_{i_t, j_t}, \text{ where } (i_t, j_t) \text{ is given in Definition 6.5.} \tag{6.11e}$$

The proof of Lemma 6.7 is similar to the proof of Lemma 6.6, using $S_+(\Lambda)$ instead of $S_-(\Lambda)$.

7. The character formula

Using the lemmas of Section 6, we are now able to prove a character formula for $V(\Lambda)$, where Λ is a singly atypical integral dominant weight.

Lemma 7.1. *Let Λ be singly atypical of type γ with $S_-(\Lambda)$ given by (6.5). Let $\gamma' = \beta_{i_{-q}, j_{-q}}$ and $q(\Lambda) = \Lambda - \sum_{\beta \in \tilde{S}_-(\Lambda)} \beta$ be the dominant weight defined in Lemma 6.6, which is singly atypical of type γ'. Then, using the notation (5.9):*

$$\chi_\gamma(\Lambda - \gamma) = \chi_{\gamma'}(q(\Lambda)). \tag{7.1}$$

Proof. As in the proof of Theorem 5.3, we can expand $\chi_\gamma(\Lambda - \gamma)$ in a series of $\chi_K(\lambda)$-terms:

$$\begin{aligned}\chi_\gamma(\Lambda - \gamma) = \chi_K(\Lambda - \gamma) - \chi_K(\Lambda - 2\gamma) + \chi_K(\Lambda - 3\gamma) - \cdots \\ + (-1)^q \chi_K(\Lambda - (q+1)\gamma) + \cdots\end{aligned} \tag{7.2}$$

But for $-q \leq t \leq -1$, (6.8b) and (6.8d) imply that $\Lambda + t\gamma$ is vanishing, hence $\chi_K(\Lambda + t\gamma) = 0$. Then (7.2) becomes:

$$\begin{aligned}\chi_\gamma(\Lambda - \gamma) &= (-1)^q \left(\chi_K(\Lambda - (q+1)\gamma) - \chi_K(\Lambda - (q+2)\gamma) + \cdots\right) \\ &= (-1)^q \chi_\gamma(\Lambda - (q+1)\gamma).\end{aligned} \tag{7.3}$$

According to (6.8d), there exists a $w \in W$ such that $w(\Lambda - (q+1)\gamma + \rho) = q(\Lambda) + \rho$ with $\gamma' = w(\gamma)$ and $\varepsilon(w) = (-1)^q$. Using (5.10) this implies that

$$\chi_{\gamma'}(q(\Lambda)) = (-1)^q \chi_\gamma(\Lambda - (q+1)\gamma). \tag{7.4}$$

Then the lemma follows from (7.3) and (7.4). ∎

Theorem 7.2. *Let Λ be singly atypical of type γ. Then*

$$chV(\Lambda) = \chi_\gamma(\Lambda). \tag{7.5}$$

Proof. In the case that $\gamma = \alpha_m$, the statement follows from Theorem 5.3. Suppose now that $\gamma > \alpha_m$. Let $\Lambda_0 = \Lambda$ and $\gamma_0 = \gamma$, and using the notation of Lemma 7.1 we define a sequence of dominant weights and elements of Δ_1^+ by

$$\Lambda_{k+1} = q(\Lambda_k), \qquad \gamma_{k+1} = \gamma_k', \qquad (k \geq 0). \tag{7.6}$$

Clearly, every $\gamma_{k+1} \leq \gamma_k$, with equality if and only if $\#S_-(\Lambda_k) = 1$, i.e. if and only if $\Lambda_k - \gamma_k$ is dominant. So $\gamma_{k+1} = \gamma_k$ can happen only a finite number of times if $\gamma_k > \alpha_m$. Therefore, there exists an s such that $\gamma_{s-1} > \gamma_s = \alpha_m$, α_m being the smallest element of Δ_1^+ according to the partial ordering (2.10). Since every Λ_k is dominant, we find, using (3.18)–(3.19) and (5.11):

$$\mathrm{ch}\overline{V}(\Lambda_k) = \chi_{\gamma_k}(\Lambda_k) + \chi_{\gamma_k}(\Lambda_k - \gamma_k). \tag{7.7}$$

Using Lemma 7.1, this becomes:

$$\mathrm{ch}\overline{V}(\Lambda_k) = \chi_{\gamma_k}(\Lambda_k) + \chi_{\gamma_{k+1}}(\Lambda_{k+1}). \tag{7.8}$$

On the other hand, since every Λ_k is singly atypical, Theorem 4.3 implies:

$$\mathrm{ch}\overline{V}(\Lambda_k) = \mathrm{ch}V(\Lambda_k) + \mathrm{ch}X(\Lambda_k), \tag{7.9}$$

where both $V(\Lambda_k)$ and $X(\Lambda_k)$ are simple G modules. Applying (7.8) and (7.9) for $k = s-1$ and using Theorem 5.3, leads to

$$\mathrm{ch}\overline{V}(\Lambda_{s-1}) = \chi_{\gamma_{s-1}}(\Lambda_{s-1}) + \mathrm{ch}V(\Lambda_s), \tag{7.10a}$$

$$\mathrm{ch}\overline{V}(\Lambda_{s-1}) = \mathrm{ch}V(\Lambda_{s-1}) + \mathrm{ch}X(\Lambda_{s-1}). \tag{7.10b}$$

From Remark 5.4, (7.7) and (7.10a), we see that $V(\Lambda_s)$ has $w_0(\Lambda_{s-1} - 2\rho_1)$ as lowest weight. Since a simple G module is characterised by its lowest weight, it follows from Lemma 3.7 and Lemma 3.8 that $V(\Lambda_s)$ is isomorphic to $X(\Lambda_{s-1})$, and (7.10) implies:

$$\mathrm{ch}V(\Lambda_{s-1}) = \chi_{\gamma_{s-1}}(\Lambda_{s-1}). \tag{7.11}$$

By iteration one finds, for all k with $0 \leq k \leq s$:

$$\mathrm{ch}V(\Lambda_k) = \chi_{\gamma_k}(\Lambda_k). \tag{7.12}$$

The theorem follows by putting $k = 0$ in (7.12). ∎

Corollary 7.3. *Let Λ be singly atypical. Then the lowest weight of $V(\Lambda)$ is given by*

$$w_0\Big(\Lambda - \sum_{\beta\notin\tilde{S}_+(\Lambda)} \beta\Big) = w_0\Big(\Lambda_- + \sum_{\beta\in\tilde{S}_+(\Lambda)} \beta\Big). \tag{7.13}$$

Proof. For given singly atypical Λ, let

$$\Omega = p(\Lambda) = \Lambda + \sum_{\beta\in\tilde{S}_+(\Lambda)} \beta. \tag{7.14}$$

Then it is a combinatorial exercise to see that $S_-(\Omega) = S_+(\Lambda)$, hence

$$\Lambda = q(\Omega) = \Omega - \sum_{\beta\in\tilde{S}_-(\Omega)} \beta. \tag{7.15}$$

From the proof of Theorem 7.2 it follows that Λ is the highest weight of $X(\Omega)$, or $X(\Omega) \cong V(\Lambda)$. But $w_0(\Omega_-)$ is the lowest weight of $\overline{V}(\Omega)$ (see Section 3), and therefore also of $X(\Omega)$. So the lowest weight of $V(\Lambda)$ is given by

$$\begin{aligned} w_0(\Omega_-) = w_0(\Omega - 2\rho_1) &= w_0\Big(\Lambda - 2\rho_1 + \sum_{\beta\in\tilde{S}_+(\Lambda)} \beta\Big) \\ &= w_0\Big(\Lambda - \sum_{\beta\notin\tilde{S}_+(\Lambda)} \beta\Big), \end{aligned} \tag{7.16}$$

since $2\rho_1 = \sum_{\beta\in\Delta_1^+} \beta$. ∎

8. Some remarks

1. In order to accommodate the characters of all simple modules of $G = sl(m/n)$, Bernstein and Leites proposed the following formula [2]:

$$\chi_L(\lambda) = L_0^{-1} \sum_{w\in W} \varepsilon(w) w \Big\{ e^{\lambda+\rho_0} \prod_{\substack{\beta\in\Delta_1^+ \\ \langle\lambda+\rho|\beta\rangle\neq 0}} (1 + e^{-\beta}) \Big\}. \tag{8.1}$$

Although for any integral dominant Λ, $\mathrm{ch}V(\Lambda) = \chi_L(\Lambda)$ if Λ is typical, as proved by Kac (Theorem 3.12 and equation (3.18) in this paper), and $\mathrm{ch}V(\Lambda) = \chi_L(\Lambda)$ if Λ is singly atypical, as proved here in Theorem 7.2, it is certainly not true in general that $\mathrm{ch}V(\Lambda) = \chi_L(\Lambda)$. A simple counterexample is the identity module $V(\mathbf{0})$ which has character $\mathrm{ch}V(\mathbf{0}) = 1$, and $\chi_L(\mathbf{0}) \neq 1$ if $m > 1$ and $n > 1$.

2. In Kac's classification of classical simple Lie superalgebras over $\mathbb{C}$ [9], the *Type I* Lie superalgebras are $A(m,n)$ and $C(n)$, where $A(m,n) = sl(m+1/n+$

1) if $m \neq n$ and $A(m,m) = sl(m+1/m+1)/\mathbb{C}I_{2m}$, and $C(n) = osp(2, 2n-2)$. It is not too difficult to verify that many lemmas given here for $sl(m/n)$ are also valid for $osp(2, 2n-2)$: the proofs in Section 4 can almost literally be transferred to the case of $osp(2, 2n-2)$; the notions in Sections 5–7 need to be slightly changed. This leads us to a proof of a character formula for singly atypical modules of $C(n)$ [16]. But $C(n)$ has only typical or singly atypical modules. We conclude that, for all integral dominant Λ for $C(n)$, $\mathrm{ch}V(\Lambda) = \chi_L(\Lambda)$, given by (8.1) but with all symbols defined for $C(n)$.

3. Let us return to the case $G = sl(m/n)$. We say that an integral dominant weight Λ is *atypical of degree d* if there are d distinct elements β in Δ_1^+ for which Λ is atypical. We shall try to give the reader an idea of the complications which arise in identifying the maximal submodule $M(\Lambda)$ if $d > 1$ by concentrating on the case of $d = 2$. For $d = 1$, Theorem 4.3 shows that $\overline{V}(\Lambda)$ always contains 2 composition factors. For $d = 2$, for example, we have calculated the composition factors of some Kac-modules in $sl(2/3)$, and their number varies : $\overline{V}([0;0;0,0])$ has 3 composition factors, $\overline{V}([1;0;1,0])$ has 5 composition factors, and $\overline{V}([2;0;2,0])$ has 4 composition factors.

We also have at least one example of a *doubly* atypical Kac-module that contains *weakly primitive* vectors (see Definition 3.3), a situation that cannot occur for typical or singly atypical Kac-modules. The example is the following: $G = sl(2/2)$ and $\Lambda = [1;0;1]$ (so $V(\Lambda)$ is the adjoint module). Using the notation of Section 3, $X(\Lambda) = U(G)v_{\Lambda_-}$ is a simple submodule of $\overline{V}(\Lambda)$. Using the basis E_{ij} described in Section 2, let v be the following vector of $\overline{V}(\Lambda)$:

$$\begin{aligned} v = (&E_{31}E_{32}E_{41}E_{43} + E_{31}E_{32}E_{42}E_{21}E_{43} \\ &+ E_{32}E_{41}E_{42}E_{21} + E_{31}E_{41}E_{42})v_\Lambda. \end{aligned} \tag{8.2}$$

One can check that $v \notin X(\Lambda)$. However $E_{14}v \neq 0$ is proportional to the highest weight vector of $X(\Lambda)$; in fact $\{0\} \neq N^+v \subseteq X(\Lambda)$, showing that v is a weakly primitive vector in $\overline{V}(\Lambda)$.

4. Despite the difficulties for *multiply atypical* modules, we have recently given a conjecture [17] for the character of all simple G modules with integral dominant highest weight Λ, and we shall briefly describe this conjecture here. Formula (8.1) can be re-expressed as an infinite alternating sum of $\chi_K(\mu)$-terms, just as in (7.2). Indeed, if Λ is atypical of degree d with respect to $\beta_1, \ldots, \beta_d$, then one defines the *cone* C_Λ with vertex at Λ as the set of lattice

points

$$C_\Lambda = \{\Lambda - \sum_{i=1}^{d} k_i\beta_i \,|\, k_i \in \mathbb{N}\,(i=1,\ldots,d)\}. \tag{8.3}$$

The expansion becomes

$$\chi_L(\Lambda) = \sum_{\mu\in C_\Lambda} (-1)^{|\Lambda-\mu|}\chi_K(\mu), \tag{8.4}$$

where $(-1)^{|\Lambda-\mu|} = (-1)^{k_1+\cdots+k_d}$ for $\mu = \Lambda - \sum_{i=1}^{d} k_i\beta_i$. The new formula is of type (8.4) with a restriction on the summation such that all terms $\chi_K(\mu)$ for which μ is a weight beyond certain truncation planes in the weight space are excluded. These truncation planes are uniquely determined, for each Λ, as symmetry planes p_{ij} under the *dot action* of elements w_{ij} $(1 \le i < j \le d)$ of the Weyl group W, where w_{ij} is the unique element such that $w_{ij}(\beta_i) = \beta_j$ and such that $w_{ij} = 1$ when restricted to the subspace of H^* orthogonal to β_i and β_j. The hyperplane p_{ij} divides the weight space H^* into two. We denote by H^*_{ij} the open half-space of H^* containing Λ. The *truncated cone* is defined to be

$$C^+_\Lambda = C_\Lambda \cap \Big(\bigcap_{critical(i,j)} H^*_{ij} \Big), \tag{8.5}$$

where the intersection is taken only with those H^*_{ij} for which (i,j) is *critical*, and the new formula becomes

$$\chi_T(\Lambda) = \sum_{\mu\in C^+_\Lambda} (-1)^{|\Lambda-\mu|}\chi_K(\mu). \tag{8.6}$$

The notion of criticality is defined elsewhere [17], and we shall content ourselves by describing it merely for doubly atypical weights. Let Λ be doubly atypical of type β_1 and β_2, with $\beta_1 > \beta_2$. Then $(1,2)$ is critical if and only if the weights in the finite set $H^*_{12} \cap \{\Lambda - t\beta_1 \,|\, t = 1,2,3,\ldots\}$ are all vanishing, or equivalently, those in $H^*_{12} \cap \{\Lambda + t\beta_2 \,|\, t = 1,2,3,\ldots\}$ are all vanishing. If Λ is not critical, no truncations occur and our conjectured character formula (8.6) coincides with (8.1). For more details concerning this conjecture and some arguments in its favour, we refer to [17].

Acknowledgements

We would like to thank S. Donkin (Queen Mary College, London) for stimulating discussions. NATO (Belgium), the Royal Society European Exchange Programme and SERC (U.K.) are acknowledged for Research Fellowships, and CNRS (France) for supporting some of us on a research visit to Paris.

REFERENCES

[1] A. Berele and A. Regev, "Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras," *Adv. Math.* **64** (1987), 118–75.

[2] I.N. Bernstein and D.A. Leites, "A formula for the characters of the irreducible finite-dimensional representations of Lie superalgebras of series gl and sl," *C.R. Acad. Bulg. Sci.* **33** (1980), 1049–51 [in Russian].

[3] L. Corwin, Y. Ne'eman and S. Sternberg, "Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry)," *Rev. Mod. Phys.* **47** (1975), 573–603.

[4] M.D. Gould, "Atypical representations for type-I Lie superalgebras," *J. Phys. A* **22** (1989), 1209–21.

[5] J.W.B. Hughes and R.C. King, "A conjectured character formula for atypical irreducible modules of the Lie superalgebra $sl(m/n)$," *J. Phys. A* **20** (1987), L1047–52.

[6] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, Berlin (1972).

[7] J.C. Jantzen, "Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren," *Math. Ann.* **226** (1977), 53–65.

[8] V.G. Kac, "Classification of simple Lie superalgebras," *Funct. Anal. Appl.* **9** (1975), 263–65.

[9] —, "Lie superalgebras," *Adv. Math.* **26** (1977), 8–96.

[10] —, "A sketch of Lie superalgebra theory," *Comm. Math. Phys.* **53** (1977), 31-64.

[11] —, "Representations of classical Lie superalgebras," in *Lecture Notes in Mathematics* **676**, Springer-Verlag, Berlin (1977), ed. Bleuler K., Petry H.

and Reetz A., pp. 579–626.

[12] —, "Characters of typical representations of classical Lie superalgebras," *Comm. Algebra* **5** (1977), 889–97.

[13] M. Scheunert, *The theory of Lie superalgebras*, Lecture Notes in Mathematics **716**, Springer-Verlag, Berlin (1979).

[14] N.N. Shapovalov, "On a bilinear form on the universal enveloping algebra of a complex semi-simple Lie algebra," *Funct. Anal. Appl.* **6** (1972), 307–12.

[15] A. Serge'ev, "The tensor algebra of the identity representation as a module over the Lie superalgebra $Gl(m/n)$ and $Q(n)$," *Math. USSR Sbornik* **51** (1985), 419–27.

[16] J. Van der Jeugt, *Character formulae for the Lie superalgebra* $C(n)$, to be published (1989).

[17] J. Van der Jeugt, J.W.B. Hughes, R.C. King and J. Thierry-Mieg, *Character formulae for irreducible modules of the Lie superalgebras* $sl(m/n)$, to be published (1989).

[18] H. Weyl, "Theorie der Darstellungen kontinuierlicher halbeinfacher Gruppen durch lineare Tranzformationen III," *Math. Z.* **24** (1926), 377–95.