A non-commutative n-particle 3D Wigner quantum oscillator

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Abstract

An n-particle 3-dimensional Wigner quantum oscillator model is constructed explicitly. It is non-canonical in that the usual coordinate and linear momentum commutation relations are abandoned in favour of Wigner's suggestion that Hamilton's equations and the Heisenberg equations are identical as operator equations. The construction is based on the use of Fock states corresponding to a family of irreducible representations of the Lie superalgebra $sl(1|3n)$ indexed by an A-superstatistics parameter p. These representations are typical for $p \geq 3n$ but atypical for $p < 3n$. The branching rules for the restriction from $sl(1|3n)$ to $gl(1) \oplus so(3) \oplus sl(n)$ are used to enumerate energy and angular momentum eigenstates. These are constructed explicitly and tabulated for $n \leq 2$. It is shown that measurements of the coordinates of the individual particles gives rise to a set of discrete values defining nests in the 3-dimensional configuration space. The fact that the underlying geometry is non-commutative is shown to have a significant impact on measurements of particle separation. In the atypical case, exclusion phenomena are identified that are entirely due to the effect of A-superstatistics. The energy spectrum and associated degeneracies are calculated for an infinite-dimensional realisation of the Wigner quantum oscillator model obtained by summing over all p. The results are compared with those applying to the analogous canonical quantum oscillator.

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1 Introduction

In a previous paper [1], hereafter referred to as I, we studied the properties of a non-canonical single particle 3-dimensional Wigner quantum oscillator (WQO). In this model physical observables, such as energy, angular momentum, position and linear momentum, were all associated with Hermitian operators acting in a Hilbert space spanned by certain Fock states [1]-[2] arising in the representation theory of the Lie superalgebra $sl(1|3)$ initiated by Kac [3]. This superalgebra was shown to arise in a natural way [1] when seeking algebraic solutions to the compatibility conditions necessary to ensure that, as suggested by Wigner [4], Hamilton's equations and the Heisenberg equations are identical as operator equations in the relevant Hilbert space. It is this suggestion, postulate (P3) in I, for a Wigner quantum system that replaces the canonical coordinate and linear momentum commutation relations of a more conventional quantum theory. This idea of Wigner has been studied by several authors from different points of view. Of the recent publications we mention [5]-[12].

Here we extend this study in I to the case of a non-canonical many particle 3-dimensional Wigner quantum oscillator. This time, in the case of n-particles the physical observables are all associated with Hermitian operators acting in a Hilbert space spanned by Fock states arising in the representation theory of the Lie superalgebra $sl(1|3n)$. The relevance of this superalgebra is established once again by looking for algebraic solutions of the n-particle compatibility conditions that ensure that as operator equations in the relevant Hilbert space, Hamilton's equations and the Heisenberg equations are identical.

The Hamiltonian itself of the n-particle WQO takes the form

$$\hat{H} = \sum_{\alpha=1}^{n} \Big(\frac{\hat{\mathbf{P}}_\alpha^2}{2m} + \frac{m\omega^2}{2} \hat{\mathbf{R}}_\alpha^2 \Big). \tag{1.1}$$

Just as in the single particle $n = 1$ case, the Hilbert spaces, $W(n,p)$, studied here in the n-particle case are associated with certain finite-dimensional irreducible representations of $sl(1|3n)$ specified by some non-negative integer p. Each of the Hilbert spaces $W(n,p)$, spanned by Fock states $|\,p,\Theta\rangle$, is generated from a unique cyclic vector, $|\,0\rangle$, by the action of the generators, $A^\pm_{\alpha i}$ of $sl(1|3n)$. As in I each generator $A^\pm_{\alpha i}$ is a certain linear combination of Hermitian coordinate and linear momentum operators, $\hat{R}_{\alpha i}(t)$ and $\hat{P}_{\alpha i}(t)$ respectively, for the particle specified by α. Again, as in I, the action of these generators is constrained by the conditions:

$$A^-_{\alpha i}\,|\,0\rangle = 0, \quad A^-_{\alpha i}A^+_{\beta j}\,|\,0\rangle = p\,\delta_{\alpha\beta}\,\delta_{ij}\,|\,0\rangle, \quad i,j = 1,2,3, \quad \alpha,\beta = 1,2,\ldots,n. \tag{1.2}$$

The resulting finite-dimensional irreducible representations are special cases of those classified by Kac [3] and include both typical and atypical representations [13]-[14] characterized here by $p \geq 3n$ and $p < 3n$, respectively.

Having set up this mathematical machinery for the WQO in Section 2, what follows is discussion of a range of physical properties of this multi-particle system. The energy spectrum and angular momentum are discussed in Section 3. This leans heavily on developments made in a sequence of previous papers [1]-[2], [15]-[16]. Here the complications associated with the determination of energy and angular momentum eigenstates in the many particle context are dealt with by the use of the branching rules appropriate to the restrictions, first from $sl(1|3n)$ to $gl(1)\oplus sl(3n)$ [17]-[21] then from $sl(3n)$ to $sl(3)\oplus sl(n)$ [22] and finally from $sl(3)$ to $so(3)$ [22]- [23], where $so(3)$ is the algebra associated with the total angular momentum of the system. By way of illustration, energy and angular momentum eigenstates are tabulated explicitly for both $n=1$ and $n=2$.

Section 4 is concerned with Wigner quantum oscillator configurations. It is found that just as energy and angular momentum are discretely quantised, so are the coordinates, however measured, of each of the n-particles. As observed previously in the 1-particle case analysed in I, the eigenvalues $r_{\alpha k}$ of the position operators $\hat{r}_{\alpha k}(t)$ of the αth particle specify nests with coordinates $\pm\sqrt{p-m}$, with the integer m now taking on the values $0,1,\ldots,\min(p,3n)-1$. The non-commutative nature of the geometry is such that once again the position of any individual particle can not be specified precisely. That is to say for each α there is no common eigenstate of $r_{\alpha k}(t)$ for all $k=1,2,3$. This observation gives notice of the fact that the interpretation of the measurement of the distance between any two particles has to be undertaken with care. This is also explored in Section 4 where it is shown that the expectation value of the square of the separation distance in any of the stationary states $|\,p,\Theta\rangle$ can be interpreted as the average of the square of the distance between the appropriate nests, weighted with respect to the probabilities of occupying each nest.

In Section 5 it is shown that the A-superstatistics of this $sl(1|3n)$ multiparticle WQO model leads to some exclusion phenomena whereby the state of one particle is influenced, or even determined, by the states of the other particles even though the original Hamiltonian (1.1) is that of n non-interacting particles. Finally a 3-dimensional n-particle WQO model based on an infinite-dimensional Hilbert space $W=\sum_{p=0}^{\infty}W(n,p)$ is compared and contrasted with an analogous canonical quantum oscillator (CQO) model. Both the WQO and the CQO models are shown to involve equally spaced energy levels, but their ground states, energy gaps and degeneracies are all shown to differ markedly in the two models.

2 Wigner quantum oscillators

Let $\hat{H}$ be the Hamiltonian of an n particle three-dimensional harmonic oscillator, that is

$$\hat{H}=\sum_{\alpha=1}^{n}\Big(\frac{\hat{\mathbf{P}}_\alpha^2}{2m}+\frac{m\omega^2}{2}\hat{\mathbf{R}}_\alpha^2\Big). \tag{2.1}$$

We proceed to view this oscillator as a Wigner quantum system. According to postulate (P3) in I the three-dimensional vector operators $\hat{\mathbf{R}}_1,\ldots,\hat{\mathbf{R}}_n$ and $\hat{\mathbf{P}}_1,\ldots,\hat{\mathbf{P}}_n$ have to be defined in such a way that Hamilton's equations

$$\dot{\hat{\mathbf{P}}}_\alpha=-m\omega^2\hat{\mathbf{R}}_\alpha,\quad \dot{\hat{\mathbf{R}}}_\alpha=\frac{1}{m}\hat{\mathbf{P}}_\alpha \ \text{ for } \ \alpha=1,2,\ldots,n, \tag{2.2}$$

and the Heisenberg equations

$$\dot{\hat{\mathbf{P}}}_\alpha = \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{P}}_\alpha], \quad \dot{\hat{\mathbf{R}}}_\alpha = \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{R}}_\alpha] \quad \text{for} \quad \alpha = 1, 2 \ldots, n, \tag{2.3}$$

are identical as operator equations. These compatibility conditions are such that

$$[\hat{H}, \hat{\mathbf{P}}_\alpha] = i\hbar m\omega^2 \hat{\mathbf{R}}_\alpha, \quad [\hat{H}, \hat{\mathbf{R}}_\alpha] = -\frac{i\hbar}{m}\hat{\mathbf{P}}_\alpha \quad \text{for} \quad \alpha = 1, 2, \ldots, n. \tag{2.4}$$

To make the connection with $sl(1|3n)$ we write the operators $\hat{\mathbf{P}}_\alpha$ and $\hat{\mathbf{R}}_\alpha$ for $\alpha = 1, 2, \ldots, n$ in terms of new operators:

$$A^\pm_{\alpha k} = \sqrt{\frac{(3n-1)m\omega}{4\hbar}}\hat{R}_{\alpha k} \pm i\sqrt{\frac{(3n-1)}{4m\omega\hbar}}\hat{P}_{\alpha k}, \quad k = 1, 2, 3. \tag{2.5}$$

The Hamiltonian $\hat{H}$ of (2.1), the single particle Hamiltonians, $\hat{H}_\alpha$, and the compatibility conditions (2.4) take the form [2]:

$$\hat{H} = \sum_{\alpha=1}^{n} \hat{H}_\alpha \quad \text{with} \quad \hat{H}_\alpha = \frac{\omega\hbar}{3n-1}\sum_{i=1}^{3}\{A^+_{\alpha i}, A^-_{\alpha i}\}, \tag{2.6}$$

$$\sum_{\beta=1}^{n}\sum_{j=1}^{3}[\{A^+_{\beta j}, A^-_{\beta j}\}, A^\pm_{\alpha i}] = \mp(3n-1)A^\pm_{\alpha i}, \quad i, j = 1, 2, 3, \quad \alpha, \beta = 1, 2, \ldots, n. \tag{2.7}$$

As a solution to (2.7) we chose operators $A^\pm_{\alpha i}$ that satisfy the following triple relations:

$$\begin{aligned}
&[\{A^+_{\alpha i}, A^-_{\beta j}\}, A^+_{\gamma k}] = \delta_{jk}\delta_{\beta\gamma}A^+_{\alpha i} - \delta_{ij}\delta_{\alpha\beta}A^+_{\gamma k},\\
&[\{A^+_{\alpha i}, A^-_{\beta j}\}, A^-_{\gamma k}] = -\delta_{ik}\delta_{\alpha\gamma}A^-_{\beta j} + \delta_{ij}\delta_{\alpha\beta}A^-_{\gamma k},\\
&\{A^+_{\alpha i}, A^+_{\beta j}\} = \{A^-_{\alpha i}, A^-_{\beta j}\} = 0.
\end{aligned} \tag{2.8}$$

Proposition 1 *The operators $A^\pm_{\alpha i}$, for $i = 1, 2, 3$ and $\alpha = 1, 2, \ldots, n$, are odd elements generating the Lie superalgebra $sl(1|3n)$. The operators $\{A^+_{\alpha i}, A^-_{\beta j}\}$ for $i, j = 1, 2, 3$ and $\alpha, \beta = 1, 2, \ldots, n$, are even elements generating the maximal even Lie subalgebra $gl(3n)$.*

The Lie superalgebra is from class A in the classification of the basic classical Lie superalgebras [3]. As we have indicated, the corresponding statistics is referred to as A-superstatistics [24]. The generators $A^\pm_{\alpha i}$ are said to be creation and annihilation operators (CAOs) of $sl(1|3n)$. These CAOs are the analogue of the Jacobsen generators for the Lie algebra $sl(3n+1)$ [25] and could also be called Jacobsen generators of $sl(1|3n)$.

We would underline the fact that all considerations here are in the Heisenberg picture. The position and momentum operators depend on time. Hence also the CAOs depend on time. Writing this time dependence explicitly, one has:

$$\text{Hamilton's equations}\quad \dot{A}^\pm_{\alpha k}(t) = \mp i\omega A^\pm_{\alpha k}(t), \tag{2.9}$$

$$\text{Heisenberg equations}\quad \dot{A}^\pm_{\alpha k}(t) = -\frac{i\omega}{3n-1}\sum_{\beta=1}^{n}\sum_{j=1}^{3}[A^\pm_{\alpha k}(t),\{A^+_{\beta j}(t),A^-_{\beta j}(t)\}]. \tag{2.10}$$

The solution of (2.9) is evident,

$$A^\pm_{\alpha k}(t) = A^\pm_{\alpha k}(0)\ e^{\mp i\omega t} \tag{2.11}$$

and therefore if the defining relations (2.8) hold at a certain time $t = 0$, i.e., for $A^\pm_{\alpha k} \equiv A^\pm_{\alpha k}(0)$, then they hold as equal time relations for any other time t. From (2.8) it follows also that the Eqs. (2.9) are identical with Eqs. (2.10). For further use we write the time dependence also of $\mathbf{R}_\alpha$ and $\mathbf{P}_\alpha$ explicitly:

$$\hat{R}_{\alpha k}(t) = \sqrt{\frac{\hbar}{(3n-1)m\omega}}(A^+_{\alpha k}e^{-i\omega t} + A^-_{\alpha k}e^{i\omega t}), \tag{2.12}$$

$$\hat{P}_{\alpha k}(t) = -i\sqrt{\frac{m\omega\hbar}{(3n-1)}}(A^+_{\alpha k}e^{-i\omega t} - A^-_{\alpha k}e^{i\omega t}). \tag{2.13}$$

Finally, the single particle angular momentum operators $\hat{M}_{\alpha j}$ defined in [2] by

$$\hat{M}_{\alpha j} = -\frac{3n-1}{2\hbar}\sum_{k,l=1}^{3}\ \epsilon_{jkl}\{\hat{R}_{\alpha k},\hat{P}_{\alpha l}\}\quad \alpha = 1,2,\ldots,n,\quad j = 1,2,3, \tag{2.14}$$

take the following form:

$$\hat{M}_{\alpha j} = -i\sum_{k,l=1}^{3}\epsilon_{jkl}\{A^+_{\alpha k},A^-_{\alpha l}\},\quad j = 1,2,3. \tag{2.15}$$

In terms of these operators the three components of the total angular momentum operator $\hat{\mathbf{M}}$ are given by

$$\hat{M}_j = \sum_{\alpha=1}^{n}\hat{M}_{\alpha j},\quad j = 1,2,3. \tag{2.16}$$

It is straightforward to verify that with respect to this choice of angular momentum operator $\hat{\mathbf{M}}$ the operators $\hat{\mathbf{R}}_\alpha$, $\hat{\mathbf{P}}_\alpha$, $\hat{\mathbf{M}}_\alpha$ and $\hat{\mathbf{M}}$ all transform as 3-vectors.

As indicated in Proposition 1 the CAO's $A^\pm_{\alpha i}$ with $\alpha = 1,2,\ldots,n$ and $i = 1,2,3$ generate the Lie superalgebra $sl(1|3n)$. This superalgebra has both finite and infinite-dimensional irreducible representations. Here we will consider only finite-dimensional irreducible representations. These have been classified by Kac [3] and are subdivided into typical and atypical irreducible representations. The typical irreducible representations coincide with the corresponding Kac-modules [13] for which there exists a rather simple character formula and a dimension formula. The same is not true of the atypical irreducible representations. Their dimensions are less than would be given by the dimensions of the corresponding Kac-modules [14].

The representations of $sl(1|3n)$ that are of interest here are those finite-dimensional covariant tensor irreducible representations, V^p, with highest weight $(p, 0, \ldots, 0)$ for some non-negative integer p. Such representations are typical if $p \geq 3n$ and atypical if $p < 3n$, and have dimension given by:

$$\dim V^p = \sum_{q=0}^{\min(p,3n)} \binom{3n}{q} = \begin{cases} \displaystyle\sum_{q=0}^{p} \binom{3n}{q} & \text{if } V^p \text{ is atypical, i.e. } p < 3n; \\ 2^{3n} & \text{if } V^p \text{ is typical, i.e. } p \geq 3n. \end{cases} \tag{2.17}$$

All of these irreducible representations, V^p, whether typical or atypical, may be constructed explicitly by means of the usual Fock space technique, as well as others for which p is not an integer. In our A-superstatistics case they may be constructed, precisely as in the parastatistics case [26], from the requirement that the corresponding representation space, $W(n,p)$, contains (up to a multiple) a unique cyclic vector $|\,0\rangle$ such that

$$A_{\alpha i}^{-}\,|\,0\rangle = 0, \quad A_{\alpha i}^{-}A_{\beta j}^{+}\,|\,0\rangle = p\delta_{\alpha\beta}\delta_{ij}\,|\,0\rangle, \quad i,j = 1,2,3, \quad \alpha,\beta = 1,2,\ldots,n. \tag{2.18}$$

The above relations are enough for the construction of the full representation space $W(n,p)$. This space defines an indecomposable finite-dimensional representation of the CAO's (2.8) and hence of $sl(1|3n)$ for any value of p. However we wish to impose the further physical requirements that:

(a) $W(n,p)$ is a Hilbert space with respect to the natural Fock space inner product;

(b) the observables, in particular the position and momentum operators (2.12)-(2.13), are Hermitian operators.

Condition (b) reduces to the requirement that the Hermitian conjugate of $A_{\alpha i}^{+}$ should be $A_{\alpha i}^{-}$, i.e.

$$(A_{\alpha i}^{\pm})^{\dagger} = A_{\alpha i}^{\mp}. \tag{2.19}$$

The condition (a) is then such that p is restricted to be a non-negative integer [27], in fact any non-negative integer. We then refer to p as the order of the statistics. As a consequence the representation space $W(n,p)$ is irreducible (and finite-dimensional). It provides a concrete realization of the irreducible representation V^p of $sl(1|3n)$, of dimension given by (2.17), as follows.

Let

$$\Theta \equiv (\theta_{11}, \theta_{12}, \theta_{13}, \theta_{21}, \theta_{22}, \theta_{23}, \ldots, \theta_{n1}, \theta_{n2}, \theta_{n3}). \tag{2.20}$$

The state space $W(n,p)$ of the system, corresponding to an order of statistics p, is spanned by the following orthonormal basis (called the Θ-basis):

$$\begin{aligned}
|\,p;\Theta\rangle &\equiv |\,p;.\theta_{\alpha 1},\theta_{\alpha 2},\theta_{\alpha 3}.\rangle \equiv |\,p;\theta_{11},\theta_{12},\theta_{13},\theta_{21},\theta_{22},\theta_{23},\ldots,\theta_{n1},\theta_{n2},\theta_{n3}\rangle \\
&= \sqrt{\frac{(p-q)!}{p!}}(A_{11}^{+})^{\theta_{11}}(A_{12}^{+})^{\theta_{12}}(A_{13}^{+})^{\theta_{13}}(A_{21}^{+})^{\theta_{21}}(A_{22}^{+})^{\theta_{22}}(A_{23}^{+})^{\theta_{23}}\cdots \\
&\times (A_{n1}^{+})^{\theta_{n1}}(A_{n2}^{+})^{\theta_{n2}}(A_{n3}^{+})^{\theta_{n3}}\,|\,0\rangle,
\end{aligned}\tag{2.21}$$

where

$$\theta_{\alpha i}\in\{0,1\}\ \text{ for all }\ \alpha=1,2,\ldots,n,\ i=1,2,3 \tag{2.22}$$

and

$$q\equiv\sum_{\alpha=1}^{n}\sum_{i=1}^{3}\theta_{\alpha i}\ \text{ with }\ 0\le q\le \min(p,3n). \tag{2.23}$$

The transformation of the basis states (2.21) under the action of the CAO's reads as follows:

$$A_{\alpha i}^{-}\,|\,p;\Theta\rangle = \theta_{\alpha i}(-1)^{\psi_{\alpha i}}\sqrt{p-q+1}\,|\,p;\Theta\rangle_{\overline{\alpha i}}, \tag{2.24}$$
$$A_{\alpha i}^{+}\,|\,p;\Theta\rangle = (1-\theta_{\alpha i})(-1)^{\psi_{\alpha i}}\sqrt{p-q}\,|\,p;\Theta\rangle_{\overline{\alpha i}}, \tag{2.25}$$

where

$$\psi_{\alpha i}=\sum_{(\beta j)<(\alpha i)}\theta_{\beta j}, \tag{2.26}$$

with the ordering on the pairs (αi) defined by (2.21), so that $(\beta j)<(\alpha i)$ if and only if either $\beta<\alpha$ or $\beta=\alpha$ and $j<i$, and $|\,p;\Theta\rangle_{\overline{\alpha i}}$ are the states obtained from $|\,p;\Theta\rangle$ after the replacement of $\theta_{\alpha i}$ by $\overline{\theta}_{\alpha i}=1-\theta_{\alpha i}$.

In what follows we shall also require the explicit action of the anticommutators $\{A_{\alpha i}^{+},A_{\beta j}^{-}\}$ on the states $|\,p;\Theta\rangle$. This is given by:

$$\{A_{\alpha i}^{+},A_{\beta j}^{-}\}\,|\,p;\Theta\rangle = \begin{cases} (p-q+\theta_{\alpha i})\,|\,p;\Theta\rangle & \text{if } (\alpha i)=(\beta j); \\ (-1)^{\psi_{\beta j}-\psi_{\alpha i}}\theta_{\beta j}(1-\theta_{\alpha i})\,|\,p;\Theta\rangle_{\overline{\alpha i},\overline{\beta j}} & \text{if } (\alpha i)<(\beta j); \\ -(-1)^{\psi_{\alpha i}-\psi_{\beta j}}\theta_{\beta j}(1-\theta_{\alpha i})\,|\,p;\Theta\rangle_{\overline{\alpha i},\overline{\beta j}} & \text{if } (\alpha i)>(\beta j). \end{cases} \tag{2.27}$$

Here $|\,p;\Theta\rangle_{\overline{\alpha i},\overline{\beta j}}$ denotes the state obtained from $|\,p;\Theta\rangle$ by replacing $\theta_{\alpha i}$ and $\theta_{\beta j}$ with $\overline{\theta}_{\alpha i}=1-\theta_{\alpha i}$ and $\overline{\theta}_{\beta j}=1-\theta_{\beta j}$, respectively.

Returning to (2.21), note the first big difference between the non-canonical WQO and the case of a conventional CQO: *each state space $W(n,p)$ of our WQO is finite-dimensional.* In fact the dimension is easily seen from (2.21) to coincide with that given for V^p by (2.17).

On the other hand in the CQO case it is well known that the corresponding (bosonic) Fock space is spanned by the states

$$
\begin{aligned}
&|\,\Phi\rangle \equiv |\,\phi_{11},\phi_{12},\phi_{13},\phi_{21},\phi_{22},\phi_{23},\ldots,\phi_{n1},\phi_{n2},\phi_{n3}\rangle = \prod_{\alpha=1}^{n}\prod_{i=1}^{3}\frac{1}{\sqrt{\phi_{\alpha i}!}} \\
&\times(B_{11}^+)^{\phi_{11}}(B_{12}^+)^{\phi_{12}}(B_{13}^+)^{\phi_{13}}(B_{21}^+)^{\phi_{21}}(B_{22}^+)^{\phi_{22}}(B_{23}^+)^{\phi_{23}}\cdots(B_{n1}^+)^{\phi_{n1}}(B_{n2}^+)^{\phi_{n2}}(B_{n3}^+)^{\phi_{n3}}\,|\,0\rangle
\end{aligned}
\tag{2.28}
$$

with

$$
\phi_{\alpha i}\in\{0,1,2,\ldots\}\ \text{ for all }\ \alpha=1,2,\ldots,n,\quad i=1,2,3. \tag{2.29}
$$

This space is clearly infinite-dimensional. The action of the bosonic operators on these states is given by:

$$
B_{\alpha i}^-\,|\,\Phi\rangle = \sqrt{\phi_{\alpha i}}\,|\,\Phi\rangle_{-\alpha i}\ \text{ and }\ B_{\alpha i}^+\,|\,\Phi\rangle = \sqrt{\phi_{\alpha i}+1}\,|\,\Phi\rangle_{+\alpha i} \tag{2.30}
$$

where $|\,\Phi\rangle_{\pm\alpha i}$ are the states obtained from $|\,\Phi\rangle$ by the replacement of $\phi_{\alpha i}$ by $\phi_{\alpha i}\pm 1$, and there is no upper bound on $\phi_{\alpha i}$.

3 Physical properties - energy spectrum and angular momentum

We now discuss some of the physical properties of the Wigner quantum oscillator (WQO), comparing them with those of the canonical quantum oscillator (CQO).

The first thing to note is that, as in the case of the CQO, the physical observables $\hat{H}$, $\hat{H}_\alpha$, $\hat{\mathbf{R}}_\alpha$, $\hat{\mathbf{P}}_\alpha$, $\hat{\mathbf{M}}$, and $\hat{\mathbf{M}}_\alpha$ for $\alpha=1,2,\ldots,n$ are, in the case of the WQO, all Hermitian operators within every Hilbert space $W(n,p)$ for each $p=0,1,\ldots$.

Secondly, in the case of the WQO the Hamiltonian $\hat{H}$ is diagonal in the basis (2.21)-(2.23), i.e. the basis vectors $|\,p;\Theta\rangle$ are stationary states of the system. In each Hilbert space $W(n,p)$ there is a finite number of equally spaced energy levels, with spacing $\hbar\omega$:

$$
\hat{H}\,|\,p;\Theta\rangle = E_q\,|\,p;\Theta\rangle\ \text{ with }\ E_q=\hbar\omega\left(\frac{3np}{3n-1}-q\right)\ \text{ for } q=0,1,2,\ldots,\min(3n,p). \tag{3.1}
$$

Similarly, for the CQO the Hamiltonian $\hat{H}$ is diagonal in the basis (2.28)-(2.29), so that the basis vectors $|\,\Phi\rangle$ are stationary states. Now however, there is an infinite number of equally spaced energy levels, but with the same spacing $\hbar\omega$:

$$
\hat{H}\,|\,\Phi\rangle = E_q\,|\,\Phi\rangle\ \text{ where }\ E_q=\hbar\omega\left(\frac{3}{2}n+q\right)\ \text{ with }\ q=\sum_{\alpha=1}^{n}\sum_{i=1}^{3}\phi_{\alpha i}=0,1,2,\ldots \tag{3.2}
$$

The fact that the energy spectrum of the WQO is as given in (3.1) can be seen by noting that under the restriction from the Lie superalgebra $sl(1|3n)$ to its reductive

Lie subalgebra $gl(1) \oplus sl(3n)$ the representation $W(n,p) = V^{p}_{sl(1|3n)}$ decomposes in accordance with the branching rule [17]-[20]:

$$
\begin{aligned}
sl(1|3n) \quad &\longrightarrow gl(1) \oplus sl(3n) \\
V^{p}_{sl(1|3n)} \quad &\longrightarrow \sum_{q=0}^{\min(p,3n)} V^{q+3n(p-q)}_{gl(1)} \otimes V^{1^q}_{sl(3n)},
\end{aligned}
\tag{3.3}
$$

where the subscripts on the representation labels indicate the relevant Lie algebra or superalgebra, and the superscripts are the highest weights of the representation written in partition notation. Here the Hamiltonian is just $\hbar\omega/(3n-1)$ times the generator, $\sum_{\alpha=1}^{n}\sum_{i=1}^{3}\{A^+_{\alpha i}, A^-_{\alpha i}\}$, of $gl(1)$, so that its eigenvalues E_q are precisely as given in (3.1).

The branching rule (3.3) gives some additional information. The notation in (3.3) is such that 1^q signifies the partition $(1,1,\ldots,1)$ all of whose q non-vanishing parts are 1. Thus $V^{1^q}_{sl(3n)}$ signifies the qth rank totally antisymmetric covariant irreducible representation of $sl(3n)$. The degeneracy of the equally spaced states of energy E_q is then given by

$$
\dim V^{1^q}_{sl(3n)} = \binom{3n}{q}, \tag{3.4}
$$

as required for consistency with (2.17).

In exactly the same way the fact that the energy spectrum of the CQO is as given by (3.2) can be seen by considering the restriction from the Lie superalgebra $osp(1|6n)$ first to its even Lie subalgebra $sp(6n)$, generated by $\{B^{\xi}_{\alpha i}, B^{\eta}_{\beta j}\}$ for all $\alpha,\beta = 1,2,\ldots,n$, $i,j = 1,2,3$ and $\xi,\eta = \pm$ [28], and then to the reductive Lie subalgebra $gl(1) \oplus sl(3n)$. Let the infinite-dimensional irreducible representation of $osp(1|6n)$ spanned by the basis states $|\,\Phi\rangle$ given by (2.28)-(2.29) be denoted by $V^{\varepsilon}_{osp(1|6n)}$, where ε is the weight vector $(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$ in the relevant $3n$-dimensional weight space. This decomposes first into the sum of the two infinite-dimensional irreducible metaplectic or oscillator representations of $sp(6n)$ [29]-[30] of $sp(6n)$, which we denote here by $V^{\varepsilon\pm}_{sp(6n)}$, and then into finite-dimensional irreducible representations of $gl(1) \oplus sl(3n)$, all in accordance with the following branching rules:

$$
\begin{aligned}
&osp(1|6n) \longrightarrow sp(6n) \longrightarrow gl(1) \oplus sl(3n) \\
&V^{\varepsilon}_{osp(1|6n)} \longrightarrow V^{\varepsilon+}_{sp(6n)} \oplus V^{\varepsilon-}_{sp(6n)} \longrightarrow \sum_{q=0}^{\infty} V^{\frac{3}{2}n+q}_{gl(1)} \otimes V^{q}_{sl(3n)}.
\end{aligned}
\tag{3.5}
$$

This time the Hamiltonian is $\hbar\omega$ times the generator, $\frac{1}{2}\sum_{\alpha=1}^{n}\sum_{i=1}^{3}\{B^+_{\alpha i}, B^-_{\alpha i}\}$, of $gl(1)$, so that its eigenvalues E_q are as given in (3.2).

Once again (3.5) carries additional information on degeneracies. This time in (3.5) the superscript q signifies the one part partition $(q,0,\ldots,0)$. Thus $V^{q}_{sl(3n)}$ signifies the qth rank totally symmetric covariant irreducible representation of $sl(3n)$. The degeneracy of the equally spaced states of energy E_q in this CQO case is then given by

$$\dim V^q_{sl(3n)} = \begin{pmatrix} 3n-1+q \\ q \end{pmatrix}. \tag{3.6}$$

This analysis shows that the energy levels of both the WQO and the CQO are equally spaced, both with separation $\hbar\omega$, with degeneracy formulae that are rather similar, albeit with the CQO having higher degeneracies than those of the WQO. In fact the analogy between them is somewhat closer if one compares the infinite-dimensional CQO spectrum not with the finite-dimensional spectrum associated with the WQO for any fixed, finite p, but with the combination of all such WQO spectra for all non-negative integer values of the order of the statistics p.

Turning now to the angular momentum, it follows from (2.15) and (2.27) that

$$\hat{M}_{\alpha 1}\,|\,p;.\theta_{\alpha 1},\theta_{\alpha 2},\theta_{\alpha 3}.\rangle = i(\theta_{\alpha 2}-\theta_{\alpha 3})\,|\,p;.\theta_{\alpha 1},\overline{\theta}_{\alpha 2},\overline{\theta}_{\alpha 3},.\rangle; \tag{3.7}$$
$$\hat{M}_{\alpha 2}\,|\,p;.\theta_{\alpha 1},\theta_{\alpha 2},\theta_{\alpha 3}.\rangle = i(-1)^{\theta_{\alpha 2}}(\theta_{\alpha 3}-\theta_{\alpha 1})\,|\,p;.\overline{\theta}_{\alpha 1},\theta_{\alpha 2},\overline{\theta}_{\alpha 3}.\rangle; \tag{3.8}$$
$$\hat{M}_{\alpha 3}\,|\,p;.\theta_{\alpha 1},\theta_{\alpha 2},\theta_{\alpha 3}.\rangle = i(\theta_{\alpha 1}-\theta_{\alpha 2})\,|\,p;.\overline{\theta}_{\alpha 1},\overline{\theta}_{\alpha 2},\theta_{\alpha 3}.\rangle. \tag{3.9}$$

By exploiting these results one then obtains

$$\hat{\mathbf{M}}^2_\alpha\,|\,p;\Theta\rangle = \delta_\alpha\, 2\,\,|\,p;\Theta\rangle, \tag{3.10}$$

where

$$\delta_\alpha = \begin{cases} 0 & \text{if } \theta_{\alpha 1}=\theta_{\alpha 2}=\theta_{\alpha 3}; \\ 1 & \text{otherwise.} \end{cases} \tag{3.11}$$

Thus the stationary states $|\,p;\Theta\rangle$ are eigenstates of the squares, $\hat{\mathbf{M}}^2_\alpha$, of the *single particle* angular momentum operator $\hat{\mathbf{M}}_\alpha$, with eigenvalues 0 or 2 for all $\alpha = 1,2,\ldots,n$. Thus the WQO behaves like a collection of spin zero and spin one particles.

However, the stationary states $|\,p;\Theta\rangle$ are not eigenstates of either $\hat{M}_3$ or $\hat{\mathbf{M}}^2$, the third component and the square, respectively, of the *total* angular momentum operator $\hat{\mathbf{M}}$, as can be seen from the following:

$$\hat{M}_3\,|\,p;\Theta\rangle = i\sum_{\alpha=1}^{n}(\theta_{\alpha 1}-\theta_{\alpha 2})\,|\,p;.\overline{\theta}_{\alpha 1},\overline{\theta}_{\alpha 2},\theta_{\alpha 3}.\rangle. \tag{3.12}$$

$$\begin{aligned}
\hat{\mathbf{M}}^2\,|\,p;\Theta\rangle &= \sum_{\alpha=1}^{n}\delta_\alpha\; 2\;\,|\,p;\Theta\rangle \\
-2 &\sum_{1\le\alpha<\beta\le n}(\theta_{\alpha 1}-\theta_{\alpha 2})(\theta_{\beta 1}-\theta_{\beta 2})\,|\,p;.\overline{\theta}_{\alpha 1},\overline{\theta}_{\alpha 2},\theta_{\alpha 3},\ldots,\overline{\theta}_{\beta 1},\overline{\theta}_{\beta 2},\theta_{\beta 3}.\rangle \\
-2 &\sum_{1\le\alpha<\beta\le n}(-1)^{\theta_{\alpha 2}+\theta_{\beta 2}}(\theta_{\alpha 3}-\theta_{\alpha 1})(\theta_{\beta 3}-\theta_{\beta 1})\,|\,p;.\overline{\theta}_{\alpha 1},\theta_{\alpha 2},\overline{\theta}_{\alpha 3},\ldots,\overline{\theta}_{\beta 1},\theta_{\beta 2},\overline{\theta}_{\beta 3}.\rangle \\
-2 &\sum_{1\le\alpha<\beta\le n}(\theta_{\alpha 2}-\theta_{\alpha 3})(\theta_{\beta 2}-\theta_{\beta 3})\,|\,p;.\theta_{\alpha 1},\overline{\theta}_{\alpha 2},\overline{\theta}_{\alpha 3},\ldots,\theta_{\beta 1},\overline{\theta}_{\beta 2},\overline{\theta}_{\beta 3}.\rangle.
\end{aligned} \tag{3.13}$$

For general values of the particle number n it is not easy to determine from these expressions all the total angular momentum eigenstates, that is the simultaneous eigenvectors of $\hat{M}_3$ and $\hat{\mathbf{M}}^2$. However, to determine the possible values of the total angular momentum, M, for the WQO we can proceed in a different way by extending further our restriction (3.3) in accordance with the chain:

$$sl(1|3n) \rightarrow gl(1)\oplus sl(3n) \rightarrow gl(1)\oplus sl(3)\oplus sl(n) \rightarrow gl(1)\oplus so(3)\oplus sl(n) \rightarrow gl(1)\oplus so(3), \tag{3.14}$$

where it is the subalgebra $so(3)$ which is associated with the total angular momentum of the system. The branching rule for $sl(3n) \rightarrow sl(3) \oplus sl(n)$ required in the second step and that for $sl(3) \rightarrow so(3)$ required in the third step are both rather well known and have been implemented for example in SCHUR[e]. Since they involve coefficients for which there is no known general formula, we content ourselves with giving the results explicitly just for the two cases $n = 1$ and $n = 2$.

In the case of $sl(1|3)$ we find:

$$\begin{aligned} sl(1|3) \quad &\longrightarrow \quad gl(1) \oplus so(3) \\ V^p_{sl(1|3)} \quad &\longrightarrow \quad \chi_{p\geq 0}\, V^{3p}_{gl(1)} \otimes V^0_{so(3)} + \chi_{p\geq 1}\, V^{3p-2}_{gl(1)} \otimes V^1_{so(3)} \\ &\qquad +\chi_{p\geq 2}\, V^{3p-4}_{gl(1)} \otimes V^1_{so(3)} + \chi_{p\geq 3}\, V^{3p-6}_{gl(1)} \otimes V^0_{so(3)}, \end{aligned} \tag{3.15}$$

where $\chi_{p\geq x}$ is 1 if $p \geq x$ and 0 otherwise. Each term of the form $V^{3p-2q}_{gl(1)} \otimes V^M_{so(3)}$ corresponds to a set of $2M+1$ states of energy $E_q = \hbar\omega(3p-2q)/2$, as given by (3.1) with $n = 1$, all having total angular momentum M.

In this one particle, $n = 1$, case it is easy to identify from (3.7)-(3.11) with $\alpha = 1$ all the angular momentum eigenstates, that is the simultaneous eigenvectors of $\hat{\mathbf{M}}^2$ and $\hat{M}_3$. They are the linear combinations of the stationary states $|\,p;\Theta\rangle$ identified in Table 1.

Table 1. One-particle eigenstates of angular momentum

$p \geq q$	$\|\,p;\Theta\rangle$	q	E_q	M	M_3
$p \geq 0$	$\|\,p;0,0,0\rangle$	0	$\frac{\hbar\omega}{2}3p$	0	0
$p \geq 1$	$\frac{1}{\sqrt{2}}(\|\,p;1,0,0\rangle + i\,\|\,p;0,1,0\rangle)$	1	$\frac{\hbar\omega}{2}(3p-2)$	1	1
$p \geq 1$	$\|\,p;0,0,1\rangle$	1	$\frac{\hbar\omega}{2}(3p-2)$	1	0
$p \geq 1$	$\frac{1}{\sqrt{2}}(\|\,p;1,0,0\rangle - i\,\|\,p;0,1,0\rangle)$	1	$\frac{\hbar\omega}{2}(3p-2)$	1	-1
$p \geq 2$	$\frac{1}{\sqrt{2}}(\|\,p;1,0,1\rangle + i\,\|\,p;0,1,1\rangle)$	2	$\frac{\hbar\omega}{2}(3p-4)$	1	1
$p \geq 2$	$\|\,p;1,1,0\rangle$	2	$\frac{\hbar\omega}{2}(3p-4)$	1	0
$p \geq 2$	$\frac{1}{\sqrt{2}}(\|\,p;1,0,1\rangle - i\,\|\,p;0,1,1\rangle)$	2	$\frac{\hbar\omega}{2}(3p-4)$	1	-1
$p \geq 3$	$\|\,p;1,1,1\rangle$	3	$\frac{\hbar\omega}{2}(3p-6)$	0	0

[e]SCHUR, an interactive program for calculating properties of Lie groups and symmetric functions, distributed by S. Christensen. E-mail: steve@scm.vnet.net; http//scm.vnet.net/Christensen.html

Thus in the atypical cases, $p=0$, $p=1$ and $p=2$ it is easy to see that the dimensions of the corresponding irreducible representations of $sl(1|3)$ are 1, 4 and 7, respectively, while for the typical cases $p\geq 3$ the dimension is 8, all in accordance with (2.17).

In the two particle case, that is for $sl(1|6)$ we find:

$$\begin{aligned}
sl(1|6) &\longrightarrow gl(1)\oplus so(3)\oplus sl(2)\\
V^p_{sl(1|6)} &\longrightarrow \chi_{p\geq 0}\, V^{6p}_{gl(1)}\otimes\big(V^0_{so(3)}\otimes V^0_{sl(2)}\big)+\chi_{p\geq 1}\, V^{6p-5}_{gl(1)}\otimes\big(V^1_{so(3)}\otimes V^1_{sl(2)}\big)\\
&+\chi_{p\geq 2}\, V^{6p-10}_{gl(1)}\otimes\big(V^2_{so(3)}\otimes V^0_{sl(2)}+V^1_{so(3)}\otimes V^2_{sl(2)}+V^0_{so(3)}\otimes V^0_{sl(2)}\big)\\
&+\chi_{p\geq 3}\, V^{6p-15}_{gl(1)}\otimes\big(V^2_{so(3)}\otimes V^1_{sl(2)}+V^1_{so(3)}\otimes V^1_{sl(2)}+V^0_{so(3)}\otimes V^3_{sl(2)}\big)\\
&+\chi_{p\geq 4}\, V^{6p-20}_{gl(1)}\otimes\big(V^2_{so(3)}\otimes V^0_{sl(2)}+V^1_{so(3)}\otimes V^2_{sl(2)}+V^0_{so(3)}\otimes V^0_{sl(2)}\big) \quad (3.16)\\
&+\chi_{p\geq 5}\, V^{6p-25}_{gl(1)}\otimes\big(V^1_{so(3)}\otimes V^1_{sl(2)}\big)+\chi_{p\geq 6}\, V^{6p-30}_{gl(1)}\otimes\big(V^0_{so(3)}\otimes V^0_{sl(2)}\big)\,.
\end{aligned}$$

Since the dimension of each irreducible representation $V^s_{sl(2)}$ of $sl(2)$ is just $s+1$, it follows that:

$$\begin{aligned}
sl(1|6) &\longrightarrow gl(1)\oplus so(3)\\
V^p_{sl(1|6)} &\longrightarrow \chi_{p\geq 0}\, V^{6p}_{gl(1)}\otimes V^0_{so(3)}+\chi_{p\geq 1}\, V^{6p-5}_{gl(1)}\otimes 2V^1_{so(3)}\\
&+\chi_{p\geq 2}\, V^{6p-10}_{gl(1)}\otimes\big(V^2_{so(3)}+3V^1_{so(3)}+V^0_{so(3)}\big)\\
&+\chi_{p\geq 3}\, V^{6p-15}_{gl(1)}\otimes\big(2V^2_{so(3)}+2V^1_{so(3)}+4V^0_{so(3)}\big)\\
&+\chi_{p\geq 4}\, V^{6p-20}_{gl(1)}\otimes\big(V^2_{so(3)}+3V^1_{so(3)}+V^0_{so(3)}\big)\\
&+\chi_{p\geq 5}\, V^{6p-25}_{gl(1)}\otimes 2V^1_{so(3)}+\chi_{p\geq 6}\, V^{6p-30}_{gl(1)}\otimes V^0_{so(3)}, \qquad (3.17)
\end{aligned}$$

where now each term of the form $V^{6p-5q}_{gl(1)}\otimes kV^M_{so(3)}$ corresponds to k sets of $2M+1$ states of energy $E_q=\hbar\omega(6p-5q)/5$, as given by (3.1) with $n=2$, all having total angular momentum M.

Of course, for any particular value of $p<6$ not all of the above terms will survive, as can be seen from the various factors $\chi_{p\geq x}$. For example if $n=2$ and $p=3$ we obtain

$$\begin{aligned}
sl(1|6) &\longrightarrow gl(1)\oplus so(3)\\
V^3_{sl(1|6)} &\longrightarrow V^{18}_{gl(1)}\otimes V^0_{so(3)}+V^{13}_{gl(1)}\otimes 2V^1_{so(3)}+V^8_{gl(1)}\otimes\big(V^2_{so(3)}+3V^1_{so(3)}+V^0_{so(3)}\big)\\
&+V^3_{gl(1)}\otimes\big(2V^2_{so(3)}+2V^1_{so(3)}+4V^0_{so(3)}\big)\,. \qquad (3.18)
\end{aligned}$$

In the two particle, $n=2$, case it is not quite so easy to identify all the eigenstates of both $\hat{\mathbf{M}}^2$ and $\hat{M}_3$. In general they are now certain linear combinations of the stationary states $|\,p;\Theta\rangle$. Rather than give all 64 such linear combinations, we content ourselves with specifying in Table 2 only those eigenstates of $\hat{\mathbf{M}}^2$ with eigenvalues $M(M+1)$ for which $M_3=M$. The remaining states may be obtained from these through the action of $\hat{M}_-$, where

$$\hat{M}_\pm=\hat{M}_1\pm i\hat{M}_2=\sum_{\alpha=1}^{n}\hat{M}_{\alpha\pm}\quad\text{with }\hat{M}_{\alpha\pm}=\hat{M}_{\alpha 1}\pm i\hat{M}_{\alpha 2}. \qquad (3.19)$$

Table 2. Two-particle eigenstates of angular momentum having the maximum value M of M_3

$p\geq q$	Orthonormal angular momentum eigenstates with $M=M_3$ as linear combinations of $\mid p;\Theta\rangle$	q	E_q	M	M_3
$p\geq 0$	$\mid p;000000\rangle$	0	$\frac{\hbar\omega}{5}6p$	0	0
$p\geq 1$	$\frac{1}{\sqrt{2}}(\mid p;010000\rangle - i\mid p;100000\rangle)$	1	$\frac{\hbar\omega}{5}(6p-5)$	1	1
$p\geq 1$	$\frac{1}{\sqrt{2}}(\mid p;000010\rangle - i\mid p;000100\rangle)$	1	$\frac{\hbar\omega}{5}(6p-5)$	1	1
$p\geq 2$	$\frac{1}{2}(\mid p;010010\rangle - i\mid p;100010\rangle - i\mid p;010100\rangle - \mid p;100100\rangle)$	2	$\frac{\hbar\omega}{5}(6p-10)$	2	2
$p\geq 2$	$\frac{1}{\sqrt{2}}(\mid p;011000\rangle - i\mid p;101000\rangle)$	2	$\frac{\hbar\omega}{5}(6p-10)$	1	1
$p\geq 2$	$\frac{1}{\sqrt{2}}(\mid p;000011\rangle - i\mid p;000101\rangle)$	2	$\frac{\hbar\omega}{5}(6p-10)$	1	1
$p\geq 2$	$\frac{1}{2}(\mid p;010001\rangle - \mid p;001010\rangle + i\mid p;001100\rangle - i\mid p;100001\rangle)$	2	$\frac{\hbar\omega}{5}(6p-10)$	1	1
$p\geq 2$	$\frac{1}{\sqrt{3}}(\mid p;100100\rangle + \mid p;010010\rangle + \mid p;001001\rangle)$	2	$\frac{\hbar\omega}{5}(6p-10)$	0	0
$p\geq 3$	$\frac{1}{2}(\mid p;010011\rangle - \mid p;100101\rangle - i\mid p;010101\rangle - i\mid p;100011\rangle)$	3	$\frac{\hbar\omega}{5}(6p-15)$	2	2
$p\geq 3$	$\frac{1}{2}(\mid p;011010\rangle - \mid p;101100\rangle - i\mid p;011100\rangle - i\mid p;101010\rangle)$	3	$\frac{\hbar\omega}{5}(6p-15)$	2	2
$p\geq 3$	$\frac{1}{2}(\mid p;001011\rangle - \mid p;100110\rangle - i\mid p;001101\rangle - i\mid p;010110\rangle)$	3	$\frac{\hbar\omega}{5}(6p-15)$	1	1
$p\geq 3$	$\frac{1}{2}(\mid p;011001\rangle - \mid p;110100\rangle - i\mid p;110010\rangle - i\mid p;101001\rangle)$	3	$\frac{\hbar\omega}{5}(6p-15)$	1	1
$p\geq 3$	$\mid p;000111\rangle$	3	$\frac{\hbar\omega}{5}(6p-15)$	0	0
$p\geq 3$	$\mid p;111000\rangle$	3	$\frac{\hbar\omega}{5}(6p-15)$	0	0
$p\geq 3$	$\frac{1}{\sqrt{3}}(\mid p;100011\rangle + \mid p;001110\rangle - \mid p;010101\rangle)$	3	$\frac{\hbar\omega}{5}(6p-15)$	0	0
$p\geq 3$	$\frac{1}{\sqrt{3}}(\mid p;011100\rangle + \mid p;110001\rangle - \mid p;101010\rangle)$	3	$\frac{\hbar\omega}{5}(6p-15)$	0	0
$p\geq 4$	$\frac{1}{2}(\mid p;011011\rangle - \mid p;101101\rangle - i\mid p;011101\rangle - i\mid p;101011\rangle)$	4	$\frac{\hbar\omega}{5}(6p-20)$	2	2
$p\geq 4$	$\frac{1}{\sqrt{2}}(\mid p;010111\rangle - i\mid p;100111\rangle)$	4	$\frac{\hbar\omega}{5}(6p-20)$	1	1
$p\geq 4$	$\frac{1}{\sqrt{2}}(\mid p;111010\rangle - i\mid p;111010\rangle)$	4	$\frac{\hbar\omega}{5}(6p-20)$	1	1
$p\geq 4$	$\frac{1}{2}(\mid p;011110\rangle - \mid p;110011\rangle - i\mid p;101110\rangle + i\mid p;110101\rangle)$	4	$\frac{\hbar\omega}{5}(6p-20)$	1	1
$p\geq 4$	$\frac{1}{\sqrt{3}}(\mid p;101101\rangle + \mid p;110110\rangle + \mid p;011011\rangle)$	4	$\frac{\hbar\omega}{5}(6p-20)$	0	0
$p\geq 5$	$\frac{1}{\sqrt{2}}(\mid p;011111\rangle - i\mid p;101111\rangle)$	5	$\frac{\hbar\omega}{5}(6p-25)$	1	1
$p\geq 5$	$\frac{1}{\sqrt{2}}(\mid p;111011\rangle - i\mid p;111101\rangle)$	5	$\frac{\hbar\omega}{5}(6p-25)$	1	1
$p\geq 6$	$\mid p;111111\rangle$	6	$\frac{\hbar\omega}{5}(6p-30)$	0	0

In the case of the CQO, the results analogous to (3.15) and (3.17) take the form

$$\begin{aligned} osp(1|6) &\longrightarrow gl(1)\oplus so(3) \\ V^{\varepsilon}_{osp(1|6)} &\longrightarrow V^{3/2}_{gl(1)}\otimes V^{0}_{so(3)} + V^{5/2}_{gl(1)}\otimes V^{1}_{so(3)} + V^{7/2}_{gl(1)}\otimes\left(V^{2}_{so(3)}+V^{0}_{so(3)}\right) \\ &\qquad +V^{9/2}_{gl(1)}\otimes\left(V^{3}_{so(3)}+V^{1}_{so(3)}\right)+\cdots \end{aligned} \tag{3.20}$$

where each term of the form $V_{gl(1)}^{(3+2q)/2}\otimes V_{so(3)}^{M}$ corresponds to a set of $2M+1$ states of energy $E_q=\hbar\omega(3+2q)/2$, as given by (3.2) with $n=1$, all having total angular momentum M.

Similarly,

$$\begin{array}{lcl} osp(1|12) & \longrightarrow & gl(1)\oplus so(3) \\ V^{\varepsilon}_{osp(1|12)} & \longrightarrow & V^3_{gl(1)}\otimes V^0_{so(3)} + V^4_{gl(1)}\otimes 2V^1_{so(3)} + V^5_{gl(1)}\otimes \left(3V^2_{so(3)}+V^1_{so(3)}+3V^0_{so(3)}\right) \\ & & +V^6_{gl(1)}\otimes\left(4V^3_{so(3)}+2V^2_{so(3)}+6V^1_{so(3)}\right)+\cdots \end{array} \tag{3.21}$$

where now each term of the form $V_{gl(1)}^{3+q}\otimes kV_{so(3)}^{M}$ corresponds to k sets of $2M+1$ states of energy $E_q=\hbar\omega(3+q)$, as given by (3.2) with $n=2$, all having total angular momentum M. It is notable that in the CQO case the degeneracies are larger than in the case of the WQO, and of course the CQO case is infinite-dimensional as compared with the fixed p finite-dimensional case of the WQO, illustrated for example by (3.18).

4 Physical properties - oscillator configurations

It is convenient to work not with the time dependent position operators $\hat{R}_{\alpha k}(t)$ themselves, but with their dimensionless version defined by

$$\hat{r}_{\alpha k}(t)=\sqrt{\frac{(3n-1)m\omega}{\hbar}}\,\hat{R}_{\alpha k}(t)=A^+_{\alpha k}\,e^{-i\omega t}+A^-_{\alpha k}\,e^{i\omega t}, \tag{4.1}$$

for $k=1,2,3$ and $\alpha=1,2,\dots,n$. It then follows from (2.8) that the squares of these operators are time independent and given by

$$\hat{r}^2_{\alpha k}=\{A^+_{\alpha k},A^-_{\alpha k}\}. \tag{4.2}$$

The first part of (2.27) then implies that

$$\hat{r}^2_{\alpha k}\,|\,p;\Theta\rangle=r^2_{\alpha k}\,|\,p;\Theta\rangle, \tag{4.3}$$

with

$$r^2_{\alpha k}=p-q+\theta_{\alpha k}. \tag{4.4}$$

Since

$$[\hat{r}^2_{\alpha i},\hat{r}^2_{\beta j}]=0 \quad \text{for all } \alpha,\beta=1,2,\dots,n \text{ and } i,j=1,2,3, \tag{4.5}$$

we are led to the following:

Conclusion 1 *If the system is in one of the Θ-basis states $|\,p;\Theta\rangle$, then measurements of the coordinates $r_{\alpha k}$ of the αth particle can yield only the values*

$$r_{\alpha k}=\pm\sqrt{p-q+\theta_{\alpha k}} \quad \textit{for } k=1,2,3. \tag{4.6}$$

These values define the position of 8 nests on a sphere of radius

$$\rho_\alpha=\sqrt{3p-3q+q_\alpha} \quad \textit{where } q_\alpha=\textstyle\sum_{k=1}^3\theta_{\alpha k}. \tag{4.7}$$

This result is completely analagous to that found in the single particle case, (cf. I Conclusion 2).

The basis states $|\,p;\Theta\rangle$ are not eigenstates of $\hat{r}_{\alpha k}(t)$. In fact

$$\hat{r}_{\alpha k}(t)\,|\,p;\ldots,\theta_{\alpha k},\ldots\rangle = (-1)^{\psi_{\alpha k}}\left(e^{-i\omega t}\overline{\theta}_{\alpha k} + e^{i\omega t}\theta_{\alpha k}\right)\sqrt{p-q+\theta_{\alpha k}}\,|\,p;\ldots,\overline{\theta}_{\alpha k},\ldots\rangle \tag{4.8}$$

However, it is not difficult to identify in $W(n,p)$ the eigenvectors of $\hat{r}_{\alpha k}(t)$ for any $\alpha = 1,\ldots,n$ and $k = 1,2,3$. These are given by:

$$v_{\alpha k}(\Theta) = \frac{1}{\sqrt{2}}(\,|\,p;\Theta_{\theta_{\alpha k}=0}\rangle + (-1)^{\psi_{\alpha k}+\theta_{\alpha k}}e^{-i\omega t}\,|\,p;\Theta_{\theta_{\alpha k}=1}\rangle), \tag{4.9}$$

where $\Theta_{\theta_{\alpha k}=0}$ stands for the Θ-value specified by the left hand side of (4.9) in which $\theta_{\alpha k}$ is replaced by 0 (and similarly for $\Theta_{\theta_{\alpha k}=1}$). The vectors $v_{\alpha k}(\Theta)$ constitute an orthonormal basis of eigenvectors of $\hat{r}_{\alpha k}(t)$ in $W(n,p)$. It is found that

$$\hat{r}_{\alpha k}(t)v_{\alpha k}(\Theta) = (-1)^{\theta_{\alpha k}}\sqrt{p-q+\theta_{\alpha k}}\,v_{\alpha k}(\Theta), \tag{4.10}$$

confirming the fact that the eigenvalues of $\hat{r}_{\alpha k}(t)$ are given by $\pm\sqrt{p-q+\theta_{\alpha k}}$. The inverse relations of (4.9) are easy to write down. They take the form:

$$|\,p;\Theta\rangle = \frac{1}{\sqrt{2}}(-1)^{\psi_{\alpha k}\theta_{\alpha k}}e^{i\omega t\theta_{\alpha k}}(v_{\alpha k}(\Theta_{\theta_{\alpha k}=0}) + (-1)^{\theta_{\alpha k}}v_{\alpha k}(\Theta_{\theta_{\alpha k}=1})). \tag{4.11}$$

As pointed out in I, the interpretation of geometric results for the WQO must be undertaken carefully since the underlying geometry is non-commutative. This can be seen by noting that for all $(\alpha i) < (\beta j)$ with $\alpha,\beta = 1,2,\ldots,n$ and $i,j = 1,2,3$

$$\begin{aligned}
&[\hat{r}_{\alpha i}(t),\hat{r}_{\beta j}(t)]\,|\,p;\ldots,\theta_{ai},\ldots,\theta_{\beta j},\ldots\rangle \\
&= (-1)^{\psi_{\beta j}-\psi_{\alpha i}}\Big(2e^{i2\omega t}\theta_{\alpha i}\theta_{\beta j}\sqrt{(p-q+1)(p-q+2)} + (\theta_{\alpha i}-\theta_{\beta j})^2(2p-2q+1) \\
&\quad +2e^{-i2\omega t}\overline{\theta}_{\alpha i}\overline{\theta}_{\beta j}\sqrt{(p-q-1)(p-q)}\Big)\,|\,p;\ldots,\overline{\theta}_{ai},\ldots,\overline{\theta}_{\beta j},\ldots\rangle.
\end{aligned} \tag{4.12}$$

The right hand side of this expression is non-zero for all $p \geq q+2$. This implies in particular in the $\alpha = \beta$ case that measurements of the ith and jth coordinates of the αth particle do not, in general, commute. Thus the position of the αth particle may not be specified precisely. The most that can be said is that for each Θ with p sufficiently large for the representation to be typical we can associate with $|\,p;\Theta\rangle$ eight nests whose coordinates serve to specify the possible outcomes of measurements of $r_{\alpha k}$ for $k = 1,2,3$.

It is of course possible to take measurements not of the coordinates $r_{\alpha k}(t)$ with respect to the original frame of reference, but of coordinates $s_{\alpha k}(t)$ associated with some alternative frame of reference whose orientation with respect to the first may be specified by means, for example, of certain Euler angles. For the sake of simplicity to illustrate the issues involved we consider an orientation obtained by rotating the frame of reference through an angle ϕ about the third axis. The relevant position operators then take the form:

$$\begin{aligned}\hat{s}_{\alpha 1}(t) &= \cos\phi\,\hat{r}_{\alpha 1}(t) + \sin\phi\,\hat{r}_{\alpha 2}(t);\\ \hat{s}_{\alpha 2}(t) &= -\sin\phi\,\hat{r}_{\alpha 1}(t) + \cos\phi\,\hat{r}_{\alpha 2}(t);\\ \hat{s}_{\alpha 3}(t) &= \hat{r}_{\alpha 3}(t).\end{aligned} \tag{4.13}$$

Once again the squares of these operators mutually commute, they are time independent and they commute with the Hamiltonian. They are given by

$$\hat{s}^2_{\alpha 1} = \cos^2\phi\,\{A^+_{\alpha 1}, A^-_{\alpha 1}\} + \cos\phi\,\sin\phi\,\left(\{A^+_{\alpha 1}, A^-_{\alpha 2}\} + \{A^+_{\alpha 2}, A^-_{\alpha 1}\}\right) + \sin^2\phi\,\{A^+_{\alpha 2}, A^-_{\alpha 2}\}; \tag{4.14}$$

$$\hat{s}^2_{\alpha 2} = \sin^2\phi\,\{A^+_{\alpha 1}, A^-_{\alpha 1}\} - \cos\phi\,\sin\phi\,\left(\{A^+_{\alpha 1}, A^-_{\alpha 2}\} + \{A^+_{\alpha 2}, A^-_{\alpha 1}\}\right) + \cos^2\phi\,\{A^+_{\alpha 2}, A^-_{\alpha 2}\}; \tag{4.15}$$

$$\hat{s}^2_{\alpha 3} = \{A^+_{\alpha 3}, A^-_{\alpha 3}\}. \tag{4.16}$$

Their action on the stationary states is such that

$$\begin{aligned}\hat{s}^2_{\alpha 1}\,|\,p; .\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}.\rangle &= \left(p - q + \theta_{\alpha 1}\cos^2\phi + \theta_{\alpha 2}\sin^2\phi\right)\,|\,p; .\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}.\rangle\\ &\quad +(\theta_{\alpha 1} - \theta_{\alpha 2})^2\cos\phi\,\sin\phi\,|\,p; .\overline{\theta}_{\alpha 1}\overline{\theta}_{\alpha 2}\theta_{\alpha 3}.\rangle; \qquad (4.17)\\ \hat{s}^2_{\alpha 2}\,|\,p; .\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}.\rangle &= \left(p - q + \theta_{\alpha 1}\sin^2\phi + \theta_{\alpha 2}\cos^2\phi\right)\,|\,p; .\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}.\rangle\\ &\quad -(\theta_{\alpha 1} - \theta_{\alpha 2})^2\cos\phi\,\sin\phi\,|\,p; .\overline{\theta}_{\alpha 1}\overline{\theta}_{\alpha 2}\theta_{\alpha 3}.\rangle; \qquad (4.18)\\ \hat{s}^2_{\alpha 3}\,|\,p; .\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}.\rangle &= (p - q + \theta_{\alpha 3})\,|\,p; .\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}.\rangle, \qquad (4.19)\end{aligned}$$

where $\overline{\theta}_{\alpha i} = 1 - \theta_{\alpha i}$ for $i = 1, 2, 3$.

Clearly the states $|\,p; \Theta\rangle$ are not eigenstates of $\hat{s}^2_{\alpha k}$. However, it is not difficult to identify the common eigenstates of these mutually commuting operators. They are given by

$$\begin{array}{rl} |\,p; \cdot\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}\cdot\rangle & \text{for } \theta_{\alpha 1} = \theta_{\alpha 2} = 0 \text{ and } \theta_{\alpha 1} = \theta_{\alpha 2} = 1;\\ \cos\phi\,|\,p; \cdot\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}\cdot\rangle + \sin\phi\,|\,p; \cdot\overline{\theta}_{\alpha 1}\overline{\theta}_{\alpha 2}\theta_{\alpha 3}\cdot\rangle & \text{for } \theta_{\alpha 1} = 1, \theta_{\alpha 2} = 0;\\ -\sin\phi\,|\,p; \cdot\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}\cdot\rangle + \cos\phi\,|\,p; \cdot\overline{\theta}_{\alpha 1}\overline{\theta}_{\alpha 2}\theta_{\alpha 3}\cdot\rangle & \text{for } \theta_{\alpha 1} = 1, \theta_{\alpha 2} = 0.\end{array} \tag{4.20}$$

The corresponding eigenvalues $s^2_{\alpha k}$ are as shown in Table 3.

Table 3. Eigenvalues and eigenvectors of $\hat{s}^2_{\alpha k}$

Eigenvectors expressed as linear combinations of $\,|\,p;\cdot\theta_{\alpha1}\theta_{\alpha2}\theta_{\alpha3}\cdot\rangle$	$s^2_{\alpha1}$	$s^2_{\alpha2}$	$s^2_{\alpha3}$	${\mathbf{s}_\alpha}^2$
$\|\,p;\cdot000\cdot\rangle$	$p-q$	$p-q$	$p-q$	$3p-3q$
$\cos\phi\,\|\,p;\cdot100\cdot\rangle+\sin\phi\,\|\,p;\cdot010\cdot\rangle$	$p-q+1$	$p-q$	$p-q$	$3p-3q+1$
$-\sin\phi\,\|\,p;\cdot100\cdot\rangle+\cos\phi\,\|\,p;\cdot010\cdot\rangle$	$p-q$	$p-q+1$	$p-q$	$3p-3q+1$
$\|\,p;\cdot001\cdot\rangle$	$p-q$	$p-q$	$p-q+1$	$3p-3q+1$
$\|\,p;\cdot110\cdot\rangle$	$p-q+1$	$p-q+1$	$p-q$	$3p-3q+2$
$\cos\phi\,\|\,p;\cdot101\cdot\rangle+\sin\phi\,\|\,p;\cdot011\cdot\rangle$	$p-q+1$	$p-q$	$p-q+1$	$3p-3q+2$
$-\sin\phi\,\|\,p;\cdot101\cdot\rangle+\cos\phi\,\|\,p;\cdot011\cdot\rangle$	$p-q$	$p-q+1$	$p-q+1$	$3p-3q+2$
$\|\,p;\cdot111\cdot\rangle$	$p-q+1$	$p-q+1$	$p-q+1$	$3p-3q+3$

Notice that we have, as required,

$$\mathbf{s}^2_\alpha=\sum_{k=1}^{3}s^2_{\alpha k}=3p-3q+\theta_{\alpha1}+\theta_{\alpha2}+\theta_{\alpha3}=\rho^2_\alpha. \tag{4.21}$$

The eigenvalues $s_{\alpha k}$ of $\hat{s}_{\alpha k}(t)$ for $k=1,2,3$ are just $\pm$ the square root of those tabulated for $\hat{s}^2_{\alpha k}$. These results indicate that the sites corresponding to possible values of measurements of the coordinates $s_{\alpha k}$ are again nests on a sphere of radius ρ_α, but the nests define a rectangular parallelepiped obtain by rotating the original one about the third axis through an angle ϕ.

It is particularly striking that the measured values of $s_{\alpha k}$ are of the form $\pm\sqrt{p-q+\theta}$ with $\theta\in\{0,1\}$, just as for $r_{\alpha k}$. They are not the values one might have expected by looking at the nests defined with respect to measurements of $r_{\alpha k}$. This is especially clear in the case of the common eigenstate $|\,p;.000.\rangle$ of all the operators $\hat{r}^2_{\alpha k}$ and $\hat{s}^2_{\alpha k}$ for $k=1,2,3$. The eigenvalues of $\hat{r}_{\alpha k}$ and $\hat{s}_{\alpha k}$ are all $\pm\sqrt{p-q}$. Thus for example in the case $\phi=\pi/4$ the nests defined with respect to measurements of $r_{\alpha k}$ for $k=1,2,3$ have coordinates $s_{\alpha k}\in\{0,\pm\sqrt{2(p-q)}\}$ for $k=1$ and 2 and all α. These are not coordinates of the nests defined with respect to measurements of $s_{\alpha k}$ for $k=1,2,3$. The explanation for this lies in the fact that the particles themselves may not be localised, since measurements of their coordinates do not mutually commute. It is the choice of coordinate to be measured that leads to the observed value corresponding to the associated eigenvalue. This is analagous to the ordinary quantum mechanical measurement of angular momentum, whereby a state of total angular momentum J is such that measurements of the third component of angular momentum gives rise to a discrete set of possible values $J_3=J,J-1,\ldots,-J$ regardless of the orientation of the third axis. For example, measurements on a particle of spin $\frac{1}{2}$ yield values for the projection of the spin in any given direction of only $\pm\frac{1}{2}$. The result is never 0 as might have been expected in a direction perpendicular to the direction in which it is observed to have spin projection $\frac{1}{2}$, nor $\frac{1}{2}\cos\phi$ for any rotation of the axes through an angle ϕ.

These observations regarding measurements of the coordinates $s_{\alpha k}$ may be generalized to the case of coordinates obtained from $r_{\alpha k}$ by means of any orthogonal transformation in the underlying 3D space. For each $g^\alpha \in O(3)$ let $g^\alpha : \hat{\mathbf{r}}_\alpha(t) \mapsto \hat{\mathbf{s}}_\alpha(t) = \hat{\mathbf{r}}_\alpha(t) g^\alpha$, so that for $g^\alpha = (g^\alpha_{ij})_{1\le i,j\le 3}$ we have

$$\hat{s}_{\alpha k}(t) = \sum_{i=1}^{3} \hat{r}_{\alpha i}(t) g^\alpha_{ik}, \tag{4.22}$$

with

$$\sum_{k=1}^{3} g^\alpha_{ik} g^\alpha_{jk} = \delta_{ij}. \tag{4.23}$$

Just as in the case of (4.13), the squares of the operators (4.22) mutually commute, they are time independent and they commute with the Hamiltonian. Their action on the stationary states $|\, p; \Theta\rangle = |\, p; .\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}.\rangle$ is such that

$$\begin{aligned}
&\hat{s}^2_{\alpha k}\,|\, p; .\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}.\rangle \\
&\quad = \big(p - q + (g^\alpha_{1k})^2\theta_{\alpha 1} + (g^\alpha_{2k})^2\theta_{\alpha 2} + (g^\alpha_{3k})^2\theta_{\alpha 3}\big)\,|\, p; .\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}.\rangle \\
&\qquad\quad + g^\alpha_{1k} g^\alpha_{2k}\,(\theta_{\alpha 1} - \theta_{\alpha 2})^2\,|\, p; .\overline{\theta}_{\alpha 1}\overline{\theta}_{\alpha 2}\theta_{\alpha 3}.\rangle \\
&\qquad\quad + g^\alpha_{1k} g^\alpha_{3k}(-1)^{\theta_{\alpha 2}}\,(\theta_{\alpha 1} - \theta_{\alpha 3})^2\,|\, p; .\overline{\theta}_{\alpha 1}\theta_{\alpha 2}\overline{\theta}_{\alpha 3}.\rangle \\
&\qquad\quad + g^\alpha_{2k} g^\alpha_{3k}\,(\theta_{\alpha 2} - \theta_{\alpha 3})^2\,|\, p; .\theta_{\alpha 1}\overline{\theta}_{\alpha 2}\overline{\theta}_{\alpha 3}.\rangle.
\end{aligned} \tag{4.24}$$

As expected the states $|\, p; \Theta\rangle$ are not eigenstates of $\hat{s}^2_{\alpha k}$. The common eigenstates of these mutually commuting operators are given in Table 4, along with the eigenvalues

Table 4. Eigenvalues and eigenvectors of $\hat{s}^2_{\alpha k}$

	Eigenvectors expressed as linear combinations of $|\, p; \cdot\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}\cdot\rangle$	$s^2_{\alpha k}$ $k = 1, 2, 3$	${\mathbf{s}_\alpha}^2$
	$|\, p; \cdot 000\cdot\rangle$	$p - q$	$3p - 3q$
$j = 1, 2, 3$	$g^\alpha_{1j}\ |\, p; \cdot 100\cdot\rangle + g^\alpha_{2j}\ |\, p; \cdot 010\cdot\rangle + g^\alpha_{3j}\ |\, p; \cdot 001\cdot\rangle$	$p - q + \delta_{jk}$	$3p - 3q + 1$
$j = 1, 2, 3$	$g^\alpha_{1j}\ |\, p; \cdot 011\cdot\rangle - g^\alpha_{2j}\ |\, p; \cdot 101\cdot\rangle + g^\alpha_{3j}\ |\, p; \cdot 110\cdot\rangle$	$p - q + 1 - \delta_{jk}$	$3p - 3q + 2$
	$|\, p; \cdot 111\cdot\rangle$	$p - q + 1$	$3p - 3q + 3$

As can be seen, the eigenvalues $s^2_{\alpha k}$ take the values $p - q + \theta$ with $\theta \in \{0, 1\}$. It follows that

$$s_{\alpha k} = \pm\sqrt{p - q + \theta} \quad \text{with } \theta \in \{0, 1\}, \tag{4.25}$$

so that we have the usual set of eight nests for the αth particle with respect this time to measurements of the coordinates $s_{\alpha k}$ for $k = 1, 2, 3$ obtained from $r_{\alpha k}$ by means of the orthogonal transformation g^α.

It is particularly noteworthy that the states $|\, p; \cdot 000\cdot\rangle$ and $|\, p; \cdot 111\cdot\rangle$ are eigenstates of $\hat{s}^2_{\alpha k}$ for all $g^\alpha \in O(3)$, that is to say for all choices of coordinates $s_{\alpha k}$. It follows that for these two particular states, $|\, p; \cdot 000\cdot\rangle$ and $|\, p; \cdot 111\cdot\rangle$, the two sets of eight nests with coordinates $\pm\sqrt{p - q}$ and $\pm\sqrt{p - q + 1}$ can appear anywhere on the spheres of radii $\pm\sqrt{3p-3q}$ and $\pm\sqrt{3p-3q+3}$, respectively. The other states $|\,p;\cdot\theta_{\alpha 1}\theta_{\alpha 2}\theta_{\alpha 3}\cdot\rangle$ with $\theta_{\alpha i}\neq\theta_{\alpha j}$ for some $i\neq j$, are not eigenstates of $\hat{s}^2_{\alpha k}$ for all $g^\alpha\in O(3)$, as can be seen from Table 4 in which the g^α dependent eigenstates are specified. This time it is these eigenstates which define two sets of eight nests on the spheres of radii $\pm\sqrt{3p-3q+1}$ and $\pm\sqrt{3p-3q+2}$, oriented in accordance with the specification of $s_{\alpha k}$ that is determined by g^α. The fact, previously noted in I, that the geometry is non-commutative and the position of a particle is not well defined, coupled with the existence of arbitrarily oriented sets of nests, makes the interpretation of measurements of the position of particle α somewhat difficult.

In these circumstances, we must expect some difficulties over the interpretation of the measurement of the distance between two particles α and β. We cannot after all simultaneously specify their positions. However, in line with classical notions of distance, we are free to call $\hat{d}^2_{\alpha\beta}(t)$ the square distance operator for particles α and β, where

$$\hat{d}^2_{\alpha\beta}(t)=\sum_{i=1}^{3}\left(\hat{r}_{\alpha i}(t)-\hat{r}_{\beta i}(t)\right)^2\,, \tag{4.26}$$

and to examine its properties, including its spectrum of eigenvalues in the space $W(n,p)$.

In the case $n=2$, with $\alpha=1$ and $\beta=2$ we find that $\hat{d}^2_{12}(t)$ has the eigenvalues listed below along with their multiplicities specified by means of subscripts:

$$\begin{array}{l|l|l}
(6p)_1 & (6p-12)_4 & (6p-22)_3\\
(6p-4)_3 & (6p-14)_9 & (6p-24)_3\\
(6p-6)_3 & (6p-16)_9 & (6p-26)_3\\
(6p-8)_3 & (6p-18)_4 & (6p-30)_1\\
(6p-10)_9 & (6p-20)_9 &
\end{array} \tag{4.27}$$

What is remarkable about these eigenvalues is that they do not coincide with the values one might have naively expected, namely the squares of the distances between the positions of the nests defined by $r_{1i}=\pm\sqrt{p-q+\theta_{1i}}$ and $r_{2j}=\pm\sqrt{p-q+\theta_{2j}}$. For example, in the state $|\,p;\Theta\rangle=|\,p;0,0,0,0,0,0\rangle$ we have $r_{1i}=\pm\sqrt{p}$ and $r_{2i}=\pm\sqrt{p}$ for $i=1,2,3$. It follows that

$$\sum_{i=1}^{3}\left(r_{1i}-r_{2i}\right)^2\in\{0,4p,8p,12p\}. \tag{4.28}$$

Thus the spectrum of eigenvalues of $\hat{d}^2_{12}(t)$ given in (4.27) does not contain for general p the values of the squares of the distances between nests as given in (4.28) for the particularly simple state $|\,p;0,0,0,0,0,0\rangle$. This apparent contradiction leads one to ask if the results we have obtained for a multi-particle Wigner quantum oscillator can possibly be self-consistent. The answer is “Yes”. The explanation of the apparent disagreement stems from the non-commutativity of measurements of the coordinates.

Even though the state $|\,p;0,0,0,0,0,0\rangle$ is a simultaneous eigenstate of $\hat{r}^2_{1i}(t)$, $\hat{r}^2_{2j}(t)$ and $\hat{d}^2_{12}(t)$, the positions of the nests are defined by the eigenvalues of $\hat{r}_{1i}(t)$ and $\hat{r}_{2j}(t)$. These operators do not commute with $\hat{d}^2_{12}(t)$. Morevover, $|\,p;0,0,0,0,0,0\rangle$ is not one of their eigenstates. Hence it is not surprising that the eigenvalues (4.27) of $\hat{d}^2_{12}(t)$ do not include the values given in (4.28) for the squares of the distances between the nests.

Returning to the general case, in the space $W(n,p)$ what does remain well defined with respect to measurements of the positions of particles α and β in the state $|\,p;\Theta\rangle$ are the radii ρ_α and ρ_β of the spheres on which they are located, namely

$$\rho_\alpha = \sqrt{3p-3q+q_\alpha} \quad \text{and} \quad \rho_\beta = \sqrt{3p-3q+q_\beta}. \tag{4.29}$$

In such a state, $|\,p;\Theta\rangle$, the expectation value of the square distance operator for particles α and β is given by

$$\overline{d^2}_{\alpha\beta}(t) = \langle p;\Theta|\ \hat{d}^2_{\alpha\beta}(t)\ \,|\,p;\Theta\rangle, \tag{4.30}$$

where we can assume $\alpha<\beta$ without loss of generality.

As in conventional quantum mechanical models and our postulate (P2) in I, we would expect to interpret this as the average value of the square of the distance between the particles α and β. To calculate this quantity is convenient to let

$$\hat{d}_{\alpha\beta k}(t) = \hat{r}_{\alpha k}(t) - \hat{r}_{\beta k}(t) \quad \text{for } k=1,2,3 \text{ and } 1\le\alpha<\beta\le n. \tag{4.31}$$

Using (4.1) the squares of these operators are given by

$$\begin{aligned}\hat{d}^2_{\alpha\beta k}(t) &= \hat{r}^2_{\alpha k} - \{\hat{r}_{\alpha k},\hat{r}_{\beta k}\} + \hat{r}^2_{\beta k}\\ &= \{A^+_{\alpha k},A^-_{\alpha k}\} - \{A^+_{\alpha k},A^-_{\beta k}\} - \{A^+_{\beta k},A^-_{\alpha k}\} + \{A^+_{\beta k},A^-_{\beta k}\},\end{aligned} \tag{4.32}$$

where use has been made of the last part of (2.8). Amongst other things this serves to eliminate the time dependence from $\hat{d}^2_{\alpha\beta k}(t)$. It then follows from (2.27) that

$$\begin{aligned}\hat{d}^2_{\alpha\beta k}\,|\,p;\ldots,\theta_{\alpha k},\ldots,\theta_{\beta k},\ldots\rangle &= (2p-2q+\theta_{\alpha k}+\theta_{\beta k})\ |\,p;\ldots,\theta_{\alpha k},\ldots,\theta_{\beta k},\ldots\rangle\\ &\quad +(-1)^{\psi_{\beta k}-\psi_{\alpha k}}(\theta_{\alpha k}-\theta_{\beta k})\,|\,p;\ldots,\overline{\theta}_{\alpha k},\ldots,\overline{\theta}_{\beta k},\ldots\rangle.\end{aligned} \tag{4.33}$$

Hense

$$\begin{aligned}\overline{d^2}_{\alpha\beta} &= \langle p;\Theta|\ \hat{d}^2_{\alpha\beta}\ \,|\,p;\Theta\rangle = \sum_{k=1}^{3}\langle p;\Theta|\ \hat{d}^2_{\alpha\beta k}\ \,|\,p;\Theta\rangle\\ &= \sum_{k=1}^{3}(2p-2q+\theta_{\alpha k}+\theta_{\beta k}) = 6p-6q+q_\alpha+q_\beta = \rho^2_\alpha+\rho^2_\beta,\end{aligned} \tag{4.34}$$

in the notation of (4.29).

For consistency of interpretation we would then expect this to coincide with $d^2_{\alpha\beta}$, the average of the square of the distance between the nests available to particles

α and β in each of the states $|\,p;\Theta\rangle$. This average will depend of course on the probabilities of occupying each nest.

Now suppose that the n particle system is in one of the Θ-basis states $|\,p;\Theta\rangle$. Then the γ-th particle ($\gamma = 1, \dots n$) could have the following coordinates:

$$r_{\gamma i} = \pm\sqrt{p - q + \theta_{\gamma i}}. \tag{4.35}$$

Let $s = (\pm \pm \pm)$ be a sequence of signs specifying the sites of the 8 possible nests of the γ-th particle associated with a particular state $|\,p;\Theta\rangle$ by signifying the signs of the corresponding coordinates $(r_{\gamma 1}, r_{\gamma 2}, r_{\gamma 3})$. Let $\mathcal{P}_\gamma(s)$ be the probability of finding the γ-th particle in the nest specified by s. Then from [31] we have

$$\begin{aligned}
&\mathcal{P}_\gamma(+++) + \mathcal{P}_\gamma(++-) + \mathcal{P}_\gamma(+-+) + \mathcal{P}_\gamma(+--) = \frac{1}{2},\\
&\mathcal{P}_\gamma(-++) + \mathcal{P}_\gamma(-+-) + \mathcal{P}_\gamma(--+) + \mathcal{P}_\gamma(---) = \frac{1}{2},\\
&\mathcal{P}_\gamma(+++) + \mathcal{P}_\gamma(++-) + \mathcal{P}_\gamma(-++) + \mathcal{P}_\gamma(-+-) = \frac{1}{2},\\
&\mathcal{P}_\gamma(+-+) + \mathcal{P}_\gamma(+--) + \mathcal{P}_\gamma(--+) + \mathcal{P}_\gamma(---) = \frac{1}{2},\\
&\mathcal{P}_\gamma(+++) + \mathcal{P}_\gamma(+-+) + \mathcal{P}_\gamma(-++) + \mathcal{P}_\gamma(--+) = \frac{1}{2},\\
&\mathcal{P}_\gamma(++-) + \mathcal{P}_\gamma(+--) + \mathcal{P}_\gamma(-+-) + \mathcal{P}_\gamma(---) = \frac{1}{2}.
\end{aligned} \tag{4.36}$$

The average square distance of the particle α, occupying the nest at $(r_{\alpha 1}, r_{\alpha 2}, r_{\alpha 3})$ with probability $\mathcal{P}_\alpha(+++)$, from the particle β, occupying the 8 nests at $(r_{\beta 1\pm}, r_{\beta 2\pm}, r_{\beta 3\pm})$, with probability $\mathcal{P}_\beta(\pm\pm\pm)$, where $r_{\alpha k\pm} = \pm\sqrt{p - q - \theta_{\alpha k}}$ and $r_{\beta k\pm} = \pm\sqrt{p - q - \theta_{\beta k}}$ for $k = 1, 2, 3$, is then given by

$$\begin{aligned}
&\mathcal{P}_\alpha(+++)\mathcal{P}_\beta(+++)((r_{\alpha 1} - r_{\beta 1})^2 + (r_{\alpha 2} - r_{\beta 2})^2 + (r_{\alpha 3} - r_{\beta 3})^2) +\\
&\mathcal{P}_\alpha(+++)\mathcal{P}_\beta(++-)((r_{\alpha 1} - r_{\beta 1})^2 + (r_{\alpha 2} - r_{\beta 2})^2 + (r_{\alpha 3} + r_{\beta 3})^2) +\\
&\mathcal{P}_\alpha(+++)\mathcal{P}_\beta(+-+)((r_{\alpha 1} - r_{\beta 1})^2 + (r_{\alpha 2} + r_{\beta 2})^2 + (r_{\alpha 3} - r_{\beta 3})^2) +\\
&\mathcal{P}_\alpha(+++)\mathcal{P}_\beta(+--)((r_{\alpha 1} - r_{\beta 1})^2 + (r_{\alpha 2} + r_{\beta 2})^2 + (r_{\alpha 3} + r_{\beta 3})^2) +\\
&\mathcal{P}_\alpha(+++)\mathcal{P}_\beta(-++)((r_{\alpha 1} + r_{\beta 1})^2 + (r_{\alpha 2} - r_{\beta 2})^2 + (r_{\alpha 3} - r_{\beta 3})^2) +\\
&\mathcal{P}_\alpha(+++)\mathcal{P}_\beta(-+-)((r_{\alpha 1} + r_{\beta 1})^2 + (r_{\alpha 2} - r_{\beta 2})^2 + (r_{\alpha 3} + r_{\beta 3})^2) +\\
&\mathcal{P}_\alpha(+++)\mathcal{P}_\beta(--+)((r_{\alpha 1} + r_{\beta 1})^2 + (r_{\alpha 2} + r_{\beta 2})^2 + (r_{\alpha 3} - r_{\beta 3})^2) +\\
&\mathcal{P}_\alpha(+++)\mathcal{P}_\beta(---)((r_{\alpha 1} + r_{\beta 1})^2 + (r_{\alpha 2} + r_{\beta 2})^2 + (r_{\alpha 3} + r_{\beta 3})^2)\\
&= \frac{1}{2}\mathcal{P}_\alpha(+++)((r_{\alpha 1} - r_{\beta 1})^2 + (r_{\alpha 2} - r_{\beta 2})^2 + (r_{\alpha 3} - r_{\beta 3})^2)\\
&+\frac{1}{2}\mathcal{P}_\alpha(+++)((r_{\alpha 1} + r_{\beta 1})^2 + (r_{\alpha 2} + r_{\beta 2})^2 + (r_{\alpha 3} + r_{\beta 3})^2)\\
&= \mathcal{P}_\alpha(+++)\sum_{i=1}^{3}(r_{\alpha i}^2 + r_{\beta i}^2) = \mathcal{P}_\alpha(+++)(\rho_\alpha^2 + \rho_\beta^2).
\end{aligned} \tag{4.37}$$

In the crucial first step all the various parts of (4.36) have been used with γ set equal to β.

Treating the other seven nests for the particle α in a similar way one can compute the average square distance of particle α, occupying the 8 nests at $(r_{\alpha 1\pm}, r_{\alpha 2\pm}, r_{\alpha 3\pm})$ with probabilities $\mathcal{P}_\alpha(\pm\pm\pm)$, from particle β, occupying the 8 nests at $(r_{\beta 1\pm}, r_{\beta 2\pm}, r_{\beta 3\pm})$ with probabilities $\mathcal{P}_\beta(\pm\pm\pm)$. We arrive at the result

$$\begin{aligned}
&\Big(\mathcal{P}_\alpha(+++) + \mathcal{P}_\alpha(++-) + \mathcal{P}_\alpha(+-+) + \mathcal{P}_\alpha(+--) + \\
&\qquad \mathcal{P}_\alpha(-++) + \mathcal{P}_\alpha(-+-) + \mathcal{P}_\alpha(--+) + \mathcal{P}_\alpha(---)\Big)(\rho_\alpha^2 + \rho_\beta^2) \\
&= \rho_\alpha^2 + \rho_\beta^2 = 6p - 6q + q_\alpha + q_\beta,
\end{aligned} \tag{4.38}$$

in perfect agreement with (4.34). It is notable, as can be seen from (4.37), that even if the particle α were located at one particular nest, the same conclusion (4.38) would be drawn about its distance from particle β. It may also be shown that this conclusion is unaltered if we use the coordinates $s_{\alpha k}$ rather than the coordinates $r_{\alpha k}$.

5 Physical properties - exclusion phenomena

In canonical quantum mechanics the Hamiltonian (2.1) corresponds to a system of n non-interacting oscillating particles, each of mass m. The allowed states of any one particular particle are free and independent of the states of the other $n-1$ particles. For the Wigner quantum oscillator this is not always the case. In fact whenever $p < 3n$, so that the corresponding irreducible representation of $sl(1|3n)$ is atypical, constraints will apply. This is a consequence of the fact that if the sum of any proper subset of the $\theta_{\alpha i}$'s takes the value p with $p < 3n$, then the remaining $\theta_{\beta j}$'s must all be zero. This implies an exclusion from certain states even in the case $n = 1$ of a single particle. For example, if $n = 1$ and $p = 2$ and $|\,p;\Theta\rangle = |\,2;1,1,\theta_{13}\rangle$ so that $\psi_{13} = \theta_{11} + \theta_{12} = 2 = p$, then $\theta_{13} = 0$. It follows that while the state $|\,2;1,1,0\rangle$ is allowed, the state $|\,2;1,1,1\rangle$ is forbidden. One might say that in the atypical case even a single particle is not "free", or equivalently that the particle is "excluded" from being in certain states.

This exclusion phenomenon is even more striking in the case of a multiparticle WQO with $n > 1$ and $p < 3n$. This can be seen even in the simplest $n = 2$ case of two particles, for example when the order of the statistics $p = 3$, as in (3.18) and Table 2. The relevant stationary states are given in the notation of (2.21) by $|\,p;\Theta\rangle = |\,3;\theta_{11},\theta_{12},\theta_{13},\theta_{21},\theta_{22},\theta_{23}\rangle$. If the first particle is in the state $\theta_{11} = \theta_{12} = \theta_{13} = 1$, then since q is constrained by the condition $q \le \min(p, 3n) = \min(3, 6) = 3$ it follows that the second particle is excluded from being in any state other than the state $\theta_{21} = \theta_{22} = \theta_{23} = 0$. This state $|\,3;1,1,1,0,0,0\rangle$ is one of the 4 states of $V^3_{gl(1)} \otimes V^0_{sl(3)}$ appearing in Table 2, having $p = q = 3$, energy $E_3 = 3\hbar\omega/5$ and angular momentum $M = 0$. The contributions to the energy E_q from each of the two particles may be calculated by noting, quite generally from (2.6), (2.27) and (3.1) that

$$E_q = \sum_{\alpha=1}^{n} E_{\alpha,q} \quad \text{with} \quad E_{\alpha,q} = \frac{\omega\hbar}{3n-1}\sum_{i=1}^{3}(p-q+\theta_{\alpha i}). \tag{5.1}$$

In the case of interest here for the state $|\,3;1,1,1,0,0,0\rangle$, we have $n=2$, $p=q=3$, $\theta_{11}=\theta_{12}=\theta_{13}=1$ and $\theta_{21}=\theta_{22}=\theta_{23}=0$, so that $E_{1,q}=3\hbar\omega/5$ and $E_{2,q}=0$. In addition, from (3.10) we have

$$\hat{\mathbf{M}}_\alpha^2\,|\,p;\Theta\rangle = M^{(\alpha)}(M^{(\alpha)}+1)\,|\,p;\Theta\rangle \quad \text{with} \quad M^{(\alpha)}(M^{(\alpha)}+1) = \delta_\alpha\; 2, \tag{5.2}$$

with δ_α defined by (3.11). For the state $|\,3;1,1,1,0,0,0\rangle$ we have $\delta_1=\delta_2=0$, so that $M^{(1)}=M^{(2)}=0$. Finally from (4.6), for this same state with $p=q=3$ we have $r_{1,k}=\pm1$ and $r_{2,k}=0$ for $k=1,2,3$. Thus the particular state, $\theta_{11}=\theta_{12}=\theta_{13}=1$, of the first particle, forces the second to be such that $\theta_{21}=\theta_{22}=\theta_{23}=0$ so that it sits at the origin contributing no energy and no angular momentum to the total system. The same phenomenon cannot occur for the CQO. In the notation of (2.28) if we have $|\,\Phi\rangle = |\,1,1,1,\phi_{21},\phi_{22},\phi_{23}\rangle$ there is no restriction on the parameters $\phi_{21},\phi_{22},\phi_{23}$ that determine the state of the second particle.

The general conclusion in the multiparticle Wigner quantum oscillator case is that despite the "free" nature of the Hamiltonian (2.1), the particles are not always "free". They may "interact", in the sense that the state of a particular particle may be constrained or even fixed by the states of the other $n-1$ particles. The above $n=2$ and $p=3$ example illustrates this. The interaction is of statistical origin, depending on the parameter p in our A-superstatistics model. This is very similar to the exclusion statistics of Haldane [32], which plays an important role in condensed matter physics.

As pointed out in Section 3, any comparison between the WQO and the CQO spectrum of energy levels is preferably based on a comparison of the direct sum of an infinite number of finite-dimensional irreducible representations of $sl(1|3n)$ and a single infinite-dimensional irreducible representations of $osp(1|6n)$, that is, in the notation of (3.3) and (3.5), a comparison of

$$\sum_{p=0}^{\infty} V^p_{sl(1|3n)} \qquad \text{and} \qquad V^\epsilon_{osp(1|6n)}. \tag{5.3}$$

From (3.1) and (3.2) the complete sets of energy levels, indexed by their level number l, are given in the two cases by

$$\frac{l\,\hbar\omega}{(3n-1)} \qquad \text{and} \qquad \frac{(3n+2l)\hbar\omega}{2} \quad \text{for} \quad l=0,1,2,\ldots\,. \tag{5.4}$$

In the WQO case the $l=0$ ground state has energy zero and the levels are equally spaced with separation $\hbar\omega/(3n-1)$ which decreases as the number of particles, n, increases. In the CQO case the $l=0$ ground state has energy $3n\hbar\omega/2$ and the levels are equally spaced with separation $\hbar\omega$, independent of the number of particles n.

In both cases let the degeneracy of the n-particle level l be denoted by $d_{n,l}$ and the corresponding generating function be denoted by

$$G_n(x) = \sum_{l=0}^{\infty} d_{n,l}\ x^l. \tag{5.5}$$

Then for the WQO, using (3.4), we find

$$d_{n,l} = \begin{cases} 1 & \text{if } l = 0; \\ 2 & \text{if } l \equiv 0 \pmod{3n} \text{ and } l > 0; \\ \binom{3n}{r} & \text{if } l \equiv r \pmod{3n} \text{ with } r > 0, \end{cases} \quad \text{and} \quad G_n(x) = \frac{(1+x)^{3n}}{(1-x^{3n})}. \tag{5.6}$$

For the CQO the n-particle level l corresponds to $q = l$ and its degeneracy is given by (3.6), so that

$$d_{n,l} = \binom{3n-1+l}{l} \quad \text{and} \quad G_n(x) = \frac{1}{(1-x)^{3n}}. \tag{5.7}$$

This degeneracy $d_{n,l}$ for the CQO increases without bound as l increases for any n, unlike the WQO case for which the degeneracy is bounded and in fact periodic in l for $l > 0$.

6 Concluding remarks

To conclude, as promised in I, we have taken the natural step of generalising a one particle 3-dimensional WQO to an n-particle 3-dimensional WQO. The relevant Fock space irreducible representations of $sl(1|3n)$, are as indicated previously, of dimension 2^{3n} for the case of typical representations, and less than this for atypical ones. With or without typicality the energy levels are equally spaced, and we have shown how to determine not just energy eigenstates but also mutual eigenstates of both energy and angular momentum, as illustrated in detail in Table 2 for the two particle case. One interesting feature of the n-particle model is that for atypical representations there is an A-superstatistics effect whereby one constituent particle may constrain the energy, angular momentum and even configuration of another. In an extreme case with $p = 3$ the existence of one particle in its lowest energy state forces all the other particles to sit at the origin contributing no further energy or angular momentum.

Moving to the n-particle case has also enabled us to explore in more detail the sometimes unexpected consequences of the non-commutative geometry arising in this WQO model. We find once again that the position of any one of the particles may not be specified precisely. However, the possible results of measurements of various coordinates lead us to identify various sites or nests whose precise positions turn out to be a function of the coordinates we choose to measure. The distances between these nests associated with different particles are not in fact what one might have expected, namely the square roots of the eigenvalues of the operators $\hat{d}^2_{\alpha\beta}(t)$ associated with the square of the distance between any two particles specified by α and β. Instead, as in more conventional quantum theory models, the expectation value $\overline{\hat{d}^2}_{\alpha\beta}(t)$ of the square of the distance operator $\hat{d}^2_{\alpha\beta}(t)$ in each stationary state $|\,p;\Theta\rangle$ gives the average square distance between the various nests of the two particles associated with the stationary state $|\,p;\Theta\rangle$, with each nest occupied with various possible probabilities.

Consideration of appropriate infinite-dimensional representations has enabled us to compare and contrast the classical canonical quantum oscillator and our non-standard Wigner quantum oscillator. The latter has the unusual feature that the equally spaced energy levels become closer together, but remain equally spaced as the number of particles is increased. In addition their degeneracy remains bounded and, above the ground state, the degeneracy is periodic in the level number for any fixed number of particles.

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