Wigner quantization and Lie superalgebra representations

J. Van der Jeugt

Dedicated to T.D. Palev on the occasion of his 75th birthday

Abstract T.D. Palev laid the foundations of the investigation of Wigner quantum systems through representation theory of Lie superalgebras. His work has been very influential, in particular on my own research. It is quite remarkable that the study of Wigner quantum systems has had some impact on the development of Lie superalgebra representations. In this review paper, I will present the method of Wigner quantization and give a short overview of systems (Hamiltonians) that have recently been treated in the context of Wigner quantization. Most attention will go to a system for which the quantization conditions naturally lead to representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$. I shall also present some recent work in collaboration with G. Regniers, where generating functions techniques have been used in order to describe the energy and angular momentum contents of 3-dimensional Wigner quantum oscillators.

1 Introduction and some history

The main ideas of Wigner quantization go back to a short paper that Wigner published in 1950 [36]. Due to the fact that his method leads to algebraic relations for operators which are in general very difficult to solve, it took many years before his work was continued. About 30 years later, when Lie superalgebra theory was developed, it was T.D. Palev who realized that particular Lie superalgebra generators satisfy the algebraic relations appearing in the Wigner quantization of certain systems. This was the real start of Wigner quantization, a program to which Palev contributed much of his scientific career. He also inspired many other scientists to

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work on the program, including myself. It has been a pleasure for me to collaborate with Tchavdar Palev and his former student Neli Stoilova, and to contribute to the theory.

In this review paper, I will give an introduction to the topic, first by presenting Wigner's original example in a contemporary context. In section 2, Palev's general method of Wigner quantization is briefly presented, and then we give a short overview of his contributions to the field, and of some other papers on Wigner quantization. Our purpose is to include also some recent work, and therefore the Wigner quantization of the n-dimensional non-isotropic oscillator is discussed in section 3. This problem stimulated the search for infinite-dimensional unitary representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$; a class of these representations were constructed only a few years ago. Using these representations, we present some interesting aspects of this Wigner quantum system in section 3, and its angular momentum contents in section 4. There is no new material in this paper: we only present and summarize some of the main ideas of Wigner quantization and some recent contributions.

In his seminal paper [36], Wigner asked the question: "Do the equations of motion determine the quantum mechanical commutation relations?" It was known at that time that, for a class of Hamiltonians written as analytic functions of the generalized position and momentum operators $\hat{q}_i$ and $\hat{p}_i$ ($i = 1,\ldots,n$), the Heisenberg equations of motion together with the canonical commutation relations (CCRs) imply formally Hamilton's equations. Vice versa, starting from the operator form of Hamilton's equations and using the CCRs, one can derive the Heisenberg equations. Since Wigner believed that the Heisenberg equations of motion and the operator form of Hamilton's classical equations of motion have a deeper physical meaning than the mathematically imposed CCRs, he wondered whether requiring the compatibility of the Heisenberg equations with Hamilton's equations would automatically lead to the CCRs. Wigner investigated this question for the Hamiltonian of the one-dimensional harmonic oscillator, given by

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2) \tag{1}$$

under the convention $m = \omega = \hbar = 1$. The Heisenberg equations are:

$$\dot{\hat{q}} = i[\hat{H},\hat{q}], \qquad \dot{\hat{p}} = i[\hat{H},\hat{p}], \tag{2}$$

and the operator form of Hamilton's equations read:

$$\dot{\hat{q}} = \mathrm{op}(\frac{\partial H}{\partial p}) = \hat{p}, \qquad \dot{\hat{p}} = -\mathrm{op}(\frac{\partial H}{\partial q}) = -\hat{q}. \tag{3}$$

So for this example the compatibility conditions become:

$$\hat{p} = i[\frac{1}{2}(\hat{p}^2 + \hat{q}^2),\hat{q}], \qquad -\hat{q} = i[\frac{1}{2}(\hat{p}^2 + \hat{q}^2),\hat{p}]. \tag{4}$$

The goal is to find (self-adjoint) operators $\hat{p}$, $\hat{q}$ satisfying these equations, without making any assumptions about the commutation relation between $\hat{p}$ and $\hat{q}$. Otherwise said, are there other operator solutions to (4) besides the canonical solution where $[\hat{q},\hat{p}] = i$? Wigner found that indeed there are other solutions. In order to describe these, let us use the language of Lie superalgebras (of course, Wigner used a different method, as Lie superalgebras were not known at that time).

Rewriting the operators $\hat{q}$ and $\hat{p}$ by the linear combinations

$$b^+ = \frac{\hat{q} - i\hat{p}}{\sqrt{2}}, \qquad b^- = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}, \tag{5}$$

the conditions (4) are equivalent to the two relations

$$[\{b^+, b^-\}, b^\pm] = \pm 2 b^\pm. \tag{6}$$

Note that these relations involve both commutators and anti-commutators. This is why it will be helpful to use Lie superalgebras. In fact, it is known that (6), the compatibility conditions to solve, are exactly the defining relations of the Lie superalgebra $\mathfrak{osp}(1|2)$ in terms of two odd generators b^+, b^- [5]. Moreover, it should hold that $\hat{p}^\dagger = \hat{p}$ and $\hat{q}^\dagger = \hat{q}$, or rewritten in terms of the new operators: $(b^\pm)^\dagger = b^\mp$. Thus, we are led to the unitary (or unitarizable) representations of $\mathfrak{osp}(1|2)$ i.e. Hilbert space representations in which $(b^\pm)^\dagger = b^\mp$ holds.

The unitary irreducible representations of $\mathfrak{osp}(1|2)$ were classified by Hughes [8]; see also [31] for a more comprehensive method. The unitary irreducible representations are labelled by a positive real number p ($p/2$ is the lowest weight); the orthonormal basis vectors are $|n\rangle$, with $n \geq 0$. The action of b^+ and b^- is given by:

$$b^+|n\rangle = \sqrt{\nu_{n+1}}\,|n+1\rangle, \quad b^-|n\rangle = \sqrt{\nu_n}\,|n-1\rangle; \quad \nu_n = n + (p-1)(1-(-1)^n)/2. \tag{7}$$

Using (5) and (7), one can deduce:

$$\hat{H}|n\rangle = \frac{1}{2}\{b^+, b^-\}|n\rangle = (n + \frac{p}{2})|n\rangle, \tag{8}$$

$$[\hat{q},\hat{p}]|2n\rangle = ip|2n\rangle, \quad [\hat{q},\hat{p}]|2n+1\rangle = i(2-p)|2n+1\rangle. \tag{9}$$

From this it is clear that only the case $p = 1$ corresponds to the CCRs. All other solutions (i.e. all other positive values of p) are non-canonical. Wigner concluded that requiring the equivalence of Hamilton's and Heisenberg's equations is a very natural approach that may lead to other quantizations besides the canonical one; and the canonical quantization solution appears as one of the more general solutions.

In the example of Wigner, the apparent difference with the canonical case is the shift in energy, as is clear from (8). It is interesting to have also a look at the wave functions for these non-canonical solutions. This was in fact not performed by Wigner, but only much later, when the above operators b^+ and b^- were studied as "parabosons" [22]. An alternative way of finding these wave functions is described in the Appendix of [9]. This is obtained by computing the (formal) eigenvectors of

$\hat{q} = (b^+ + b^-)/\sqrt{2}$ in the above Hilbert space. Writing these formal eigenvectors of $\hat{q}$ as

$$v(q) = \sum_{n=0}^{\infty} \Psi_n^{(p)}(q)\,|n\rangle, \tag{10}$$

and expressing $\hat{q}\,v(q) = q\,v(q)$ by means of the action (7) yields a set of recurrence relation for the coefficients $\Psi_n^{(p)}(q)$. The solution leads to the conclusion that the spectrum of $\hat{q}$ is $\mathbb{R}$, and that

$$\begin{aligned}
\Psi_{2n}^{(p)}(x) &= (-1)^n \sqrt{\frac{n!}{\Gamma(n+p/2)}}\,|x|^{(p-1)/2}\,e^{-x^2/2} L_n^{(p/2-1)}(x^2),\\
\Psi_{2n+1}^{(p)}(x) &= (-1)^n \sqrt{\frac{n!}{\Gamma(n+p/2+1)}}\,|x|^{(p-1)/2}\,e^{-x^2/2} x L_n^{(p/2)}(x^2),
\end{aligned} \tag{11}$$

in terms of generalized Laguerre polynomials. These coefficients $\Psi_n^{(p)}(q)$ have an interpretation as the position wave functions of the Wigner oscillator. Alternatively, one can work in the position representation, where the operator $\hat{q}$ is still represented by "multiplication by x", and the operator $\hat{p}$ has a realization as $-i\frac{d}{dx} + i\frac{p-1}{2x}R$, where $Rf(x) = f(-x)$ is a reflection operator [23, Chapter 23]. Using this realization the time-independent Schrödinger equation can be solved, also yielding the expressions (11) [22]. For $p = 1$, the Laguerre polynomials reduce to Hermite polynomials, and one gets the commonly known wave functions. It is interesting to compare the plots of the wave functions for $p \neq 1$ with those of the canonical case $p = 1$, see Figure 1.

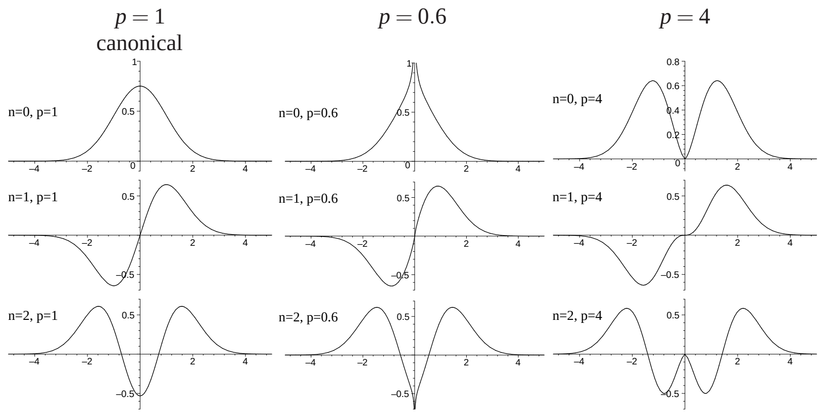

Fig. 1 Plots of the wave functions $\Psi_n^{(p)}(x)$. The three figures on the left are for $p = 1$ and correspond to the canonical case; the figures in the middle are for $p = 0.6$, and three figures on the right are for $p = 4$. In each case, we plot the wave functions for $n = 0, 1, 2$.

2 Wigner quantum systems and Palev's contributions

Wigner's work on this alternative quantization method for the one-dimensional oscillator did not receive much attention originally. This was mainly because of the mathematical difficulties when trying to apply it to a Hamiltonian different from (1). In fact, trying to solve Wigner's compatibility conditions for other systems leads to complicated operator relations, for which often no general solutions are known. By 1980 however, Lie superalgebra theory and their representations became well understood [10, 11]. T.D. Palev had worked with Lie superalgebras, mainly in the context of parabosons and parafermions [24]. He was the first to realize the importance of Lie superalgebra representations in the context of Wigner quantization. It is also to him that we owe the term "Wigner quantum system" or "Wigner quantization". In one of his first papers on the topic [25], however, he used the term "Dynamical quantization", referring to the fact that quantization follows from compatibility conditions related to the equations of motion.

Let us briefly summarize the main principles of Wigner quantization, as developed by Palev. Consider a quantum system with n degrees of freedom and a Hamiltonian of the form

$$\hat{H} = \sum_{j=1}^{n} \frac{\hat{p}_j^2}{2m_j} + \mathscr{V}(\hat{q}_1, \ldots, \hat{q}_n). \tag{12}$$

In Wigner quantization, one keeps all axioms of quantum mechanics, only the axiom on the CCRs is replaced. The canonical commutation relations

$$[\hat{q}_k, \hat{q}_l] = [\hat{p}_k, \hat{p}_l] = 0, \qquad [\hat{q}_k, \hat{p}_l] = i\hbar\delta_{kl} \tag{13}$$

are *replaced* by a different set of operator relations between position and momentum operators. This set consists of the (operator) Compatibility Conditions (CC) between the Heisenberg equations and the operator form of Hamilton's equations.

So, in short, Wigner quantization for a system described by (12) consists of the following three steps:

1. Rewrite the Hamiltonian $\hat{H}$ appropriately in terms of operators $\hat{p}_k$ and $\hat{q}_k$ (in some symmetric form, not assuming any commutativity between the operators).
2. Determine the Compatibility Conditions (CC). This gives rise to a (non-linear) set of operator relations for the $\hat{p}_k$ and $\hat{q}_k$. The $\star$-algebra $\mathscr{A}$ is then defined as an algebra with generators $\hat{p}_k$ and $\hat{q}_k$ and defining relations (CC), subject to the $\star$-conditions $\hat{p}_k^\star = \hat{p}_k$ and $\hat{q}_k^\star = \hat{q}_k$.
3. Find $\star$-representations (unitary representations) of $\mathscr{A}$.

Very often, it is difficult to identify $\mathscr{A}$ as a known algebra, and hence it is too difficult to find all $\star$-representations. So instead of trying to work with $\mathscr{A}$, one looks for a *known* algebra $\mathscr{B}$ whose generators also satisfy (CC). Then it remains to construct the $\star$-representations of $\mathscr{B}$ and to determine physical properties (energy, spectrum of observables,...) in these representations. This gives rise to a subset of solutions.

Note that this approach leads quite naturally to *non-commutative coordinate operators*, without any forced or external input as is sometimes done in other approaches of “non-commutative quantum mechanics”.

In the first main paper on Wigner quantization [25], Palev investigated two particles interacting via a harmonic potential. After removal of the center of mass, the remaining Hamiltonian is essentially that of the 3-dimensional isotropic harmonic oscillator (HO). Palev investigated the CCs, and found that these were satisfied by certain generators of the Lie superalgebra $\mathfrak{gl}(1|3)$. In other words, he chose $\mathscr{B} = \mathfrak{gl}(1|3)$. Then, he went on to study properties in a particular class of $\star$-representations, namely the so-called Fock space representations. A remarkable feature here is the finite-dimensionality of these $\star$-representations, implying that all physical operators have a *discrete* spectrum. In the same year, Palev showed [26] that the CCs for the n-dimensional HO are satisfied by generators of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$; however, no representations were considered. Later, Kamupingene, Palev and Tsaneva [12] considered in more detail the 2-dimensional HO with $\mathscr{B} = \mathfrak{sl}(1|2)$. Interesting physical properties were obtained by Palev and Stoilova for the $\mathfrak{osp}(3|2)$ solutions of the 3-dimensional HO. Here, one could make use of a classification of the $\star$-representations of $\mathfrak{osp}(3|2)$ [35]. Palev and Stoilova [27, 28] later compared the solutions of the 3-dimensional isotropic Wigner HO provided by $\mathfrak{sl}(1|3)$, $\mathfrak{osp}(1|6)$ and $\mathfrak{osp}(3|2)$. The postulates of Wigner quantum systems were more carefully described in [29]. In this paper, the n-dimensional isotropic HO is revisited, and for the first time angular momentum operators are discussed (for $n = 3N$). In a review paper, Palev and Stoilova [30] describe the algebraic solutions for the n-particle 3-dimensional isotropic HO in terms of the Lie superalgebras $\mathfrak{sl}(1|3n)$, $\mathfrak{osp}(1|6n)$ and $\mathfrak{sl}(3|n)$. Further physical properties for the $\mathfrak{sl}(1|3)$ or $\mathfrak{sl}(1|3n)$ solutions, in particular related to the discrete spacial structure, were investigated in [14, 15]. Then, a few years ago, Stoilova and Van der Jeugt [34] made a quite general classification of Lie superalgebra solutions of the CCs for the n-dimensional isotropic HO.

Lievens *et al* [16, 17] applied Wigner quantization to more complicated Hamiltonians, such as a linear chain of coupled particles. They show how this reduces to the Hamiltonian for an n-dimensional *non-isotropic* HO, and obtain new solutions in terms of $\mathfrak{gl}(1|n)$.

There appeared also a number of papers related to the fundamentals of Wigner quantization, or related algebraic quantizations. We mention here in particular the work of Man’ko, Marmo, Zaccaria and Sudarshan [21], Blasiak, Horzela, Kapuscik [13, 7, 4], and that of Atakishiyev, Wolf and collaborators [1, 2, 3] in the context of finite oscillator models.

More recently, Regniers and Van der Jeugt [32] investigated one-dimensional Hamiltonians with continuous energy spectra as Wigner quantum systems.

All these papers make it clear that Wigner quantization has given rise to challenging mathematical problems, and to interesting physical properties. Wigner quantization has also raised questions in Lie superalgebra representation theory, and stimulated further research into specific classes of Lie superalgebra representations.

In the following section we shall review the treatment of the n-dimensional non-isotropic harmonic oscillator in Wigner quantization as our main example. This has given rise to the study of a new class of representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$.

3 Main example: the n-dimensional non-isotropic harmonic oscillator

For this example, we drop the previous convention with $m = \omega = \hbar = 1$, and consider the n-dimensional non-isotropic harmonic oscillator with Hamiltonian:

$$\hat{H} = \frac{1}{2m}\sum_{j=1}^{n} \hat{p}_j^2 + \frac{m}{2}\sum_{j=1}^{n} \omega_j^2 \hat{q}_j^2, \tag{14}$$

where m stands for the mass of the oscillator and ω_j for the frequency in direction j. Let us construct the compatibility conditions CC. Clearly, the operator form of Hamilton's equations reads:

$$\dot{\hat{q}}_j = \mathrm{op}(\frac{\partial H}{\partial p_j}) = \frac{1}{m}\hat{p}_j, \qquad \dot{\hat{p}}_j = -\mathrm{op}(\frac{\partial H}{\partial q_j}) = -m\omega_j^2 \hat{q}_j, \qquad j = 1,\ldots,n. \tag{15}$$

The Heisenberg equations are:

$$\dot{\hat{q}}_j = \frac{i}{\hbar}[\hat{H},\hat{q}_j], \qquad \dot{\hat{p}}_j = \frac{i}{\hbar}[\hat{H},\hat{p}_j], \qquad j = 1,\ldots,n. \tag{16}$$

So the compatibility conditions become:

$$[\hat{H},\hat{q}_j] = -i\frac{\hbar}{m}\hat{p}_j, \qquad [\hat{H},\hat{p}_j] = i\hbar m\omega_j^2 \hat{q}_j, \qquad j = 1,\ldots,n, \tag{17}$$

where $\hat{H}$ is given by (14).

It is useful to write these compatibility conditions in a different form. For this purpose, introduce the following linear combinations of the operators $\hat{q}_j$ and $\hat{p}_j$:

$$a_j^{\mp} = \sqrt{\frac{m\omega_j}{2\hbar}}\hat{q}_j \pm \frac{i}{\sqrt{2m\hbar\omega_j}}\hat{p}_j, \qquad j = 1,\ldots,n. \tag{18}$$

Now the expression of the Hamiltonian becomes

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^{n} \omega_j (a_j^+ a_j^- + a_j^- a_j^+) = \frac{\hbar}{2}\sum_{j=1}^{n} \omega_j \{a_j^+, a_j^-\}. \tag{19}$$

The new form of the compatibility conditions can be written as:

$$\Big[\sum_{j=1}^{n}\omega_j\{a_j^+,a_j^-\},a_k^\pm\Big]=\pm 2\omega_k a_k^\pm, \qquad k=1,\ldots,n. \tag{20}$$

In terms of the notation of the previous section, $\mathscr{A}$ is the $\star$-algebra generated by $2n$ generators $a_j^\pm$ ($j=1,\ldots,n$) with $\star$-relations $(a_j^\pm)^\star = a_j^\mp$ and with defining relations (20).

Quite surprisingly, the structure of $\mathscr{A}$ and its unitary Hilbert space representations is known completely only for $n=1$ (in which case it is just Wigner's example of section 1). For $n>1$, only some classes of unitary Hilbert space representations are known.

We shall now describe an algebraic solution for the conditions (20), in other words we shall determine an algebra $\mathscr{B}$ whose generators also satisfy (20) (but for which (20) are not the defining relations). This is provided by the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$. In fact, it were Ganchev and Palev [5] who discovered – in the context of parabosons – that $\mathfrak{osp}(1|2n)$ can be defined as an algebra with $2n$ generators $b_j^\pm$ subject to the following triple relations:

$$[\{b_j^\xi,b_k^\eta\},b_l^\varepsilon]=(\varepsilon-\xi)\delta_{jl}b_k^\eta+(\varepsilon-\eta)\delta_{kl}b_j^\xi, \tag{21}$$

where $j,k,l\in\{1,\ldots,n\}$, and $\eta,\varepsilon,\xi\in\{+,-\}$ (to be interpreted as $+1$ or -1 in algebraic expressions such as $\varepsilon-\xi$). It is indeed very easy to verify that the operators

$$a_j^- = b_j^-, \qquad a_j^+ = b_j^+ \tag{22}$$

satisfy the compatibility conditions (20). Otherwise said, the triple relations (21) imply the relations (20). Furthermore, the $\star$-relations for the generators of $\mathscr{A}$ imply the following $\star$-relations for the $\mathfrak{osp}(1|2n)$ generators:

$$(b_j^\pm)^\dagger = b_j^\mp. \tag{23}$$

So we are led to investigating unitary representation of $\mathfrak{osp}(1|2n)$ for these $\star$-conditions.

In order to study the $\mathfrak{osp}(1|2n)$ solutions, it will be useful to identify some subalgebras of $\mathfrak{osp}(1|2n)$. First of all, note that due to the triple relations (21), a basis of $\mathfrak{osp}(1|2n)$ is given by the $2n$ *odd* elements $b_j^\pm$ and by the $2n^2+n$ *even* elements $\{b_j^\xi,b_k^\eta\}$ ($j,k\in\{1,\ldots,n\}$; $\eta,\xi\in\{+,-\}$). The even subalgebra of $\mathfrak{osp}(1|2n)$ is the symplectic Lie algebra $\mathfrak{sp}(2n)$, so a basis of $\mathfrak{sp}(2n)$ consists of all even elements $\{b_j^\xi,b_k^\eta\}$ ($j,k\in\{1,\ldots,n\}$; $\eta,\xi\in\{+,-\}$). A subalgebra of $\mathfrak{sp}(2n)$ is the general linear Lie algebra $\mathfrak{gl}(n)$, whose standard basis is given by the n^2 even elements $\frac{1}{2}\{b_j^+,b_k^-\}$ ($j,k\in\{1,\ldots,n\}$). Finally, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(1|2n)$ is that of its even subalgebra $\mathfrak{sp}(2n)$. A basis of $\mathfrak{h}$ is given by the n elements $h_j=\frac{1}{2}\{b_j^-,b_j^+\}$ ($j=1,\ldots,n$). So we have, in this realization of $\mathfrak{osp}(1|2n)$, a natural chain of subalgebras:

$$\mathfrak{osp}(1|2n)\supset\mathfrak{sp}(2n)\supset\mathfrak{gl}(n)\supset\mathfrak{h}. \tag{24}$$

Note that for this algebraic solution the Hamiltonian is written as

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\{a_j^-,a_j^+\} = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\{b_j^-,b_j^+\} = \hbar\sum_{j=1}^{n}\omega_j h_j, \tag{25}$$

so it is an element of the Cartan subalgebra. This will facilitate the problem of determining the spectrum of $\hat{H}$.

It should be mentioned that a second algebraic solution of the conditions (20) can be given by means of generators of the Lie superalgebra $\mathfrak{gl}(1|n)$ [19]. This class of solutions also gives rise to many interesting properties, but these cannot be presented in this short review.

The algebraic $\mathfrak{osp}(1|2n)$ solution to (20) is easy to describe. In fact, it was already known since 1982 for the simpler isotropic case with $\omega_1 = \cdots = \omega_n = \omega$ [26]. The reason why it was not studied further was because no class of unitary representations was known (for the $\star$-condition (23)). This changed in 2008, when Lievens *et al* [18] managed to construct a class of unitary representations. These are the infinite-dimensional lowest weight representations $V(p)$ of $\mathfrak{osp}(1|2n)$ with lowest weight $(\frac{p}{2},\ldots,\frac{p}{2})$. For these representations, the authors obtained an appropriate Gelfand-Zetlin basis, explicit actions of the generators on the basis vectors, and a character formula [18]. For these results, the subalgebra chain (24) plays an important role, in particular the decomposition with respect to the $\mathfrak{gl}(n)$ subalgebra. Irreducible characters of $\mathfrak{gl}(n)$ are given as a Schur function $s_\lambda(x_1,\ldots,x_n)$, where λ is a partition of length $\ell(\lambda)$ at most n (see the standard book [20] for notations of partitions, Schur functions, etc.). In such character formulas, the exponents of $(x_1,\ldots,x_n)$ carry the components of the corresponding weight of the representation according to the basis $(h_1,\ldots,h_n)$ of the Cartan subalgebra $\mathfrak{h}$. In other words, a term $x_1^{\nu_1}\cdots x_n^{\nu_n}$ corresponds to the weight $(\nu_1,\ldots,\nu_n)$.

The character determined in [18] can be described as follows:
The $\mathfrak{osp}(1|2n)$ representation $V(p)$ with lowest weight $(\frac{p}{2},\ldots,\frac{p}{2})$ is a unitary irreducible representation if and only if $p \in \{1,2,\ldots,n-1\}$ or $p > n-1$.

- For $p > n-1$, one has

$$\begin{aligned}\operatorname{char} V(p) &= \frac{(x_1\cdots x_n)^{p/2}}{\prod_i(1-x_i)\prod_{j<k}(1-x_jx_k)} \\ &= (x_1\cdots x_n)^{p/2}\sum_{\lambda} s_\lambda(x_1,\ldots,x_n).\end{aligned} \tag{26}$$

- For $p \in \{1,2,\ldots,n-1\}$, the character of $V(p)$ is given by

$$\operatorname{char} V(p) = (x_1\cdots x_n)^{p/2}\sum_{\lambda,\ \ell(\lambda)\le p} s_\lambda(x_1,\ldots,x_n) \tag{27}$$

where $\ell(\lambda)$ is the *length* of the partition λ.

Such characters can be used to determine the spectrum of $\hat{H}$ in the $\mathfrak{osp}(1|2n)$ representation $V(p)$. Indeed, as noted earlier, the character is a weight generating function:

$$\mathrm{char}\, V(p) = \sum_{\nu_1,\ldots,\nu_n} d_{\nu_1,\ldots,\nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n}, \tag{28}$$

where $(\nu_1,\ldots,\nu_n)$ is a weight from the representation and $d_{\nu_1,\ldots,\nu_n}$ stands for the multiplicity of this weight. Recall that in the current solution

$$\hat{H} = \sum_{j=1}^{n} \hbar\omega_j h_j, \tag{29}$$

i.e. $\hat{H}$ is an element from the Cartan subalgebra $\mathfrak{h}$. Hence, to get a *spectrum generating function* one must make the substitution $x_j \to t^{\hbar\omega_j}$ in the character (27). Expressions like $s_\lambda(t^{\hbar\omega_1}, t^{\hbar\omega_2}, \ldots, t^{\hbar\omega_n})$ simplify a lot in the isotropic case $\omega_1 = \cdots = \omega_n = \omega$, when $x_j \to t^{\hbar\omega} \equiv z$, since

$$s_\lambda(z,\ldots,z) = z^{|\lambda|} s_\lambda(1,\ldots,1). \tag{30}$$

Formulas for $s_\lambda(1,\ldots,1)$ are well known [20]; after all, $s_\lambda(1,\ldots,1)$ stands for the dimension of the $\mathfrak{gl}(n)$ representation characterized by the partition λ. So from (27) one obtains a "spectrum generating function" in the representation $V(p)$:

$$\begin{aligned} \mathrm{spec}\, \hat{H} &= z^{np/2} \sum_{\lambda,\, \ell(\lambda)\leq p} s_\lambda(z,\ldots,z) \\ &= \sum_{k\geq 0} \sum_{\lambda,\, |\lambda|=k,\, \ell(\lambda)\leq p} s_\lambda(1,\ldots,1) t^{\hbar\omega(np/2+k)}. \end{aligned} \tag{31}$$

In this series expansion, the power of t gives the energy level E, and the coefficient in front of t^E gives the multiplicity $\mu(E)$ of the energy level E. Clearly, we have equidistant energy levels

$$E_k^{(p)} = \hbar\omega(np/2+k), \quad k = 0,1,2,3,\cdots \tag{32}$$

with spacing $\hbar\omega$ and with multiplicities (degeneracies)

$$\mu(E_k^{(p)}) = \sum_{\lambda,\, |\lambda|=k,\, \ell(\lambda)\leq p} s_\lambda(1,\ldots,1). \tag{33}$$

In the representation $V(1)$ of $\mathfrak{osp}(1|2n)$, the CCRs are satisfied, so this is the representation corresponding to the canonical solution. One finds indeed that:

$$\mu(E_k^{(p=1)}) = \binom{n+k-1}{k}, \tag{34}$$

and (with $z = t^{\hbar\omega}$)

$$\mathrm{spec}\, \hat{H} = \frac{z^{n/2}}{(1-z)^n} = \sum_{k\geq 0} \binom{n+k-1}{k} t^{\hbar\omega(n/2+k)}, \tag{35}$$

which is a classical result.

For a more detailed analysis of spectrum generating functions for the other representations $V(p)$, we refer to [19]. Let us just give the results for the 3-dimensional oscillator, i.e. the case $n=3$. Then there are essentially three distinct cases to be considered for the $\mathfrak{osp}(1|6)$ representations $V(p)$, namely $p=1$, $p=2$ and $p>2$. The spectrum generating functions, the energy levels, and the energy multiplicities are given in the following table [19]:

	GF	levels	multiplicity
$p=1$	$\dfrac{z^{1/2}}{(1-z)^3}$	$\hbar\omega(\frac{3}{2}+k)$	$\mu(E_k^{(1)})=\binom{k+2}{2}$
$p=2$	$\dfrac{z^3(1+z+z^2)}{(1-z^2)^3(1-z)^2}$	$\hbar\omega(3+k)$	$\mu(E_{2k}^{(2)})=\binom{k+2}{2}^2$
			$\mu(E_{2k+1}^{(2)})=\binom{k+2}{2}\binom{k+3}{2}$
$p>2$	$\dfrac{z^{3p/2}}{(1-z^2)^3(1-z)^3}$	$\hbar\omega(\frac{3p}{2}+k)$	$\mu(E_{2k}^{(p)})=\frac{4k+5}{5}\binom{k+4}{4}$
			$\mu(E_{2k+1}^{(p)})=\frac{4k+15}{5}\binom{k+4}{4}$

Expanding the above generating functions, or alternatively working out the above multiplicities, one finds for the first few energy levels the following results:

	$\mu(E_0^{(p)})$	$\mu(E_1^{(p)})$	$\mu(E_2^{(p)})$	$\mu(E_3^{(p)})$	$\mu(E_4^{(p)})$
$p=1$	1	3	6	10	15
$p=2$	1	3	9	18	36
$p>2$	1	3	9	19	39

So, just as for the one-dimensional Wigner oscillator, this $\mathfrak{osp}(1|6)$ approach to the 3-dimensional Wigner oscillator leads to a shift in energy compared to the canonical case ($p=1$). Moreover, the degeneracies increase from the 3rd energy level onwards.

4 Recent advances: angular momentum operators and their spectrum

Now that the structure of the representations $V(p)$ of $\mathfrak{osp}(1|2n)$ is well known, we can also consider the angular momentum contents in the case that n is a multiple of 3 (i.e. if we work in 3-dimensional space). Let us first concentrate on the simple case that $n=3$, i.e. a 3-dimensional harmonic oscillator.

In the canonical case, the components of the angular momentum operator $\mathbf{M}$ are determined by $\mathbf{M}=\mathbf{q}\times\mathbf{p}$, in other words, $M_j=\sum_{k,l}\varepsilon_{jkl}\hat{q}_k\hat{p}_l$ ($j=1,2,3$), where ε_{jkl} is the Levi-Civita symbol. Since the position and momentum operators cannot be assumed to commute in Wigner quantization, and since we want Wigner quantization to coincide with canonical quantization when the CCRs do hold, it is logical to define the angular momentum operators in Wigner quantization by

$$M_j = \frac{1}{2}\sum_{k,l}\varepsilon_{jkl}\{\hat{q}_k,\hat{p}_l\} = \frac{-i\hbar}{2}\sum_{k,l}\varepsilon_{jkl}\{a_k^+,a_l^-\} \qquad (j=1,2,3). \tag{36}$$

The last expression follows from (18). Now one can investigate whether these operators satisfy any particular commutation relations. It turns out that using the CCs (20) do *not* lead to closed commutation relations between the operators M_1, M_2 and M_3. In other words, in the algebra $\mathscr{A}$, the commutation relations between the M_j do not close. Next, consider the $\mathfrak{osp}(1|6)$ solution with $a_j^\pm = b_j^\pm$ satisfying (21). Once again, the commutation relations between the M_j do not close, except when all ω_j are equal, i.e. except one works in the isotropic case. In that case, one finds:

$$[M_1,M_2] = i\hbar M_3 \qquad \text{(+ cyclic)}, \tag{37}$$

just as in the canonical case. For this reason, we shall now continue with the isotropic case. The above relations imply that we have identified an $\mathfrak{so}(3)$ subalgebra in our chain of subalgebras:

$$\mathfrak{osp}(1|6) \supset \mathfrak{sp}(6) \supset \mathfrak{gl}(3) \supset \mathfrak{so}(3)\oplus\mathfrak{u}(1). \tag{38}$$

Herein $\mathfrak{so}(3)$ is generated by M_1, M_2 and M_3, and $\mathfrak{u}(1)$ by the Hamiltonian $\hat{H} = \hbar\omega(h_1+h_2+h_3)$. In other words, the $\mathfrak{so}(3)\oplus\mathfrak{u}(1)$ decomposition of a representation gives us the angular momentum and energy contents of the Wigner quantum system in that representation.

In the current situation, the question is: how does the representation $V(p)$ of $\mathfrak{osp}(1|6)$ decompose with respect to these subalgebras? As before, the answer follows from the expression of the character of $V(p)$,

$$\operatorname{char} V(p) = (x_1x_2x_3)^{p/2} \sum_{\lambda,\ell(\lambda)\le\lceil p\rceil} s_\lambda(x_1,x_2,x_3) \tag{39}$$

where there are three distinct cases to be considered: $p=1$, $p=2$ or $p>2$. Since this character already gives in a straightforward way the decomposition of $V(p)$ with respect to $\mathfrak{gl}(3)$, it remains to determine the next step in the “branching”, from $\mathfrak{gl}(3)$ to $\mathfrak{so}(3)\oplus\mathfrak{u}(1)$. This step has a well known solution and is known as the $U(3)$ to $SO(3)$ branching [6]. In our notation, where the $\mathfrak{gl}(3)$ representation is characterized by a partition $\lambda = (\lambda_1,\lambda_2,\lambda_3)$, this $\mathfrak{gl}(3)$ to $\mathfrak{so}(3)\oplus\mathfrak{u}(1)$ branching rule generating function reads

$$G = \frac{1+A_1^2A_2z^3J}{(1-A_1A_2A_3z^3)(1-A_1zJ)(1-A_1A_2z^2J)(1-A_1^2z^2)(1-A_1^2A_2^2z^4)}. \tag{40}$$

In the expansion of G as a power series, the coefficient of $A_1^{\lambda_1}A_2^{\lambda_2}A_3^{\lambda_3}$ is a polynomial $p_\lambda(J,z) = \sum \mu^{\lambda}_{j,E} J^j z^E$ in J and z. The coefficient $\mu^{\lambda}_{j,E}$ is the multiplicity of the $\mathfrak{so}(3)\oplus\mathfrak{u}(1)$ representation $(j,E) = (j)\oplus(E)$ in the decomposition of the $\mathfrak{gl}(3)$ representation (characterized by) λ.

Using (39) and (40), it now follows that we have generating functions for the angular momentum and energy contents for the representations $V(p)$ of $\mathfrak{osp}(1|6)$. For $V(1)$:

$$G_1 = \frac{z^{3/2}}{(1-zJ)(1-z^2)}.$$

For $V(2)$:

$$G_2 = \frac{z^3(1+z^3J)}{(1-zJ)(1-z^2J)(1-z^2)(1-z^4)}.$$

For $V(p)$, $p>2$:

$$G_p = \frac{z^{3p/2}(1+z^3J)}{(1-zJ)(1-z^2J)(1-z^2)(1-z^3)(1-z^4)}.$$

Clearly, one can use these generating functions to derive the $\mathfrak{so}(3)$ representations that emerge at energy level $E_k^{(p)}$. This information can be made accessible by means of a table in which the element in row $k+1$ and column $j+1$ (counted from the bottom) marks the number of representations (j) at energy level $E_k^{(p)}$ in the angular momentum decomposition of $\mathfrak{osp}(1|6)$. We call this the (j,E)-diagram of $\mathfrak{osp}(1|6)$ for $V(p)$. For G_1, the expansion gives

$$G_1 = z^{3/2} + Jz^{5/2} + (1+J^2)z^{7/2} + (J+J^3)z^{9/2} + (1+J^2+J^4)z^{11/2} + \cdots,$$

yielding the following (j,E)-diagram:

$$\begin{array}{c|cccccc}
\vdots & & & & & & \iddots \\
11/2 & 1 & & 1 & & 1 & \\
9/2 & & 1 & & 1 & & \\
7/2 & 1 & & 1 & & & \\
5/2 & & 1 & & & & \\
3/2 & 1 & & & & & \\
\hline
E_k & & & & & & \\
\quad j & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}$$

Of course, this result is already known because $p=1$ represents the canonical case. This (j,E)-diagram for instance appears in [37].

For $p=2$, the expansion of G_2 gives rise to the following diagram:

$$
\begin{array}{c|cccccc}
\vdots & & & & & & \iddots \\
7 & 2 & 1 & 3 & 1 & 1 & \\
6 & & 2 & 1 & 1 & & \\
5 & 1 & 1 & 1 & & & \\
4 & & 1 & & & & \\
3 & 1 & & & & & \\
\hline
E_k & & & & & & \\
\quad j & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}
$$

and for $p > 2$ the expansion of G_p gives:

$$
\begin{array}{c|cccccc}
\vdots & & & & & & \iddots \\
3p/2+4 & 2 & 2 & 3 & 1 & 1 & \\
3p/2+3 & 1 & 2 & 1 & 1 & & \\
3p/2+2 & 1 & 1 & 1 & & & \\
3p/2+1 & & 1 & & & & \\
3p/2 & 1 & & & & & \\
\hline
E_k & & & & & & \\
\quad j & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}
$$

Note that for the lower energy levels, the cases $p = 2$ and $p > 2$ do not differ very much from the canonical case. The larger discrepancies are found in higher energy regions.

So far, we have considered only the 3-dimensional Wigner harmonic oscillator and its angular momentum contents in the $\mathfrak{osp}(1|6)$ representations $V(p)$. In the more general case of $\mathfrak{osp}(1|2n)$ with $n = 3N$, the Hamiltonian $\hat{H}$ can be interpreted as an N-particle 3-dimensional oscillator. It is common to write the position operators and momentum operators by a multi-index: e.g. the position operators are $\hat{q}_{j,\alpha}$, with $j = 1,2,3$ (referring to the 3 dimensions) and $\alpha = 1,2,\ldots,N$ (referring to the N particles). The angular momentum operators of particle α are given by

$$M_{j,\alpha} = \frac{1}{2}\sum_{k,l}\varepsilon_{jkl}\{\hat{q}_{k,\alpha},\hat{p}_{l,\alpha}\} = \frac{-i\hbar}{2}\sum_{k,l}\varepsilon_{jkl}\{a^+_{k,\alpha},a^-_{l,\alpha}\}$$

and then the components of the total angular momentum operator are

$$M_j = \sum_{\alpha=1}^{N} M_{j,\alpha} \qquad (j = 1,2,3).$$

If we want these operators to satisfy the commutation relations (37), we need again to work in the $\mathfrak{osp}(1|6N)$ picture where $a^\pm_{k,\alpha} = b^\pm_{k,\alpha}$ and moreover in the case that all ω_j's are equal to ω (i.e. N identical isotropic oscillators). In $\mathfrak{osp}(1|6N) \supset \mathfrak{sp}(6N) \supset \mathfrak{gl}(3N)$, the $\mathfrak{gl}(3N)$ basis elements are $\{b^+_{j,\alpha}, b^-_{k,\beta}\}$. The relevant subalgebras of $\mathfrak{gl}(3N)$ are $\mathfrak{gl}(3)$ and $\mathfrak{gl}(N)$, with basis elements:

$$\mathfrak{gl}(3): \qquad E_{jk} = \frac{1}{2}\sum_{\alpha}\{b^+_{j,\alpha}, b^-_{k,\alpha}\} \qquad (j,k = 1,2,3),$$
$$\mathfrak{gl}(N): \qquad \mathscr{E}_{\alpha,\beta} = \frac{1}{2}\sum_{j}\{b^+_{j,\alpha}, b^-_{j,\beta}\} \qquad (\alpha,\beta = 1,2,\ldots,N).$$

So the total angular momentum operators M_1, M_2, M_3 are the basis elements of an $\mathfrak{so}(3)$ subalgebra of $\mathfrak{gl}(3)$, and one needs to decompose the representations $V(p)$ of $\mathfrak{osp}(1|6N)$ according to

$$\mathfrak{osp}(1|6N) \supset \mathfrak{sp}(6N) \supset \mathfrak{gl}(3N) \supset \mathfrak{gl}(3) \oplus \mathfrak{gl}(N) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1)$$

Although the decomposition of $\mathfrak{gl}(3N)$ representations according to $\mathfrak{gl}(3N) \supset \mathfrak{gl}(3) \oplus \mathfrak{gl}(N)$ is in principle known, it turns out to be computationally quite involved when $N \geq 2$. For details, and results with angular momentum and energy decompositions, see [33].

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