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# Unitary representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$ and parabosons

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**Abstract**

It is known that there is a close connection between the Fock space of $n$ pairs of boson operators $B_i^\pm$ ($i = 1, 2, \ldots, n$) and the so-called *metaplectic* representation $V(1)$ of the Lie superalgebra $\mathfrak{osp}(1|2n)$ with lowest weight $(1/2, 1/2, \ldots, 1/2)$. On the other hand, the defining relations of $\mathfrak{osp}(1|2n)$ are equivalent to the defining relations of $n$ pairs of paraboson operators $b_i^\pm$. In particular, with the usual star conditions, this implies that the "parabosons of order $p$" correspond to a unitary irreducible (infinite-dimensional) lowest weight representation $V(p)$ of $\mathfrak{osp}(1|2n)$ with lowest weight $(p/2, p/2, \ldots, p/2)$. Apart from the simple cases $p = 1$ or $n = 1$, these representations had never been constructed due to computational difficulties, despite their importance.

We have now managed to give an explicit and elegant construction of these representations $V(p)$, and can present explicit actions or matrix elements of the $\mathfrak{osp}(1|2n)$ generators. Essentially, $V(p)$ is constructed as a quotient module of an induced module. In all steps of the construction and for the chosen basis vectors, the subalgebra $\mathfrak{u}(n)$ of $\mathfrak{osp}(1|2n)$ plays a crucial role.

## 1 Introduction

Green [1] generalized the classical notion of Bose operators or bosons to parabose operators or parabosons. The first applications were in quantum field theory [2–4] and quantum statistics (para-statistics) [1, 5]. The generalization of the usual boson Fock space is characterized by a parameter $p$, referred to as the order of the paraboson. For a single paraboson, $n = 1$, the structure of the paraboson Fock space is well known [6]. Surprisingly, for a system of $n$ parabosons with $n > 1$, the structure of the paraboson Fock space is not known, even though it can in principle be constructed by means of the so-called Green ansatz [1, 5].

In this paper, we analyse the structure of the paraboson Fock space, for arbitrary $p$ and $n$. An important step in the solution was made many years ago by Ganchev and Palev [7], who discovered the relation between $n$ pairs of parabosons and the defining relations for the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [8]. From their result it follows that the paraboson Fock space of order $p$ is a certain infinite-dimensional unitary irreducible representation (unirrep) $V(p)$ of $\mathfrak{osp}(1|2n)$. Here, we construct this representation $V(p)$ explicitly. Our solution uses group theoretical techniques, in particular the branching $\mathfrak{osp}(1|2n) \supset \mathfrak{sp}(2n) \supset \mathfrak{u}(n)$. This allows us to construct a proper Gelfand-Zetlin (GZ) basis for some induced representation [9], from which the basis for the irreducible representation $V(p)$ follows. For the representation $V(p)$ we give an orthogonal GZ-basis, the action (matrix elements) of the paraboson operators in this basis, and also the character. More details of the analysis presented here can be found in the extended paper [10].

## 2 The paraboson Fock space $V(p)$

For a single pair ($n = 1$) of paraboson operators $b^+$, $b^-$ [1], the defining relation is a triple relation (with anticommutator $\{.,.\}$ and commutator $[.,.]$) given by

$$[\{b^-, b^+\}, b^\pm] = \pm 2b^\pm. \tag{1}$$

The paraboson Fock space [6] is a Hilbert space with vacuum vector $|0\rangle$, characterized by the following conditions:

$$\begin{aligned} &\langle 0|0\rangle = 1, \qquad b^-|0\rangle = 0, \qquad (b^\pm)^\dagger = b^\mp, \\ &\{b^-, b^+\}|0\rangle = p\,|0\rangle, \end{aligned} \tag{2}$$

and by irreducibility under the action of $b^+$, $b^-$. Herein, $p$ is a parameter, known as the order of the paraboson. In order to have a genuine inner product for the Hilbert space, $p$ should be positive and real: $p > 0$. A set of basis vectors for this space, which will be denoted by $V(p)$, is given by

$$|2k\rangle = \frac{(b^+)^{2k}}{2^k\sqrt{k!(p/2)_k}}|0\rangle, \qquad |2k+1\rangle = \frac{(b^+)^{2k+1}}{2^k\sqrt{k!2(p/2)_{k+1}}}|0\rangle. \tag{3}$$

This basis is orthogonal and normalized; the symbol $(a)_k = a(a+1)\cdots(a+k-1)$ is the common Pochhammer symbol.

If one considers $b^+$ and $b^-$ as odd generators of a Lie superalgebra, then the elements $\{b^+, b^+\}$, $\{b^+, b^-\}$ and $\{b^-, b^-\}$ form a basis for the even part of this superalgebra. Using the relations (1) it is easy to see that this superalgebra is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$, with even part $\mathfrak{sp}(2) = \mathfrak{sp}(2, \mathbb{R})$. The paraboson Fock space $V(p)$ is then a unirrep of $\mathfrak{osp}(1|2)$. It splits as the direct sum of two positive discrete series representations of $\mathfrak{sp}(2)$: one with lowest weight vector $|0\rangle$ (lowest weight $p/2$) and basis vectors $|2k\rangle$, and one with lowest weight vector $|1\rangle$ (lowest weight $1+p/2$) and basis vectors $|2k+1\rangle$. For $p = 1$ the paraboson Fock space coincides with the ordinary boson Fock space.

Let us now consider the case of $n$ pairs of paraboson operators $b_j^\pm$ $(j = 1,\ldots,n)$. The defining triple relations for such a system are given by [1]

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi, \tag{4}$$

where $j,k,l \in \{1,2,\ldots,n\}$ and $\eta,\epsilon,\xi \in \{+,-\}$ (to be interpreted as $+1$ and $-1$ in the algebraic expressions $\epsilon - \xi$ and $\epsilon - \eta$). The paraboson Fock space $V(p)$ is the Hilbert space with vacuum vector $|0\rangle$, defined by means of $(j,k = 1,2,\ldots,n)$

$$\begin{aligned} &\langle 0|0\rangle = 1, \qquad b_j^-|0\rangle = 0, \qquad (b_j^\pm)^\dagger = b_j^\mp, \\ &\{b_j^-, b_k^+\}|0\rangle = p\,\delta_{jk}\,|0\rangle, \end{aligned} \tag{5}$$

and by irreducibility under the action of the algebra spanned by the elements $b_j^+$, $b_j^-$ $(j = 1,\ldots,n)$, subject to (4). The parameter $p$ is referred to as the order of the paraboson system. In general $p$ is thought of as a positive integer, and for $p = 1$ the paraboson Fock space $V(p)$ coincides with the ordinary $n$-boson Fock space. We shall see that also certain non-integer $p$-values are allowed.

Constructing a basis for the Fock space $V(p)$ turns out to be a difficult problem. Even the simpler question of finding the structure of $V(p)$ (weight structure) was not solved until now. In [10] we unravel the structure of $V(p)$, determine for which $p$-values $V(p)$ is actually a Hilbert space, construct an orthogonal (normalized) basis for $V(p)$, and give the actions of the generators $b_j^\pm$ on the basis vectors. A summary will be presented here.

## 3 Relation with the Lie superalgebra $\mathfrak{osp}(1|2n)$

The orthosymplectic superalgebra $\mathfrak{osp}(1|2n)$ is one of the basic classical Lie superalgebras [8]. It consists of matrices of the form

$$\begin{pmatrix} 0 & a & a_1 \\ a_1^t & b & c \\ -a^t & d & -b^t \end{pmatrix}, \tag{6}$$

where $a$ and $a_1$ are $(1 \times n)$-matrices, $b$ is any $(n \times n)$-matrix, and $c$ and $d$ are symmetric $(n \times n)$-matrices. The even elements have $a = a_1 = 0$ and the odd elements are those with $b = c = d = 0$. It will be convenient to have the row and column indices running from 0 to $2n$ (instead of 1 to $2n+1$), and to denote by $e_{ij}$ the matrix with zeros everywhere except a 1 on position $(i,j)$. Then the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(1|2n)$ is spanned by the diagonal elements

$$h_j = e_{jj} - e_{n+j,n+j} \qquad (j = 1,\ldots,n). \tag{7}$$

In terms of the dual basis $\delta_j$ of $\mathfrak{h}^*$, the odd root vectors and corresponding roots of $\mathfrak{osp}(1|2n)$ are given by:

$$\begin{aligned} &e_{0,k} - e_{n+k,0} \leftrightarrow -\delta_k, \qquad k = 1,\ldots,n, \\ &e_{0,n+k} + e_{k,0} \leftrightarrow \delta_k, \qquad k = 1,\ldots,n. \end{aligned}$$

The even roots and root vectors are

$$\begin{aligned} e_{j,k} - e_{n+k,n+j} &\leftrightarrow \delta_j - \delta_k, \qquad j \neq k = 1, \ldots, n, \\ e_{j,n+k} + e_{k,n+j} &\leftrightarrow \delta_j + \delta_k, \qquad j \leq k = 1, \ldots, n, \\ e_{n+j,k} + e_{n+k,j} &\leftrightarrow -\delta_j - \delta_k, \qquad j \leq k = 1, \ldots, n. \end{aligned}$$

If we introduce the following multiples of the odd root vectors

$$b_k^+ = \sqrt{2}(e_{0,n+k} + e_{k,0}), \qquad b_k^- = \sqrt{2}(e_{0,k} - e_{n+k,0}) \qquad (k = 1, \ldots, n) \quad (8)$$

then it is easy to verify that these operators satisfy the triple relations (4). Since all even root vectors can be obtained by anticommutators $\{b_j^\xi, b_k^\eta\}$, the following holds [7]

**Theorem 1 (Ganchev and Palev)** *As a Lie superalgebra defined by generators and relations,* $\mathfrak{osp}(1|2n)$ *is generated by* $2n$ *odd elements* $b_k^\pm$ *subject to the following (paraboson) relations:*

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi. \quad (9)$$

The paraboson operators $b_j^+$ are the positive odd root vectors, and the $b_j^-$ are the negative odd root vectors.

Recall that the paraboson Fock space $V(p)$ is characterized by (5). Furthermore, it is easy to verify that

$$\{b_j^-, b_j^+\} = 2h_j \qquad (j = 1, \ldots, n). \quad (10)$$

Hence we have the following:

**Corollary 2** *The paraboson Fock space* $V(p)$ *is the unitary irreducible representation of* $\mathfrak{osp}(1|2n)$ *with lowest weight* $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$.

In order to construct the representation $V(p)$ [11] one can use an induced module construction. The relevant subalgebras of $\mathfrak{osp}(1|2n)$ are easy to describe by means of the odd generators $b_j^\pm$.

**Proposition 3** *A basis for the even subalgebra* $\mathfrak{sp}(2n)$ *of* $\mathfrak{osp}(1|2n)$ *is given by the* $2n^2 + n$ *elements*

$$\{b_j^\pm, b_k^\pm\} \quad (1 \leq j \leq k \leq n), \quad \{b_j^+, b_k^-\} \quad (1 \leq j, k \leq n). \quad (11)$$

*The* $n^2$ *elements*

$$\{b_j^+, b_k^-\} \qquad (j, k = 1, \ldots, n) \quad (12)$$

*are a basis for the* $\mathfrak{sp}(2n)$ *subalgebra* $\mathfrak{u}(n)$.

Note that with $\{b_j^+, b_k^-\} = 2E_{jk}$, the triple relations (9) imply the relations $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$. In other words, the elements $\{b_j^+, b_k^-\}$ form, up to a factor 2, the standard $\mathfrak{u}(n)$ or $\mathfrak{gl}(n)$ basis elements.

So the odd generators $b_j^\pm$ clearly reveal the subalgebra chain $\mathfrak{osp}(1|2n) \supset \mathfrak{sp}(2n) \supset \mathfrak{u}(n)$. Note that $\mathfrak{u}(n)$ is, algebraically, the same as the general linear Lie algebra $\mathfrak{gl}(n)$. But the conditions $(b_j^\pm)^\dagger = b_j^\mp$ imply that we are dealing here with the "compact form" $\mathfrak{u}(n)$.

The subalgebra $\mathfrak{u}(n)$ can be extended to a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{osp}(1|2n)$ [11]:

$$\mathcal{P} = \text{span}\{\{b_j^+, b_k^-\}, b_j^-, \{b_j^-, b_k^-\} \mid j,k = 1,\ldots,n\}. \tag{13}$$

Recall that $\{b_j^-, b_k^+\}|0\rangle = p\,\delta_{jk}\,|0\rangle$, with $\{b_j^-, b_j^+\} = 2h_j$. This means that the space spanned by $|0\rangle$ is a trivial one-dimensional $\mathfrak{u}(n)$ module $\mathbb{C}|0\rangle$ of weight $(\frac{p}{2},\ldots,\frac{p}{2})$. Since $b_j^-|0\rangle = 0$, the module $\mathbb{C}|0\rangle$ can be extended to a one-dimensional $\mathcal{P}$ module. Now we are in a position to define the induced $\mathfrak{osp}(1|2n)$ module $\overline{V}(p)$:

$$\overline{V}(p) = \text{Ind}_{\mathcal{P}}^{\mathfrak{osp}(1|2n)}\mathbb{C}|0\rangle. \tag{14}$$

This is an $\mathfrak{osp}(1|2n)$ representation with lowest weight $(\frac{p}{2},\ldots,\frac{p}{2})$. By the Poincaré-Birkhoff-Witt theorem [9, 11], it is easy to give a basis for $\overline{V}(p)$:

$$\begin{aligned}&(b_1^+)^{k_1}\cdots(b_n^+)^{k_n}(\{b_1^+,b_2^+\})^{k_{12}}(\{b_1^+,b_3^+\})^{k_{13}}\cdots(\{b_{n-1}^+,b_n^+\})^{k_{n-1,n}}|0\rangle,\\ &\qquad k_1,\ldots,k_n,k_{12},k_{13}\ldots,k_{n-1,n}\in\mathbb{Z}_+.\end{aligned} \tag{15}$$

The difficulty comes from the fact that in general $\overline{V}(p)$ is not a simple module (i.e. not an irreducible representation) of $\mathfrak{osp}(1|2n)$. Let $M(p)$ be the maximal nontrivial submodule of $\overline{V}(p)$. Then the simple module (irreducible module), corresponding to the paraboson Fock space, is

$$V(p) = \overline{V}(p)/M(p). \tag{16}$$

The purpose is now to determine the vectors belonging to $M(p)$, and hence to find the structure of $V(p)$. This is not our only goal: we also want to find explicit matrix elements of the $\mathfrak{osp}(1|2n)$ generators $b_j^\pm$ in an appropriate basis of $V(p)$.

## 4 Paraboson Fock representations of $\mathfrak{osp}(1|2n)$

We shall start our analysis by considering the induced module $\overline{V}(p)$. Computing the action of the generators $b_j^\pm$ in the basis (15) turns out to be very difficult: we managed to do this for $\mathfrak{osp}(1|4)$ (i.e. $n = 2$) but not for larger values of $n$. Furthermore, from this action itself it was still very hard to determine the vectors belonging to $M(p)$. For this reason, we have introduced a new basis for $\overline{V}(p)$. In this new basis, matrix elements can be computed, and from these expressions it will be clear which vectors belong to $M(p)$.

The "old" basis for $\overline{V}(p)$ has been given in (15). From this basis, it is easy to write down the character of $\overline{V}(p)$: this is a formal infinite series of terms $\mu x_1^{j_1}\cdots x_n^{j_n}$, with $(j_1,\ldots,j_n)$ a weight of $\overline{V}(p)$ and $\mu$ the dimension of this weight space. So the vacuum vector $|0\rangle$ of $\overline{V}(p)$, of weight $(\frac{p}{2},\ldots,\frac{p}{2})$, yields a term $x_1^{\frac{p}{2}} \cdots x_n^{\frac{p}{2}} = (x_1 \cdots x_n)^{p/2}$ in the character $\mathrm{char}\, \overline{V}(p)$. The weight of a general vector follows from (15), and contributes a term

$$x_1^{k_1} \cdots x_n^{k_n} (x_1 x_2)^{k_{12}} (x_1 x_3)^{k_{13}} \cdots (x_{n-1} x_n)^{k_{n-1,n}} (x_1 \cdots x_n)^{p/2}$$

in the character of $\overline{V}(p)$. Summing over all basis vectors (i.e. over all $k_1, \ldots, k_n, k_{12}, k_{13} \ldots, k_{n-1,n}$) yields

$$\mathrm{char}\, \overline{V}(p) = \frac{(x_1 \cdots x_n)^{p/2}}{\prod_{i=1}^n (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_j x_k)}. \tag{17}$$

Such expressions have an interesting expansion in terms of Schur functions.

**Proposition 4 (Cauchy, Littlewood)** *Let $x_1, \ldots, x_n$ be a set of $n$ variables. Then [12]*

$$\frac{1}{\prod_{i=1}^n (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_j x_k)} = \sum_\lambda s_\lambda(x_1, \ldots, x_n) = \sum_\lambda s_\lambda(x) \tag{18}$$

*In the right hand side, the sum is over all partitions $\lambda$ and $s_\lambda(x)$ is the Schur symmetric function [13].*

For $n$ variables, $s_\lambda(x) = 0$ if the length $\ell(\lambda)$ is greater than $n$, so in practice the sum is over all partitions of length less than or equal to $n$.

The characters of finite dimensional $\mathfrak{u}(n)$ representations are given by such Schur functions $s_\lambda(x)$. Hence such expansions are useful since they yield the branching to $\mathfrak{u}(n)$ of the $\mathfrak{osp}(1|2n)$ representation $\overline{V}(p)$. But for such finite dimensional $\mathfrak{u}(n)$ representations labelled by a partition $\lambda$, there is a known basis: the Gelfand-Zetlin basis (GZ) [14, 15]. We shall use the $\mathfrak{u}(n)$ GZ basis vectors as our new basis for $\overline{V}(p)$. Thus the new basis of $\overline{V}(p)$ consists of vectors of the form

$$|m) \equiv |m)^n \equiv \left| \begin{array}{lllll} m_{1n} & \cdots & \cdots & m_{n-1,n} & m_{nn} \\ m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & \\ \vdots & \iddots & & & \\ m_{11} & & & & \end{array} \right) = \left| \begin{array}{l} [m]^n \\ |m)^{n-1} \end{array} \right) \tag{19}$$

where the top line of the pattern, also denoted by the $n$-tuple $[m]^n$, is any partition $\lambda$ (consisting of non increasing nonnegative numbers) with $\ell(\lambda) \leq n$. The label $p$ itself is dropped in the notation of the vectors $|m)$; strictly speaking we should denote our vectors by $|p; m)$. The remaining $n - 1$ lines of the pattern will sometimes be denoted by $|m)^{n-1}$. So all $m_{ij}$ in the above GZ-pattern are nonnegative integers, satisfying the *betweenness conditions*

$$m_{i,j+1} \geq m_{ij} \geq m_{i+1,j+1} \qquad (1 \leq i \leq j \leq n-1). \tag{20}$$

Note that, since the weight of $|0\rangle$ is $(\frac{p}{2}, \ldots, \frac{p}{2})$, the weight of the above vector is determined by

$$h_k |m) = \left( \frac{p}{2} + \sum_{j=1}^{k} m_{jk} - \sum_{j=1}^{k-1} m_{j,k-1} \right) |m). \tag{21}$$

Now we develop a technique to compute the matrix elements of $b_i^\pm$ in this basis. By the triple relations, one obtains

$$[\{b_i^+, b_j^-\}, b_k^+] = 2\delta_{jk} b_i^+.$$

With the identification $\{b_i^+, b_j^-\} = 2E_{ij}$ in the standard $\mathfrak{u}(n)$ basis, this is equivalent to the adjoint action $E_{ij} \cdot e_k = \delta_{jk} e_i$. In other words, the triple relations imply that $(b_1^+, b_2^+, \ldots, b_n^+)$ is a standard $\mathfrak{u}(n)$ tensor of rank $(1, 0, \ldots, 0)$. This means that one can attach a unique GZ-pattern with top line $10 \cdots 0$ to every $b_j^+$, corresponding to the weight $+\delta_j$. Explicitly:

$$b_j^+ \sim \begin{array}{l} 10\cdots 000 \\ 10\cdots 00 \\ \cdots \\ 0\cdots 0 \\ \cdots \\ 0 \end{array}, \tag{22}$$

where the pattern consists of $j-1$ zero rows at the bottom, and the first $n-j+1$ rows are of the form $10\cdots 0$. The tensor product rule in $\mathfrak{u}(n)$ reads

$$([m]^n) \otimes (10\cdots 0) = ([m]^n_{+1}) \oplus ([m]^n_{+2}) \oplus \cdots \oplus ([m]^n_{+n}) \tag{23}$$

where $([m]^n) = (m_{1n}, m_{2n}, \ldots, m_{nn})$ and a subscript $\pm k$ indicates an increment of the $k$th label by $\pm 1$:

$$([m]^n_{\pm k}) = (m_{1n}, \ldots, m_{kn} \pm 1, \ldots, m_{nn}). \tag{24}$$

In the right hand side of (23), only those components which are still partitions (i.e. consisting of nondecreasing integers) survive.

A general matrix element of $b_j^+$ can now be written as follows:

$$\begin{aligned}(m'|b_j^+|m) &= \left( \begin{array}{l} [m]^n_{+k} \\ |m')^{n-1} \end{array} \middle| b_j^+ \middle| \begin{array}{l} [m]^n \\ |m)^{n-1} \end{array} \right) \\ &= \left( \begin{array}{l} [m]^n \\ |m)^{n-1} \end{array} ; \begin{array}{l} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \middle| \begin{array}{l} [m]^n_{+k} \\ |m')^{n-1} \end{array} \right) \times ([m]^n_{+k}||b^+||[m]^n).\end{aligned} \tag{25}$$

The first factor in the right hand side is a $\mathfrak{u}(n)$ Clebsch-Gordan coefficient [16], the second factor is a reduced matrix element. By the tensor product rule, the first line of $|m')$ has to be of the form (24), i.e. $[m']^n = [m]^n_{+k}$ for some $k$-value.

The special $\mathfrak{u}(n)$ CGCs appearing here are well known, and have fairly simple expressions. They can be found, e.g. in [16]. They can be expressed by means of $\mathfrak{u}(n)$-$\mathfrak{u}(n-1)$ *isoscalar factors* and $\mathfrak{u}(n-1)$ CGC's, which on their turn are written by means of $\mathfrak{u}(n-1)$-$\mathfrak{u}(n-2)$ isoscalar factors and $\mathfrak{u}(n-2)$ CGC's, etc. The explicit form of the special $\mathfrak{u}(n)$ CGCs appearing here is given in Appendix A of [10]. The actual problem is now converted into finding expressions for the reduced matrix elements, i.e. for the functions $F_k([m]^n)$, for arbitrary $n$-tuples of non increasing nonnegative integers $[m]^n = (m_{1n}, m_{2n}, \ldots, m_{nn})$:

$$F_k([m]^n) = F_k(m_{1n}, m_{2n}, \ldots, m_{nn}) = ([m]^n_{+k}||b^+||[m]^n). \tag{26}$$

So one can write:

$$b_j^+|m) = \sum_{k,m'} \left( \begin{array}{l} [m]^n \\ \\ |m)^{n-1} \end{array} ; \begin{array}{l} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \left| \begin{array}{l} [m]^n_{+k} \\ \\ |m')^{n-1} \end{array} \right. \right) F_k([m]^n) \left| \begin{array}{l} [m]^n_{+k} \\ |m')^{n-1} \end{array} \right), \tag{27}$$

$$b_j^-|m) = \sum_{k,m'} \left( \begin{array}{l} [m]^n_{-k} \\ \\ |m')^{n-1} \end{array} ; \begin{array}{l} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \left| \begin{array}{l} [m]^n \\ \\ |m)^{n-1} \end{array} \right. \right) F_k([m]^n_{-k}) \left| \begin{array}{l} [m]^n_{-k} \\ |m')^{n-1} \end{array} \right). \tag{28}$$

For $j = n$, the CGCs in (27)-(28) take a simple form [16], and we have

$$\begin{aligned} b_n^+|m) = & \sum_{i=1}^n \left( \frac{\prod_{k=1}^{n-1}(m_{k,n-1} - m_{in} - k + i - 1)}{\prod_{k\neq i=1}^{n}(m_{kn} - m_{in} - k + i)} \right)^{1/2} \\ & \times F_i(m_{1n}, m_{2n}, \ldots, m_{nn})\ |m)_{+in}; \end{aligned} \tag{29}$$

$$\begin{aligned} b_n^-|m) = & \sum_{i=1}^n \left( \frac{\prod_{k=1}^{n-1}(m_{k,n-1} - m_{in} - k + i)}{\prod_{k\neq i=1}^{n}(m_{kn} - m_{in} - k + i + 1)} \right)^{1/2} \\ & \times F_i(m_{1n}, \ldots, m_{in} - 1, \ldots, m_{nn})\ |m)_{-in}. \end{aligned} \tag{30}$$

In order to determine the $n$ unknown functions $F_k$, one can start from the following action:

$$\{b_n^-, b_n^+\}|m) = 2h_n|m) = (p + 2(\sum_{j=1}^n m_{jn} - \sum_{j=1}^{n-1} m_{j,n-1}))|m). \tag{31}$$

Expressing the left hand side by means of (29)-(30), one finds a system of coupled recurrence relations for the functions $F_k$. Taking the appropriate boundary conditions into account, we have been able to solve this system of relations. The calculations were still very hard, and without the use of Maple we would not have found the solution. We only present the final result here:

**Proposition 5** *The reduced matrix elements $F_k$ appearing in the actions of $b_j^\pm$ on vectors $|m)$ of $\overline{V}(p)$ are given by:*

$$\begin{aligned} F_k(m_{1n}, m_{2n}, \ldots, m_{nn}) = & (-1)^{m_{k+1,n}+\cdots+m_{nn}} \\ & \times (m_{kn} + n + 1 - k + \mathcal{E}_{m_{kn}}(p-n))^{1/2} \\ & \times \prod_{j\neq k=1}^{n} \left( \frac{m_{jn} - m_{kn} - j + k}{m_{jn} - m_{kn} - j + k - \mathcal{O}_{m_{jn}-m_{kn}}} \right)^{1/2}, \end{aligned} \tag{32}$$

*where $\mathcal{E}$ and $\mathcal{O}$ are the* even *and* odd *functions defined by*

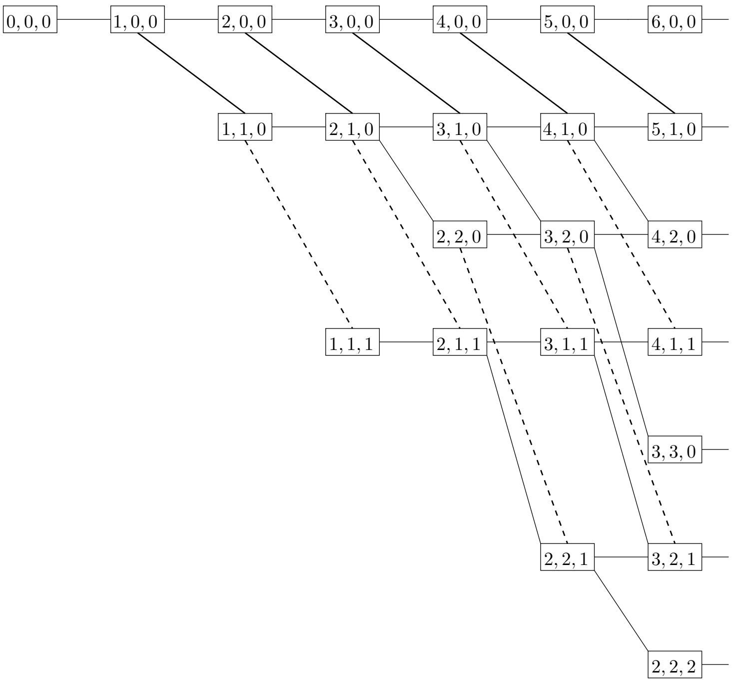
$$\mathcal{E}_j = 1 \text{ if } j \text{ is even and 0 otherwise},$$
$$\mathcal{O}_j = 1 \text{ if } j \text{ is odd and 0 otherwise}. \tag{33}$$

The proof consists of verifying that all triple relations (9) hold when acting on any vector $|m)$. Each such verification leads to an algebraic identity in $n$ variables $m_{1n}, \ldots, m_{nn}$. In these computations, there are some intermediate verifications: e.g. the action $\{b_j^+, b_k^-\}|m)$ should leave the top row of the GZ-pattern $|m)$ invariant (since $\{b_j^+, b_k^-\}$ belongs to $\mathfrak{u}(n)$). Furthermore, it must yield (up to a factor 2) the known action of the standard $\mathfrak{u}(n)$ matrix elements $E_{jk}$ in the classical GZ-basis.

The next task is to deduce the structure of $V(p)$ from the general expression of the matrix elements in $\overline{V}(p)$. For this purpose, it is essential to investigate for which basis elements $|m')$ the reduced matrix element $([m']^n||b^+||[m]^n)$ becomes zero. This will be governed by the factor

$$(m_{kn} + n + 1 - k + \mathcal{E}_{m_{kn}}(p-n))$$

in the expression of $F_k([m]^n)$, since this is the only factor in the right hand side of (32) that may become zero. If this factor is zero or negative, the assigned vector $|m')$ belongs to $M(p)$. Recall that the integers $m_{jn}$ satisfy $m_{1n} \geq m_{2n} \geq \cdots \geq m_{nn} \geq 0$. If $m_{kn} = 0$ (its smallest possible value), then this factor in $F_k$ takes the value $(p - k + 1)$. So the $p$-values $1, 2, \ldots, n-1$ will play a special role. Let us depict these factors in a scheme (here shown for $n = 3$):



In this diagram, there is an edge between two partitions $([m]^n)$ and $([m']^n) = ([m]^n_{+k})$ if the reduced matrix element $([m]^n_{+k}||b^+||[m]^n) = F_k([m]^n)$ is in general nonzero. For the boldface lines, the relevant factor $(m_{kn}+n+1-k+\mathcal{E}_{m_{kn}}(p-n))$ is equal to $p-1$; for the dotted lines, this factor is equal to $p-2$. As a consequence, for $p=1$ the irreducible module $V(p)$ corresponds to the first line of the scheme only, i.e. only partitions $([m]^n)$ of length 1 appear (the length of a partition is the number of nonzero parts). For $p=2$, the irreducible module $V(p)$ is composed of partitions $([m]^n)$ of length 1 or 2 only. This observation holds in general, due to the factor $(p-k+1)$ for $F_k$. This finally leads to the following result:

**Theorem 6** *The $\mathfrak{osp}(1|2n)$ representation $V(p)$ with lowest weight $(\frac{p}{2},\ldots,\frac{p}{2})$ is a unirrep if and only if $p\in\{1,2,\ldots,n-1\}$ or $p>n-1$.*
*For $p>n-1$, $V(p)=\overline{V}(p)$ and*

$$\operatorname{char} V(p) = \frac{(x_1\cdots x_n)^{p/2}}{\prod_i(1-x_i)\prod_{j<k}(1-x_jx_k)} \tag{34}$$

$$= (x_1\cdots x_n)^{p/2}\sum_\lambda s_\lambda(x) \tag{35}$$

*For $p\in\{1,2,\ldots,n-1\}$, $V(p)=\overline{V}(p)/M(p)$ with $M(p)\neq 0$. The structure of $V(p)$ is determined by*

$$\operatorname{char} V(p) = (x_1\cdots x_n)^{p/2}\sum_{\lambda,\ \ell(\lambda)\leq p} s_\lambda(x) \tag{36}$$

*where $\ell(\lambda)$ is the* length *of the partition $\lambda$.*

The explicit action of the $\mathfrak{osp}(1|2n)$ generators in $V(p)$ is given by (27)-(28), and the basis is orthogonal and normalized. For $p\in\{1,2,\ldots,n-1\}$ this action remains valid, provided one keeps in mind that all vectors with $m_{p+1,n}\neq 0$ must vanish.

Note that the first line of Theorem 6 can also be deduced from [17], where all lowest weight unirreps of $\mathfrak{osp}(1|2n)$ are classified by means of their lowest weight.

In [10], we discuss an alternative expression for $\operatorname{char} V(p)$, when $p\in\{1,2,\ldots,n-1\}$. This expression takes the following form:

$$\operatorname{char} V(p) = (x_1\cdots x_n)^{p/2}\frac{\sum_\eta(-1)^{c_\eta}s_\eta(x)}{\prod_i(1-x_i)\prod_{j<k}(1-x_jx_k)}. \tag{37}$$

In the numerator of the right hand side, the sum is over all partitions $\eta$ of the form

$$\eta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1+p & a_2+p & \cdots & a_r+p \end{pmatrix} \tag{38}$$

in Frobenius notation (see [13] for this notation), and

$$c_\eta = a_1+a_2+\cdots+a_r+r. \tag{39}$$

## 5 Conclusions

In this contribution, we have presented a solution to a problem that has been open for many years, namely giving the explicit structure of paraboson Fock representations. In order to solve this problem, we have used a combination of known techniques and new computational power. We used in particular: the relation with unirreps of the Lie superalgebra $\mathfrak{osp}(1|2n)$, the decomposition of the induced module $\overline{V}(p)$ with respect of the compact subalgebra $\mathfrak{u}(n)$, the known GZ-basis for $\mathfrak{u}(n)$ representations, the method of reduced matrix elements for $\mathfrak{u}(n)$ tensor operators and known expressions for certain $u(n)$ CGCs and isoscalar factors.

The solution given here is also the explicit solution that would be obtained by means of Green's ansatz [1]. The method of Green's ansatz is easy to describe, but difficult to perform, and has not lead to the explicit solution of the paraboson Fock representations, as presented here. In representation theoretic terms, Green's ansatz amounts to considering the $p$-fold tensor product of the ordinary boson Fock space, $V(1)^{\otimes p}$, and extracting in this tensor product the irreducible component with lowest weight $(\frac{p}{2}, \ldots, \frac{p}{2})$.

Since the actions of all $\mathfrak{osp}(1|2n)$ generators $b_j^{\pm}$ are known explicitly, one can also determine the action of all $\mathfrak{sp}(2n)$ basis elements $\{b_j^{\pm}, b_k^{\pm}\}$. Under this $\mathfrak{sp}(2n)$ action, $V(p)$ is in general not irreducible. It is not very difficult to determine its irreducible $\mathfrak{sp}(2n)$ components. For $p > n-1$, $V(p)$ has $n+1$ irreducible $\mathfrak{sp}(2n)$ components. For $p \in \{1, 2, \ldots, n-1\}$, $V(p)$ has only $p+1$ irreducible $\mathfrak{sp}(2n)$ components. The characters of these $\mathfrak{sp}(2n)$ unirreps can be obtained from the character of $V(p)$, see [10].

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