The non-commutative and discrete spatial structure of a 3D Wigner quantum oscillator

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Abstract

The properties of a non-canonical 3D Wigner quantum oscillator, whose position and momentum operators generate the Lie superalgebra $sl(1|3)$, are further investigated. Within each state space $W(p)$, $p = 1, 2, \ldots$, the energy E_q, $q = 0, 1, 2, 3$, takes no more than 4 different values. If the oscillator is in a stationary state $\psi_q \in W(p)$ then measurements of the non-commuting Cartesian coordinates of the particle are such that their allowed values are consistent with it being found at a finite number of sites, called "nests". These lie on a sphere centered on the origin of fixed, finite radius ϱ_q. The nests themselves are at the vertices of a rectangular parallelepiped. In the typical cases ($p > 2$) the number of nests is 8 for $q = 0$ and 3, and varies from 8 to 24, depending on the state, for $q = 1$ and 2. The number of nests is less in the atypical cases ($p = 1, 2$), but it is never less than two. In certain states in $W(2)$ (resp. in $W(1)$) the oscillator is "polarized" so that all the nests lie on a plane (resp. on a line). The particle cannot be localized in any one of the available nests alone since the coordinates do not commute. The probabilities of measuring particular values of the coordinates are discussed. The mean trajectories and the standard deviations of the coordinates and momenta are computed, and conclusions are drawn about uncertainty relations.

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1 Introduction

In the present paper we continue the investigation of new quantum systems originating from the representation theory of basic classical Lie superalgebras. In particular, we study further the properties of a 3-dimensional (3D) Wigner quantum oscillator whose mathematical background involves the Lie superalgebra $sl(1|3)$ [1, 2].

The idea itself behind these investigations stems from the 1950's paper of Wigner *Do the equations of motion determine the quantum mechanical commutation relations?* [3]. In this paper Wigner has generalized a result of Ehrenfest [4]. The latter stated (up to ordering details, which are irrelevant in our case) that in the Heisenberg picture of quantum mechanics Hamilton's (resp. the Heisenberg) equations are a unique consequence of the canonical commutation relations (CCRs) and the Heisenberg (resp. Hamilton's) equations. Wigner has proved a stronger statement. He has shown through an example that Hamilton's equations can be identical to the Heisenberg equations even if the position and momentum operators do not satisfy the CCRs.

Wigner's example was a one-dimensional oscillator with a Hamiltonian $\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)$, in units such that $m = \omega = \hbar = 1$. Abandoning the requirement $[\hat{q}, \hat{p}] = i$, Wigner searched for all operators $\hat{q}$ and $\hat{p}$ such that Hamilton's equations $\dot{\hat{q}} = \hat{p}$ and $\dot{\hat{p}} = -\hat{q}$ are identical with the Heisenberg equations $\dot{\hat{q}} = i[\hat{H}, \hat{q}]$ and $\dot{\hat{p}} = i[\hat{H}, \hat{p}]$. In addition to the canonical solution he found infinitely many other solutions, that is infinitely many solutions for the pair $\hat{q}$ and $\hat{p}$. He interpreted these as position and momentum operators despite the fact that they do not satisfy the CCRs $[\hat{q}, \hat{p}] = i$.

Different aspects of Wigner's idea were studied by several authors. Among the earlier references we mention [5]-[11], but the subject still remains of interest [12]-[20].

The key motivation for a generalization of the concept of a quantum system [1, 2] comes from the observation of Wigner [3] that the Heisenberg equations and Hamilton's equations have a more immediate physical significance than the canonical commutation relations. From this point of view it is logically justified to postulate as the starting point the Heisenberg equations and Hamilton's equations instead of the CCRs.

The conjecture that the CCRs have to be modified, including the possibility that the configuration space coordinates may not mutually commute, originated recently from string theory (we refer to [21] for a survey on the subject) and also from quantum groups (see [22] for a review). However, the idea itself was already suggested by Heisenberg in the late 1930's (as explained in [23]), and perhaps the first example of this kind was given by Snyder [24].

The above observations justify the consideration of what we call a Wigner quantum system [1, 2]. As in [2], a Wigner quantum system (WQS) differs from a canonical quantum system only by the replacement of the postulate on canonical commutation relations by a new postulate. In particular, consider an n-particle system in three dimensions with Hamiltonian

$$\hat{H} = \sum_{\alpha=1}^{n} \frac{\hat{\mathbf{P}}_\alpha^2}{2m_\alpha} + V(\hat{\mathbf{R}}_1, \hat{\mathbf{R}}_2, \ldots, \hat{\mathbf{R}}_n), \tag{1.1}$$

which depends on the $6n$ variables $\hat{\mathbf{R}}_\alpha$ and $\hat{\mathbf{P}}_\alpha$, with $\alpha = 1, 2, \ldots, n$, to be interpreted as (Cartesian) coordinates and momenta, respectively. Just as in ordinary quantum mechanics, the following conditions should hold:

P1 The state space W is a Hilbert space. To every physical observable $\mathcal{O}$ there corresponds a Hermitian (self-adjoint) operator $\hat{O}$ acting in W.

P2 The observable $\mathcal{O}$ can take on only those values which are eigenvalues of $\hat{O}$. The expectation value of the observable $\mathcal{O}$ in a state ψ is given by $\langle \hat{O} \rangle_\psi = (\psi, \hat{O}\psi)/(\psi, \psi)$, where (ψ, ϕ) denotes the scalar product of $\psi, \phi \in W$.

Together with others, these postulates are common to any quantum system. We repeat only **P1** and **P2** here because we will explicitly make use of them. The difference with canonical quantum mechanics comes from postulating **P3**:

P3 Hamilton's equations and the Heisenberg equations hold and are identical (as operator equations) in W.

In the canonical case instead of **P3** one postulates the validity of the Heisenberg equations and the CCRs. Then, as mentioned above, Hamilton's equations hold too.

Here, we shall no longer use the CCRs, but rely on **P3** instead. The corresponding system is called a Wigner quantum system. Although $\hat{\mathbf{R}}_\alpha$ and $\hat{\mathbf{P}}_\alpha$ no longer satisfy the CCRs, following Wigner (and the related papers [5]-[20], [24]), we shall still interpret them as the operators corresponding to measurements of the physical position and momentum of the WQS and refer to them as position and momentum operators.

Let us point out again that the above postulates do not provide a complete description of Wigner quantum systems nor of Wigner quantum oscillators to be studied in the present paper. On the ground of these postulates alone one cannot determine the expressions for the operators of the angular momentum, for instance.

It is perhaps worth mentioning that postulate **P3** can be weakened (so far only in the 1D case) in a manner consistent with Wigner's ideas [11] so that deformed quantum oscillators [25, 26] and, more generally, Daskaloyannis oscillators [11, 27] can be viewed as (generalized) Wigner oscillators.

Our approach to Wigner quantum oscillators is essentially based on two observations. The first one, due to Kamefuchi and Ohnuki [28], is the proof that all solutions found by Wigner are different representations of *just one pair* of para-Bose (pB) creation and annihilation operators (CAOs) $B^\pm = (q \mp p)/\sqrt{2}$. More generally, we recall that the (representation independent) pB operators, which generalize Bose statistics, are defined by the relations [29]:

$$[\{B_i^\xi, B_j^\eta\}, B_k^\zeta] = (\zeta - \xi)\delta_{ik} B_j^\eta + (\zeta - \eta)\delta_{jk} B_i^\xi, \quad i,j,k = 1,2,\dots,N, \quad \xi,\eta,\zeta = \pm, \qquad (1.2)$$

(where, by convention, ξ, η, ζ are written as $\pm$ when used as superscripts, and as ± 1 when used algebraically in the factors $(\zeta - \xi)$ and $(\zeta - \eta)$). Here and throughout the paper $\{x, y\} = xy + yx$ and $[x, y] = xy - yx$ for any x, y.

The second relevant observation is that any N pairs of pB operators $B_1^\pm, \dots, B_N^\pm$ are odd elements, generating a Lie superalgebra [30], isomorphic to the orthosymplectic Lie superalgebra $osp(1|2N)$ [31]. The Fock spaces of any N pairs of parabosons and in particular of bosons are irreducible $osp(1|2N)$ modules. In this terminology the oscillator of Wigner can be called an $osp(1|2)$ oscillator, its position and momentum operators are the odd generators of $osp(1|2)$, the Hamiltonian is a simple polynomial of the position and momentum operators and the solutions found by Wigner are different irreducible representations of this Lie superalgebra.

The $osp(1|2N)$ Lie superalgebra is a basic classical Lie superalgebra from class B in the classification of Kac [32]. In fact (1.2) yields one possible definition of $osp(1|2N)$: the associative superalgebra with unity, subject to the relations (1.2) is the universal enveloping algebra of $osp(1|2N)$.

The results of Wigner can be easily extended to any N-dimensional harmonic oscillator, turning it into a Wigner quantum oscillator (WQO). To this end one has first to express the Hamiltonian via Bose operators

$$[b_i^-, b_j^+] = \delta_{ij} \quad [b_i^+, b_j^+] = [b_i^-, b_j^-] = 0, \quad i, j = 1, \ldots, N, \tag{1.3}$$

and their anticommutators and subsequently replace them with pB operators (1.2). The corresponding solutions are now associated with infinite-dimensional irreducible representation of the Lie superalgebra $osp(1|2N)$. Since this superalgebra is of class B we refer to the statistics of the canonical quantum oscillator (CQO) (and its pB generalization) as being B-superstatistics.

Having observed all this, it was natural to ask whether one can satisfy the postulates **P1-P3** with position and momentum operators which generate algebras from the other, different from B, classes of basic Lie superalgebras. A positive answer to this question was given in [1, 2] with operators $A_i^\pm$, $i = 1, \ldots, N$, which satisfy certain relations that we will specify in the next section. These ensure that they generate the Lie superalgebra $sl(1|N)$. The corresponding solutions are this time associated with a Wigner quantum system that takes the form of an N-dimensional *non-canonical* Wigner quantum oscillator. Since the special linear Lie superalgebra $sl(1|N)$ is of class A we refer to the statistics of this WQO as being A-superstatistics.

In a similar approach Barut and Bracken [33] have described the internal dynamics (*Zitterbewegung*) of Dirac's electron. Their creation and annihilation operators satisfy similar triple relations as in our case (Eqs. (2.11)), but instead of a Lie superalgebra they generate the Lie algebra $so(5)$.

In the present paper we study further the properties of the 3D WQO, related to the $sl(1|3)$ superalgebra and initiated in [1, 2]. The paper is organized as follows.

In Section 2 we outline the mathematical structure of the 3D non-canonical oscillator. The compatibility between the Heisenberg equations and the Hamilton's equations is achieved with operators $A_1^\pm, A_2^\pm, A_3^\pm$ which satisfy triple relations similar to those for the para-Bose case (1.2), but this time they generate the Lie superalgebra $sl(1|3)$. The Fock spaces $W(p)$ of these operators are defined. The inequivalent representations are labeled by one positive integer p. For $p > 2$ all Fock spaces are 8-dimensional, whereas in the case $p = 1$ (resp. $p = 2$) dim $W(1) = 4$ (resp. dim $W(2) = 7$). In the terminology of Kac [32] the $p = 1, 2$ representations are called atypical representations. For this reason we refer to the Fock spaces $W(1)$ and $W(2)$ as to atypical, to the corresponding oscillator as to atypical etc. We shall see in the next sections that the properties of the atypical oscillators are very different from those with $p > 2$.

In Section 3 we recall the known [1, 2] physical properties of the $sl(1|3)$ WQOs. Firstly, the oscillator has finite space dimensions and the Hamiltonian has no more than 4 different eigenvalues. In the stationary states the distance of the particle to the origin is quantized so that the particle is constrained to move on one of 4 possible spheres. Secondly, the geometry of the oscillator is non-commutative. The various coordinates do not commute with one another, nor do the various components of the momentum. Therefore, in particular, the position of the particle on the corresponding sphere cannot be localized. In this respect the WQO belongs to the class of models of non-commutative quantum oscillators [34]-[38] and, more generally, to theories with non-commutative geometry [39, 40].

All results after Section 3 are new. In Section 4 the probabilistic distribution of the particle is analyzed. The basis consists of stationary states. The main result is the following: if the particle is in one of the basis states with $p > 2$, then measurements of its coordinates are consistent with it only being found at 8 particular points on a sphere which form the vertices of a rectangular parallelepiped (see Figure 1). Thus as in [41], the coordinates of the particle are observables with a quantized spectrum just like energy, angular momentum, etc. The number of points, called "nests", can be even less in the atypical cases. In certain states with $p = 2$ the oscillator is becoming a flat object with 4 vertices (Figure 2). There are three states in the $p = 1$ case when the oscillator is even one-dimensional (Figure 3). In Section 5 the mean trajectories and the standard deviations of the position and momenta operators for an arbitrary state are written down. It is shown (Conclusion 3) that there exists no nontrivial analogue of the Heisenberg uncertainty relations since one can always find a state x for which either $(\Delta r_k)_x = 0$ or $(\Delta p_k)_x = 0$.

In Section 6 we show that despite the fact that the $sl(1|3)$ oscillator is very different from the 3D canonical oscillator, they still have some features in common. In particular we show that to each $p = 1$ mean trajectory of the $sl(1|3)$ oscillator there corresponds exactly the same trajectory of the 3-dimensional canonical oscillator.

2 Mathematical structure of the 3D WQO

Let $\hat{H}$ be the Hamiltonian of a three-dimensional harmonic oscillator, that is

$$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2m} + \frac{m\omega^2}{2}\hat{\mathbf{R}}^2. \tag{2.1}$$

We proceed to view this oscillator as a Wigner quantum system and work throughout in the Heisenberg picture in which the operators are, in general, time dependent. According to postulate **P3** the operators $\hat{\mathbf{R}}$ and $\hat{\mathbf{P}}$ have to be defined in such a way that Hamilton's equations

$$\dot{\hat{\mathbf{P}}} = -m\omega^2\hat{\mathbf{R}}, \quad \dot{\hat{\mathbf{R}}} = \frac{1}{m}\hat{\mathbf{P}} \tag{2.2}$$

and the Heisenberg equations

$$\dot{\hat{\mathbf{P}}} = \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{P}}], \quad \dot{\hat{\mathbf{R}}} = \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{R}}] \tag{2.3}$$

are both valid, and are identical as operator equations. These equations are compatible only if

$$[\hat{H},\hat{\mathbf{P}}]=i\hbar m\omega^2\hat{\mathbf{R}},\quad [\hat{H},\hat{\mathbf{R}}]=-\frac{i\hbar}{m}\hat{\mathbf{P}}. \tag{2.4}$$

The most general solution of (2.2) and (2.3) is not known. Here we mention the canonical Bose solution. Expressed via boson creation and annihilation operators it reads:

$$r_k(t)=\sqrt{\frac{\hbar}{2m\omega}}\,(b_k^+e^{\,i\omega t}+b_k^-\,e^{-i\omega t}),\quad p_k(t)=i\,\sqrt{\frac{m\omega\hbar}{2}}\,(b_k^+e^{\,i\omega t}-\,b_k^-\,e^{-i\omega t}). \tag{2.5}$$

In this setting r_k and p_k are canonical position and momentum operators, defined in a Bose Fock space Φ with orthonormal basis states

$$|n_1,n_2,n_3)=\frac{(b_1^+)^{n_1}(b_2^+)^{n_2}\,(b_3^+)^{n_3}}{\sqrt{n_1!\;n_2!\;n_3!}}|0\rangle,\quad n_1,\;n_2,\;n_3\in\mathbb{Z}_+ \tag{2.6}$$

subject to the known transformation relations:

$$b_k^+|\ldots,n_k,\ldots)=\sqrt{n_k+1}|\ldots,n_k+1,\ldots),\quad b_k^-|\ldots,n_k,\ldots)=\sqrt{n_k}|\ldots,n_k-1,\ldots). \tag{2.7}$$

As mentioned already in the Introduction, this Bose solution belongs to the class of B-superstatistics.

In the present paper we deal with solutions of (2.2) and (2.3) for which the operators $\hat{R}_i$ and $\hat{P}_i$, $i=1,2,3$, generate a Lie superalgebra from the class A, more precisely $sl(1|3)$. We emphasize again that despite the fact that $\hat{R}_i$ and $\hat{P}_i$ do not satisfy the CCRs, i.e. they are only "position-like" and "momentum-like" operators, we interpret them as the operators describing the position and momentum (i.e. physical properties). Accordingly we refer to them as position and momentum operators in the rest of this paper. To make the connection with $sl(1|3)$ we write the operators $\hat{\mathbf{P}}\equiv(\hat{P}_1,\hat{P}_2,\hat{P}_3)$ and $\hat{\mathbf{R}}\equiv(\hat{R}_1,\hat{R}_2,\hat{R}_3)$ in terms of new operators:

$$A_k^{\pm}=\sqrt{\frac{m\omega}{2\hbar}}\hat{R}_k\pm i\sqrt{\frac{1}{2m\omega\hbar}}\hat{P}_k,\qquad k=1,2,3. \tag{2.8}$$

The Hamiltonian $\hat{H}$ of (2.1) and the compatibility conditions (2.4) then take the form:

$$\hat{H}=\frac{\omega\hbar}{2}\sum_{i=1}^{3}\{A_i^+,A_i^-\}, \tag{2.9}$$

$$\sum_{i=1}^{3}[\{A_i^+,A_i^-\},A_k^{\pm}]=\mp 2A_k^{\pm},\quad i,k=1,2,3. \tag{2.10}$$

As a solution to (2.10) we chose operators $A_i^{\pm}$ that satisfy the following triple relations:

$$[\{A_i^+,A_j^-\},A_k^+]=\delta_{jk}A_i^+-\delta_{ij}A_k^+, \tag{2.11a}$$
$$[\{A_i^+,A_j^-\},A_k^-]=-\delta_{ik}A_j^-+\delta_{ij}A_k^-, \tag{2.11b}$$
$$\{A_i^+,A_j^+\}=\{A_i^-,A_j^-\}=0. \tag{2.11c}$$

In our case $i,j,k=1,2,3$. Equations (2.11) are defined however for $i,j,k=m,m+1,\ldots,n$, where m and n are any integers (including $m=-\infty$ and $n=\infty$).

Proposition 1 *The operators $A_i^\pm$, $i = 1, \ldots, n$, satisfying (2.11), are odd elements generating the Lie superalgebra $sl(1|n)$ [42].*

The generators $A_i^\pm$, $i = 1, \ldots, n$ are said to be creation and annihilation operators of $sl(1|n)$. They resemble ordinary Fermi operators (see (2.11c)) and can be interpreted as quasi-fermions in the context of generalized statistics [43]. These CAOs are the analogue of the Jacobson generators for the Lie algebra $sl(n+1)$ [44] and could also be called Jacobson generators of $sl(1|n)$.

Coming back to the 3D oscillator, we emphasize again that all considerations here are in the Heisenberg picture. The position and momentum operators depend on time. Hence also the CAOs depend on time. Writing this time dependence explicitly, one has:

$$\text{Hamilton's equations} \quad \dot{A}_k^\pm(t) = \mp i\omega A_k^\pm(t), \tag{2.12}$$

$$\text{Heisenberg equations} \quad \dot{A}_k^\pm(t) = \frac{i\omega}{2} \sum_{i=1}^{3} [\{A_i^+(t), A_i^-(t)\}, A_k^\pm(t)]. \tag{2.13}$$

The solution of (2.12) is evident,

$$A_k^\pm(t) = \exp(\mp i\omega t) A_k^\pm(0) \tag{2.14}$$

and therefore if the defining relations (2.11) hold at a certain time $t = 0$, i.e., for $A_k^\pm \equiv A_k^\pm(0)$, then they hold as equal time relations for any other time t. From (2.11) it follows also that the equations (2.12) are identical with equations (2.13). For further use we write the time dependence of $\hat{\mathbf{R}} = (\hat{R}_1, \hat{R}_2, \hat{R}_3)$ and $\hat{\mathbf{P}} = (\hat{P}_1, \hat{P}_2, \hat{P}_3)$ explicitly:

$$\hat{R}_k(t) = \sqrt{\frac{\hbar}{2m\omega}} (A_k^+ e^{-i\omega t} + A_k^- e^{i\omega t}), \tag{2.15a}$$

$$\hat{P}_k(t) = -i\sqrt{\frac{m\omega\hbar}{2}} (A_k^+ e^{-i\omega t} - A_k^- e^{i\omega t}), \tag{2.15b}$$

where $k = 1, 2, 3$.

Next, we wish to introduce the concept of angular momentum and the related space rotations. In order to define the angular momentum operator $\hat{\mathbf{M}} = (\hat{M}_1, \hat{M}_2, \hat{M}_3)$, we assume (as in canonical quantum mechanics, for observables having a classical analogue) that its components are in the enveloping algebra of $\hat{\mathbf{R}} = (\hat{R}_1, \hat{R}_2, \hat{R}_3)$ and $\hat{\mathbf{P}} = (\hat{P}_1, \hat{P}_2, \hat{P}_3)$ and moreover that they are linear with respect the components of these operators. We also require that $\hat{M}_1$, $\hat{M}_2$ and $\hat{M}_3$ commute with the Hamiltonian, and that they span a basis of the Lie algebra $so(3)$. The operators with the required properties are:

$$\hat{M}_j = -\frac{1}{\hbar} \sum_{k,l=1}^{3} \epsilon_{jkl} \{\hat{R}_k, \hat{P}_l\}, \qquad j = 1, 2, 3, \tag{2.16}$$

(ϵ_{jkl} is the antisymmetric unit tensor of rank 3), which take the following form in terms of the CAOs (2.11)

$$\hat{M}_j = -i \sum_{k,l=1}^{3} \epsilon_{jkl} \{A_k^+, A_l^-\}, \qquad j = 1, 2, 3. \tag{2.17}$$

It is straightforward to verify that these operators satisfy the required commutation relations:

$$[\hat{M}_j, \hat{M}_k] = i\epsilon_{jkl}\hat{M}_l, \qquad [\hat{M}_j, \hat{H}] = 0. \tag{2.18}$$

Hence $\hat{M}_1$, $\hat{M}_2$, $\hat{M}_3$ are the generators of the Lie algebra $so(3)$, i.e. they generate the $so(3)$ subalgebra of $sl(1|3)$. Moreover they are integrals of motion: they do not depend on time. We shall interpret $\hat{\mathbf{M}}$ as the operator describing the angular momentum of the WQO, and simply refer to it as the angular momentum. As in the canonical case, we shall also identify the components of $\hat{\mathbf{M}}$ as the operators of infinitesimal rotations. Then the relations

$$[\hat{M}_j, \hat{R}_k] = i\epsilon_{jkl}\hat{R}_l, \qquad [\hat{M}_j, \hat{P}_k] = i\epsilon_{jkl}\hat{P}_l, \tag{2.19}$$

together with (2.18), show that $\hat{\mathbf{R}}$, $\hat{\mathbf{P}}$ and $\hat{\mathbf{M}}$ all transform as vector operators with respect to space rotations.

The state spaces which we consider here are those irreducible $sl(1|3)$ modules that may be constructed by means of the usual Fock space technique precisely as in the parastatistics case [29]. To this end we require that the representation space, $W(p)$, contains (up to a multiple) a unique cyclic vector $|0\rangle$ such that

$$A_i^- \,|\, 0\rangle = 0, \quad A_i^- A_j^+ \,|\, 0\rangle = p\delta_{ij} \,|\, 0\rangle, \quad i,j = 1,2,3. \tag{2.20}$$

The above relations are enough for the construction of the full representation space $W(p)$. This space defines an indecomposable finite-dimensional representation of the CAOs (2.11) and hence of $sl(1|3)$ for any value of p. However, following **P1** and **P2**, we wish to impose the further physical requirements that:

(a) $W(p)$ is a Hilbert space with respect to the natural Fock space inner product;

(b) the observables, in particular the position and momentum operators (2.15), are Hermitian operators.

Condition (b) reduces to the requirement that the Hermitian conjugate of A_i^+ should be A_i^-, i.e.

$$(A_i^{\pm})^\dagger = A_i^{\mp}. \tag{2.21}$$

The condition (a) is then such that p is restricted to be a positive integer [42], in fact any positive integer.

Let $\Theta \equiv (\theta_1, \theta_2, \theta_3)$. The state space $W(p)$ of the system is spanned by the following orthonormal basis (called the Θ-basis):

$$|p;\Theta\rangle \equiv |p;\theta_1,\theta_2,\theta_3\rangle = \sqrt{\frac{(p-q)!}{p!}}\;(A_1^+)^{\theta_1}(A_2^+)^{\theta_2}(A_3^+)^{\theta_3}\;|0\rangle, \tag{2.22}$$

where

$$\theta_i \in \{0,1\} \;\text{ for all }\; i = 1,2,3 \tag{2.23}$$

and

$$0 \le q \equiv \theta_1 + \theta_2 + \theta_3 \le \min(p,3). \tag{2.24}$$

The transformation of the basis states (2.22) under the action of the CAOs reads as follows:

$$A_i^- \,|p;\ldots,\theta_i,\ldots\rangle = \theta_i(-1)^{\theta_1+\cdots+\theta_{i-1}}\sqrt{p-q+1}\;|p;\ldots,\theta_i-1,\ldots\rangle, \tag{2.25a}$$

$$A_i^+ \,|p;\ldots,\theta_i,\ldots\rangle = (1-\theta_i)(-1)^{\theta_1+\cdots+\theta_{i-1}}\sqrt{p-q}\;|p;\ldots,\theta_i+1,\ldots\rangle. \tag{2.25b}$$

The factors θ_i and $(1-\theta_i)$ ensure that the only non-vanishing cases are those for which $|p;\ldots,\theta_i\pm 1,\ldots\rangle$ do indeed belong to the set of basis states defined by (2.22)-(2.24).

Note the first big difference between this non-canonical WQO and the case of a conventional CQO:

Observation 1 *Contrary to the CQO with an infinite-dimensional state space, each state space $W(p)$ of the WQO is finite-dimensional.*

In fact $\dim W(p) = 8$ for $p > 2$, whereas $\dim W(1) = 4$ and $\dim W(2) = 7$.

3 Known properties of 3D WQOs

Here we recall the physical properties of the Wigner quantum oscillators as given in [1, 2].

The first thing we note is that the representation of $sl(1|3)$ was chosen such that, as in the case of a 3D CQO, the physical observables $\hat{H}$, $\hat{\mathbf{R}}$, $\hat{\mathbf{P}}$ and $\hat{\mathbf{M}}$ are, in the case of the WQO, all Hermitian operators within every Hilbert space $W(p)$ for each $p = 0, 1, \ldots$ (in accordance with postulate **P1**).

Secondly, in the case of the WQO the Hamiltonian $\hat{H}$ is diagonal in the basis (2.22)-(2.24), i.e. the basis vectors $|p;\Theta\rangle$ are stationary states of the system. As in the 3D CQO the energy levels are equally spaced with the same spacing $\hbar\omega$. Contrary to the CQO each Hilbert space $W(p)$ has no more than four equally spaced energy levels, with spacing $\hbar\omega$. More precisely,

$$\hat{H}\,|p;\Theta\rangle = E_q\,|p;\Theta\rangle \quad \text{with} \quad E_q = \frac{\hbar\omega}{2}\,(3p-2q)\,. \tag{3.1}$$

So we can define stationary states ψ_q as superpositions of states $|p;\Theta\rangle$ with the same q:

$$\psi_0 = |p;0,0,0\rangle, \tag{3.2a}$$

$$\psi_1 = \alpha(1,0,0)|p;1,0,0\rangle + \alpha(0,1,0)|p;0,1,0\rangle + \alpha(0,0,1)|p;0,0,1\rangle, \tag{3.2b}$$

$$\psi_2 = \alpha(1,1,0)|p;1,1,0\rangle + \alpha(1,0,1)|p;1,0,1\rangle + \alpha(0,1,1)|p;0,1,1\rangle, \tag{3.2c}$$

$$\psi_3 = |p;1,1,1\rangle, \tag{3.2d}$$

where $\alpha(\theta_1,\theta_2,\theta_3)$ are complex numbers. The stationary states satisfy $\hat{H}\psi_q = E_q\psi_q$. Only the states with $q \le p$ belong to the space $W(p)$. Note that in the atypical cases ($p = 1, 2$) the lowest energy level is degenerate: there are three linearly independent states with the same ground state energy.

Perhaps the most striking difference between the WQO and the CQO is that the geometry of the Wigner oscillators is non-commutative: the position operators $\hat{R}_1$, $\hat{R}_2$, $\hat{R}_3$ of the oscillating particle do not commute with each other,

$$[\hat{R}_i, \hat{R}_j] \neq 0 \quad \text{for} \quad i \neq j = 1, 2, 3. \tag{3.3}$$

Hence for the Wigner oscillators *a coordinate representation (x-representation) does not exist.* Similarly,

$$[\hat{P}_i, \hat{P}_j] \neq 0 \quad \text{for} \quad i \neq j = 1, 2, 3 \tag{3.4}$$

and therefore also *a momentum representation (p-representation) cannot be defined.*

The relations (3.3) and (3.4) imply that the WQO belongs to the class of models of non-commutative quantum oscillators [34]-[38] and, more generally, to theories with non-commutative geometry [39, 40]. The literature on this subject is vast. Moreover the subject is not any longer of purely theoretical interest. Most recently papers predicting (experimentally) measurable deviations from the commutativity of the coordinates have been published [45]-[49]. Here however, we only deal with a purely theoretical description.

Following the non-commutativity, it is natural to ask about the nature and value of the commutator $[\hat{R}_i, \hat{R}_j]$ (or $[\hat{P}_i, \hat{P}_j]$, or $[\hat{R}_i, \hat{P}_j]$). The answer to this question is relevant since it is used in the derivation of the uncertainty relations between the coordinates for instance. To answer this, note that the commutators between the operators $\hat{R}_i$ and $\hat{P}_j$ do not belong to the Lie superalgebra $sl(1|3)$ ($\hat{R}_i$ and $\hat{P}_i$ are odd elements of the algebra), so they cannot be rewritten in a simpler form. Of course, one can compute the action of these commutators on basis vectors of the considered $sl(1|3)$ modules $W(p)$. We have actually made these computations. However the resulting formulas are rather complicated. Later, we content ourselves with just one illustrative example in (5.18).

It is worth pointing out that the commutators $[\hat{R}_1, \hat{R}_2]$, $[\hat{R}_1, \hat{R}_3]$ and $[\hat{R}_2, \hat{R}_3]$ are themselves operators that do not commute with each other in any one of the state spaces $W(p)$, $p = 1, 2, \ldots$ and therefore cannot be diagonalized simultaneously. For this reason the non-commutativity between our position and momentum operators is very different from the non-commutativity of generalized Heisenberg commutation relations

$$[\hat{r}_i, \hat{r}_j] = i\theta_{ij}, \quad [\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\bar{\theta}_{ij} \tag{3.5}$$

often adopted in the literature on non-commutative quantum mechanics (see for instance [34]-[38]). In the right hand side of (3.5) θ_{ij} and $\bar{\theta}_{ij}$ are numbers (which are often further simplified, e.g., $\bar{\theta}_{ij} = 0$, $\theta_{ij} = \theta$, etc.) and therefore all operators $[\hat{r}_i, \hat{r}_j]$, $[\hat{r}_i, \hat{p}_j]$, $[\hat{p}_i, \hat{p}_j]$ are simultaneously diagonal (in any basis of the state space).

Turning back to the WQO, there are two interesting integrals of motion, namely $\hat{\mathbf{R}}^2$ and $\hat{\mathbf{P}}^2$. Furthermore, they are proportional to $\hat{H}$:

$$\hat{\epsilon} \equiv \frac{2}{\omega\hbar}\hat{H} = \frac{2m\omega}{\hbar}\hat{\mathbf{R}}^2 = \frac{2}{m\omega\hbar}\hat{\mathbf{P}}^2 = \sum_{i=1}^{3}\{A_i^+, A_i^-\}. \tag{3.6}$$

And thus:

$$\hat{\mathbf{R}}^2 \,|p;\Theta\rangle = \frac{\hbar}{2m\omega}\,(3p-2q)\,\,|p;\Theta\rangle, \tag{3.7a}$$

$$\hat{\mathbf{P}}^2 \,|p;\Theta\rangle = \frac{m\omega\hbar}{2}\,(3p-2q)\,\,|p;\Theta\rangle, \tag{3.7b}$$

for $0 \leq q \equiv \theta_1+\theta_2+\theta_3 \leq \min(p,3)$. Eq. (3.7a) indicates that if the oscillator is in a stationary state ψ_q with energy $E_q = \frac{\hbar\omega}{2}\,(3p-2q)$, then the distance ϱ_q between the oscillating particle and the origin of the coordinate system is

$$\varrho_q = \sqrt{\frac{\hbar}{2m\omega}(3p - 2\theta_1 - 2\theta_2 - 2\theta_3)} \tag{3.8}$$

and this distance is an integral of motion, it is preserved in time. For further references we formulate the following observation.

Conclusion 1 *Each stationary state ψ_q, which is a superposition of states $|p;\Theta\rangle$ with one and the same $q = \theta_1 + \theta_2 + \theta_3$, corresponds to a configuration in which the particle is somewhere at a distance ϱ_q from the centre of the coordinate system. However, the position of the particle on the sphere of radius ϱ_q cannot be localized because the coordinates do not commute with one another.*

The maximum distance of the particle from the centre is

$$\varrho_{max} \equiv \varrho_0 = \sqrt{\frac{3\hbar p}{2m\omega}} \tag{3.9}$$

and this corresponds to the state $|p;0,0,0\rangle$, which carries also the maximal energy $E_{max} = \frac{3}{2}\hbar\omega p$. Thus the WQO occupies a finite volume. The oscillating particle is locked in a sphere with radius (3.9), which is another property very different from the CQO for which there is no finite upper bound on the radial distance.

Finally we note that

$$\hat{\mathbf{M}}^2|p;\Theta\rangle = \begin{cases} 0 & \text{if } \theta_1 = \theta_2 = \theta_3; \\ 2|p;\Theta\rangle & \text{otherwise.} \end{cases} \tag{3.10}$$

Therefore each state $|p;\Theta\rangle$ carries angular momentum 0 or 1. If $p > 2$, $W(p)$ decomposes as $(1) \oplus (3) \oplus (3) \oplus (1)$ with respect to the $so(3)$ subalgebra of $sl(1|3)$; herein (1) is a 1-dimensional subspace with $M = 0$ and (3) is a 3-dimensional subspace with $M = 1$. For $p = 2$, $W(2) = (1) \oplus (3) \oplus (3)$, and for $p = 1$ one has $W(1) = (1) \oplus (3)$.

4 On the position and momentum of the oscillating particle

The results in the previous section are not very precise about the position of the oscillating particle in one of its stationary states ψ_q or $|p;\Theta\rangle$: the only conclusion is that the particle is localized on a sphere with radius ϱ_q.

We shall first investigate the probabilistic distribution of the particle on the sphere corresponding to the states $|p;\Theta\rangle$ or ψ_q. In particular we shall show, with respect to measurements of $\hat{R}_1$, $\hat{R}_2$ and $\hat{R}_3$, that in the stationary states $|p;\Theta\rangle$ the particle can be found at only 8 points on the sphere (we call them "nests") with radius ϱ_q, see (3.8), and the number of such nests is even less in the atypical cases $p = 1$ and $p = 2$.

The main tool to obtain these results is based on the observation that the set of operators

$$\hat{H},\ \hat{\mathbf{R}}^2, \hat{\mathbf{P}}^2,\ \hat{R}_1^2,\ \hat{R}_2^2,\ \hat{R}_3^2,\ \hat{P}_1^2,\ \hat{P}_2^2,\ \hat{P}_3^2, \tag{4.1}$$

mutually commute and therefore can be diagonalized simultaneously. Observe that $\hat{R}_k^2$ and $\hat{P}_k^2$ and more generally all even elements are independent of the time t, which is why we do not write $\hat{R}_k(t)^2$ and $\hat{P}_k(t)^2$.

The sequence (4.1) contains in fact only 3 independent integral of motions, for instance $\hat{R}_1^2$, $\hat{R}_2^2$,$\hat{R}_3^2$, since

$$\hat{P}_k^2 = m^2\omega^2\hat{R}_k^2,\ \hat{H} = m\omega^2\hat{\mathbf{R}}^2,\ \hat{\mathbf{P}}^2 = m^2\omega^2\hat{\mathbf{R}}^2,\ \text{and}\ \ \hat{\mathbf{R}}^2 = \hat{R}_1^2 + \hat{R}_2^2 + \hat{R}_3^2. \tag{4.2}$$

All these are Hermitian operators in $W(p)$. Hence we can choose a basis consisting of common eigenvectors to all of them. In this case, we are lucky in the sense that all these operators are already diagonal in the Θ-basis.

At this point it is convenient to introduce dimensionless notation for the energy, the coordinates and the momenta:

$$\hat{\varepsilon} = \frac{2}{\omega\hbar}\hat{H}, \quad \hat{r}_i(t) = \sqrt{\frac{2m\omega}{\hbar}}\hat{R}_i(t), \quad \hat{p}_i(t) = \sqrt{\frac{2}{m\omega\hbar}}\hat{P}_i(t), \quad i = 1, 2, 3. \tag{4.3}$$

Then $\hat{r}_i^2 = \hat{p}_i^2$, $i = 1, 2, 3$ and

$$\hat{\varepsilon} = \hat{\mathbf{r}}^2 = \hat{\mathbf{p}}^2 \ = \sum_{i=1}^{3}\{A_i^+, A_i^-\}. \tag{4.4}$$

4.1 The basis vectors of $W(p)$ with $p > 2$ (typical case)

For $p > 2$ all state spaces $W(p)$ of the system are 8-dimensional. The following holds:

$$\hat{r}_k^2|p;\Theta\rangle = \hat{p}_k^2|p;\Theta\rangle = (p - q + \theta_k)|p;\Theta\rangle, \quad k = 1, 2, 3. \tag{4.5}$$

What are the conclusions, which we can draw from Eqs. (4.5)? Let us answer this question first for one particular state, e.g. $|p;1,1,0\rangle$. If measurements of the observables corresponding to $\hat{\mathbf{r}}^2$, $\hat{r}_1^2$, $\hat{r}_2^2$, $\hat{r}_3^2$ are performed, then according to postulate **P2** they will give the eigenvalues of these operators, namely

$$r^2 = 3p - 4,\ r_1^2 = r_2^2 = p - 1,\ r_3^2 = p - 2. \tag{4.6}$$

Moreover since the operators $\hat{\mathbf{r}}^2$, $\hat{r}_1^2$, $\hat{r}_2^2$, $\hat{r}_3^2$ commute the results (4.6) can be measured simultaneously. The latter means that if several measurements of the coordinates are performed, then they will discover all of the time that the particle is accommodated in one of 8 nests with coordinates

$$r_1 = \pm\sqrt{p-1}, \quad r_2 = \pm\sqrt{p-1}, \quad r_3 = \pm\sqrt{p-2}, \tag{4.7}$$

of a sphere with radius $\rho = \sqrt{3p-4}$.

Similarly, the measurements of the projections of the momenta will give (due to (4.5)):

$$p_1 = \pm\sqrt{p-1},\ \ p_2 = \pm\sqrt{p-1},\ \ p_3 = \pm\sqrt{p-2}. \tag{4.8}$$

The generalization of this result to any Θ-state is evident:

Conclusion 2 *If the system is in one of the* Θ*-basis states* $|p;\Theta\rangle$ *then measurements of* r_1*,* r_2 *and* r_3 *imply that the oscillating particle can be found in no more than 8 nests with coordinates*

$$r_1 = \pm\sqrt{p-q+\theta_1}, \quad r_2 = \pm\sqrt{p-q+\theta_2}, \quad r_3 = \pm\sqrt{p-q+\theta_3}, \tag{4.9}$$

on a sphere with radius $\rho_q = \sqrt{3p-2q}$*. The measured values of the momenta can take also only 8 different values,*

$$p_1 = \pm\sqrt{p-q+\theta_1}, \quad p_2 = \pm\sqrt{p-q+\theta_2}, \quad p_3 = \pm\sqrt{p-q+\theta_3}. \tag{4.10}$$

Conclusion 2 significantly enhances the properties of the WQO known so far, and collected in Conclusion 1. The particle is not just anywhere on the sphere. In every Θ-state $|p;\Theta\rangle$ the particle can be spotted in no more than 8 points of the sphere with radius ρ_q. This is what we can say so far. What we cannot say yet is whether some of these nests are not forbidden for "visits" or what is the probability of finding the particle in any one of them.

In order to investigate this last question we shall need the eigenvectors and the eigenvalues of all the operators of the coordinates and of the momenta. Before that a short remark related to the properties of any WQS will be in order.

Let $\mathcal{O}$ be an observable and let $x_1, \ldots, x_n$ be an orthonormed basis of eigenvectors of $\hat{O}$: $\hat{O}x_i = O_i x_i$. Assume that the system is in a state $\psi = \alpha_1 x_1 + \ldots + \alpha_n x_n$ normalized to 1. Postulate **P2** tells us that the expectation value $\langle \hat{O}\rangle_\psi$ of the observable $\mathcal{O}$ in the state ψ is

$$\langle \hat{O}\rangle_\psi = (\psi, \hat{O}\psi) = |\alpha_1|^2 O_1 + \ldots + |\alpha_n|^2 O_n. \tag{4.11}$$

It follows that $|\alpha_i|^2$ gives the probability of measuring the eigenvalue O_i of the operator $\hat{O}$. This is just *the superposition principle* of quantum mechanics. The conclusion is that this principle holds for any WQS.

Thus in order to examine the probability for the particle to be in one of the 8 nests, one has to introduce as a first step an $\hat{r}_k$-basis, namely an orthonormal basis of eigenvectors of $\hat{r}_k$ for any $k = 1, 2, 3$. The second step is to express the Θ-basis via the $\hat{r}_k$-basis for any $k = 1, 2, 3$, and to apply the superposition principle.

One has to proceed in a similar way in order to examine the probability for the particle to have each one of the possible values of momentum.

Let us define, for any $k \in \{1, 2, 3\}$ and any Θ satisfying (2.23), the following vectors in $W(p)$:

$$v_k(\Theta) = \frac{1}{\sqrt{2}}(|p;\Theta_{\theta_k=0}\rangle + (-1)^{\theta_1+\cdots+\theta_k} e^{-i\omega t}|p;\Theta_{\theta_k=1}\rangle). \tag{4.12}$$

Herein, $\Theta_{\theta_k=0}$ stands for the Θ-value specified by the left hand side of (4.12) in which θ_k is replaced by 0 (and similarly for $\Theta_{\theta_k=1}$). Thus $v_k(\Theta)$ depends on θ_k only through the sign factor $(-1)^{\theta_1+\cdots+\theta_k}$. A careful computation shows that these (time-dependent) vectors $v_k(\Theta)$ constitute an orthonormal basis of eigenvectors of $\hat{r}_k(t)$ in $W(p)$:

$$\hat{r}_k(t)v_k(\Theta) = (-1)^{\theta_k}\sqrt{p-q+\theta_k}\, v_k(\Theta). \tag{4.13}$$

The physical interpretation of each eigenvector $v_k(\Theta)$ is clear (Postulate **P2**): if (at the time t) the oscillating particle is in a state $v_k(\Theta)$ then its k-th coordinate is $(-1)^{\theta_k}\sqrt{p-q+\theta_k}$.

The inverse relations of (4.12) are also easy to write down:

$$|p;\Theta\rangle = \frac{1}{\sqrt{2}}(-1)^{(\theta_1+\cdots+\theta_{k-1})\theta_k} e^{i\omega t\theta_k}(v_k(\Theta_{\theta_k=0}) + (-1)^{\theta_k} v_k(\Theta_{\theta_k=1})). \tag{4.14}$$

The main observation needed is that in the inverse transformations (4.14) only two different vectors v_k appear, each with a coefficient of which the square modulus is 1/2. In order to understand the importance of this observation, consider an example, say $|p;1,1,0\rangle$. The expansion of this vector in the $\hat{r}_k(t)$ eigenvectors (for $k=1,2,3$) reads:

$$\begin{aligned} |p;1,1,0\rangle &= \frac{e^{i\omega t}}{\sqrt{2}}(v_1(0,1,0) - v_1(1,1,0)) &\quad (4.15a)\\ &= -\frac{e^{i\omega t}}{\sqrt{2}}(v_2(1,0,0) - v_2(1,1,0)) &\quad (4.15b)\\ &= \frac{1}{\sqrt{2}}(v_3(1,1,0) + v_3(1,1,1)). &\quad (4.15c) \end{aligned}$$

We see that the coefficients of $v_1(0,1,0)$ and $v_1(1,1,0)$ are equal up to a sign, and moreover their square modulus is 1/2. Therefore the superposition principle asserts that with equal probability 1/2 the first coordinate r_1 of the particle is either $+\sqrt{p-1}$ or $-\sqrt{p-1}$. In other words, the probability of finding the particle somewhere in the four nests above the r_2r_3-plane is 1/2; and the probability to find the particle somewhere in the four nests below the r_2r_3-plane is also 1/2. Let us underline that this conclusion is time independent. The time dependent basis which we have used in order to derive it was playing only an intermediate role.

By means of the same arguments, using (4.15b), (4.15c) and (4.13), one concludes that also with probability 1/2 the second coordinate r_2 and the third coordinate r_3 of the particle take values $\pm\sqrt{p-1}$ and $\pm\sqrt{p-2}$, respectively for the state $|p;1,1,0\rangle$.

Taking the three results (about the probabilities for r_1, r_2 and r_3) together does however not lead to a unique solution for the probabilities to find the particle in a particular nest. Indeed, there are 8 probabilities to be determined (one for each nest). From (4.15a) we have deduced that the sum of four of them (above the r_2r_3-plane) is 1/2, and the sum of the remaining four (below the r_2r_3-plane) is also 1/2; so this yields 2 linear relations for the 8 unknown probabilities. Similarly, (4.15b) and (4.15c) each yield 2 linear relations. So in total there are 6 linear relations in 8 unknowns. A more detailed investigation even shows that only 4 of the 6 linear relations are independent. This leads to the conclusion that the probability of the particle being found in each nest cannot be determined by the present considerations: there remain certain degrees of freedom.

We have made this analysis for the example $|p;1,1,0\rangle$, but from (4.14) it is clear that this conclusion generalizes to the case of all Θ-states. This follows from the fact that in the inverse transformations (4.14) only two different vectors v_k appear, each with a coefficient of which the square modulus is 1/2.

We summarize the results in the next proposition.

Proposition 2 *If the system is in one of the Θ-basis states $|p;\Theta\rangle$, then measurements of the position of the oscillating particle are such that it can only be observed to occupy one of the 8 nests with coordinates*

$$r_{k\pm} = \pm\sqrt{p-q+\theta_k}, \quad k=1,2,3, \tag{4.16}$$

on a sphere of dimensionless radius $\rho_q = \sqrt{3p-2q}$. The probability $\mathcal{P}(\pm\pm\pm)$ of finding the particle in the nest with coordinates $(r_{1\pm}, r_{2\pm}, r_{3\pm})$ cannot be determined. However, the probability of finding the particle somewhere in the four nests with first coordinate equal to r_{1+} is 1/2, and of finding it somewhere in the four nests with first coordinate equal to r_{1-} is also 1/2. The same holds for the second and third coordinates.

The measurement of the momentum of the particle can take one of the eight values

$$p_{k\pm} = \pm\sqrt{p-q+\theta_k}, \quad k=1,2,3. \tag{4.17}$$

Again, the individual probabilities for each of the eight possible values of the momenta cannot be determined; but the the probability of having a fixed component p_{k+} is 1/2, and that of a fixed component p_{k-} is also 1/2 ($k=1,2,3$).

The proof of the second part of Proposition 2, related to the probabilities of the 8 possible values (4.17) for the measurement of the momentum of the particle, is essentially the same as for the coordinates. This time however, one has to use the orthonormal basis of eigenvectors of $\hat{p}_k(t)$, given by:

$$\tilde{v}_k(\Theta) = \frac{1}{\sqrt{2}}(|p;\Theta_{\theta_k=0}\rangle - i(-1)^{\theta_1+\cdots+\theta_k}e^{-i\omega t}|p;\Theta_{\theta_k=1}\rangle), \tag{4.18}$$

with

$$\hat{p}_k(t)\tilde{v}_k(\Theta) = (-1)^{\theta_k}\sqrt{p-q+\theta_k}\tilde{v}_k(\Theta). \tag{4.19}$$

The inverse relations of (4.18) are:

$$|p;\Theta\rangle = \frac{i^{\theta_k}}{\sqrt{2}}(-1)^{(\theta_1+\cdots+\theta_{k-1})\theta_k}e^{i\omega t\theta_k}(\tilde{v}_k(\Theta_{\theta_k=0}) + (-1)^{\theta_k}\tilde{v}_k(\Theta_{\theta_k=1})). \tag{4.20}$$

The properties deduced in this subsection are summarized in Figure 1.

4.2 The basis vectors of $W(p)$ for $p \leq 2$ (atypical cases)

So far we have considered the properties of almost all state spaces $W(p)$. There are only two more cases left, namely those with $p=1$ and $p=2$. We shall see in this section that some of their physical properties are very different from those of the typical cases, considered above.

4.2.1 The state space $W(2)$

For $p = 2$, the state space $W(2)$ is 7-dimensional, since $|p; 1, 1, 1\rangle = 0$. Equations (4.5) remain valid for all admissible Θ-values (that is, for all Θ with $\Theta \neq (1, 1, 1)$). This implies that also Conclusion 2 (with equations (4.9) and (4.10)) remains valid for the admissible Θ-values. In this case, it is interesting to note that for the states with $q = p = 2$, one of the operators $\hat{r}_k^2$ has zero eigenvalue. For example, for the state $|2; 1, 1, 0\rangle$ one finds

$$r_1^2 = r_2^2 = 1, \qquad r_3^2 = 0. \tag{4.21}$$

Thus the third coordinate of the particle is zero. Also p_3, the third component of the momentum, is zero. So the system becomes flat, and the particle is "polarized" so as to lie in the r_1r_2-plane. The oscillator behaves as a two-dimensional object. The coordinates of the possible nests for this state are $(r_1, r_2, r_3) = (\pm 1, \pm 1, 0)$. So there are 4 nests where the particle can be found (see Figure 2, where a complete picture of the possibe $|p; \Theta\rangle$ states is given); similarly, it can have only four different momenta.

The conclusions about the probabilities, formulated in Proposition 2, remain valid, but should be modified appropriately for the lowest energy states with $q = 2$.

4.2.2 The state space $W(1)$

The state space $W(1)$ is 4-dimensional. The admissible Θ-values have $\theta_1 + \theta_2 + \theta_3 \leq 1$. For these admissible Θ-values, (4.5) and Conclusion 2 remain valid. In this case, the interesting states are those with lowest energy with $q = p = 1$. For these states, two of the operators $\hat{r}_k^2$ have zero eigenvalue. For example, for the state $|1; 1, 0, 0\rangle$ one finds

$$r_1^2 = 1, \qquad r_2^2 = r_3^2 = 0. \tag{4.22}$$

The coordinates of the two possible nests for this state are $(r_1, r_2, r_3) = (\pm 1, 0, 0)$. Similarly, $p_2 = p_3 = 0$ for this state. So the oscillating system becomes one-dimensional, the particle is "polarized" along the r_1-axis (see Figure 3, where a copmplete picture of all $|p; \Theta\rangle$ states is given).

For $q = 1$ the considerations about probabilities of finding the particle in one of the two nests lead to a unique solution. For each of the states $|1; \Theta\rangle$ with $q = p = 1$, this probability is $1/2$. These nests are at the opposite poles on a sphere with radius 1.

4.3 Arbitrary vectors of $W(p)$

So far we were studying mainly the properties of the Θ-states. Here we proceed to consider some properties of the coordinates and momenta for an arbitrary state $x \in W(p)$ and for any representation label p.

An arbitrary vector x from the state space $W(p)$ can be represented as

$$x = \sum_{\theta_{123} \leq p} \alpha(\theta_1, \theta_2, \theta_3) |p; \theta_1, \theta_2, \theta_3\rangle, \tag{4.23}$$

where $\alpha(\theta_1,\theta_2,\theta_3)$ are any complex numbers, such that

$$\sum_{\theta_{123}\leq p} |\alpha(\theta_1,\theta_2,\theta_3)|^2 = 1. \tag{4.24}$$

Above and throughout

$$\theta_{ijk} = \theta_i + \theta_j + \theta_k \tag{4.25}$$

and $\sum_{\theta_{123}\leq p}$ denotes a sum over all $\theta_1,\theta_2,\theta_3 \in \{0,1\}$ with the additional restriction $\theta_1+\theta_2+\theta_3 \leq p$. We shall be using also the polar form of $\alpha(\theta_1,\theta_2,\theta_3)$

$$\alpha(\Theta) = |\alpha(\Theta)|e^{i\varphi(\Theta)}, \tag{4.26}$$

where $\alpha(\Theta)=\alpha(\theta_1,\theta_2,\theta_3)$ and $\varphi(\Theta)=\varphi(\theta_1,\theta_2,\theta_3)$.

The possible coordinates (and momenta) of the oscillator, in an arbitrary state x, follows from the previous discussions and the superposition principle. For clarity, let us formulate it for $W(p)$ with $p>2$. Then a measurement of the position of the particle in the state x will yield one of the 64 possible nests (see Figure 1)

$$(r_1,r_2,r_3) = (\pm\sqrt{p-q+\theta_1}, \pm\sqrt{p-q+\theta_2}, \pm\sqrt{p-q+\theta_3}), \quad \text{with} \quad q=\theta_1+\theta_2+\theta_3.$$

The probability of finding the particle somewhere in the eight nests associated with $|p;\theta_1,\theta_2,\theta_3\rangle$ is given by $|\alpha(\theta_1,\theta_2,\theta_3)|^2$, but the probability for each nest separately cannot be determined. Similarly, an arbitrary state x of $W(2)$ can be in 44 possible nests (see Figure 2); an arbitrary state x of $W(1)$ can be found in 14 possible nests (see Figure 3).

In order to give more properties of the position probabilities, it is again necessary to expand the general state x in terms of the orthonormalized eigenstates of $\hat{r}_k(t)$. Let us do it here explicitly for $p>2$; (4.23) and (4.14) imply:

$$x = \sum_{\Theta} \frac{1}{\sqrt{2}}\Big(\alpha(\Theta_{\theta_k=0}) + (-1)^{\theta_1+\cdots+\theta_{k-1}} e^{i\omega t}\alpha(\Theta_{\theta_k=1})\Big) v_k(\Theta). \tag{4.27}$$

Then the square modulus of the coefficient in front of $v_k(\Theta)$ yields the probability of the particle in the state x being observed to have $\hat{r}_k(t)$ eigenvalue $(-1)^{\theta_k}\sqrt{p-q+\theta_k}$.

Many of our formulas to be presented later will look rather complicated in the general state x, so sometimes we shall concentrate on a particular example of such a normalized state which carries all the main features of the general picture. We take as our standard example one of the simplest non-stationary states (we assume that $p>2$, but most of the results, apart from the number of the nests hold for $p=1$ and 2)

$$z = \frac{1}{\sqrt{2}}|p;0,0,0\rangle + \frac{1}{\sqrt{2}}|p;0,0,1\rangle. \tag{4.28}$$

The consideration of such an example will help to understand some of the peculiar features of the WQO in a general state. Let us explicitly deduce what can be said about the position of the particle when the system is in the state z. First of all, only two Θ-states are involved in (4.28), each of these Θ-states corresponding to 8 nests. All of these nests are different, so the particle can be detected in 16 possible nests. The probability of detecting the particle somewhere in the 8 nests corresponding to $|p;0,0,0\rangle$ or to $|p;0,0,1\rangle)$ is $1/2$; these probabilities are just the square moduli of the coefficients in (4.28).

The state (4.28) of the oscillator corresponds to a configuration in which 8 nests have value of $r_3 = \sqrt{p}$ and the other 8 states have $r_3 = -\sqrt{p}$ (see Figure 1). We cannot determine the probabilities of the 16 nests separately, but we can draw conclusions about the probability $\mathcal{P}(r_3 = \pm\sqrt{p})$ of detecting the particle in the nests with a given $r_3 = \pm\sqrt{p}$. To this end consider the expansion of the state z in terms of the eigenvectors of $\hat{r}_3(t)$:

$$z = \frac{1}{2}(1+e^{i\omega t})v_3(0,0,0) + \frac{1}{2}(1-e^{i\omega t})v_3(0,0,1). \tag{4.29}$$

Then the square moduli of the coefficients give the probabilities of finding the particle in the nests with a particular r_3-value. So we find:

$$r_3 = -\sqrt{p} \quad \text{with probability} \quad \mathcal{P}(r_3 = -\sqrt{p}) = \frac{1-\cos(\omega t)}{2}, \tag{4.30a}$$

$$r_3 = \sqrt{p} \quad \text{with probability} \quad \mathcal{P}(r_3 = \sqrt{p}) = \frac{1+\cos(\omega t)}{2}. \tag{4.30b}$$

Equations (4.30) describe an interesting new phenomenon, which does not show up whenever the oscillator is in one of the Θ-basis states or more generally in any stationary state (3.2). As it should be $\mathcal{P}(r_3 = -\sqrt{p}) + \mathcal{P}(r_3 = \sqrt{p}) = 1$. But the probabilities are time dependent. There is an oscillation of the probabilities: the probabilities for the particle to be found in the nests either with $r_3 = \sqrt{p}$ or with $r_3 = -\sqrt{p}$ vary from zero to one.

Contrary to this the probabilities of finding the particle in the nests with a fixed r_1-value, or with a fixed r_2-value are time independent. This follows from the expansion of the z-state in terms of the eigenvectors of $\hat{r}_1(t)$ and $\hat{r}_2(t)$, which yields:

$$\mathcal{P}(r_1 = \sqrt{p}) = \mathcal{P}(r_1 = -\sqrt{p}) = \mathcal{P}(r_1 = \sqrt{p-1}) = \mathcal{P}(r_1 = -\sqrt{p-1}) = \frac{1}{4}, \tag{4.31}$$

and

$$\mathcal{P}(r_2 = \sqrt{p}) = \mathcal{P}(r_2 = -\sqrt{p}) = \mathcal{P}(r_2 = \sqrt{p-1}) = \mathcal{P}(r_2 = -\sqrt{p-1}) = \frac{1}{4}. \tag{4.32}$$

In the case of $p = 1$ (4.29) and (4.30) still hold, so the oscillations of the probabilities along the r_3-axis remain unaltered. In this case however two of the 10 nests, those associated with $|1;0,0,1\rangle$, are on the third axis, which yields:

$$\mathcal{P}(r_1 = 1) = \mathcal{P}(r_1 = -1) = \frac{1}{4}, \quad \mathcal{P}(r_1 = 0) = \frac{1}{2}, \tag{4.33}$$

and

$$\mathcal{P}(r_2 = 1) = \mathcal{P}(r_2 = -1) = \frac{1}{4}, \quad \mathcal{P}(r_2 = 0) = \frac{1}{2}. \tag{4.34}$$

Based on (4.30)-(4.32) we can compute the average values of the coordinates in the state z:

$$\langle \hat{r}_3(t)\rangle_z = -\sqrt{p}\,\frac{1-\cos(\omega t)}{2} + \sqrt{p}\,\frac{1+\cos(\omega t)}{2} = \sqrt{p}\cos(\omega t). \tag{4.35a}$$

$$\langle \hat{r}_1(t)\rangle_z = \langle \hat{r}_2(t)\rangle_z = 0. \tag{4.35b}$$

5 Mean trajectories and standard deviations of positions and momenta

Now that we have discussed some properties of the position operators in more general states, let us next compute the mean trajectories or time dependent expectation values of the coordinates and momenta and their standard deviations in a general state x. We shall then specify our results to the stationary states ψ_q. We shall indicate also (Conclusion 3) that for the WQS there exists no (nontrivial) Heisenberg uncertainty relations.

For the mean trajectory of the coordinates in an arbitrary state x we obtain:

$$\begin{aligned}\langle \hat{r}_k(t)\rangle_x &= (x, \hat{r}_k(t)x) = \sum_{\theta_{123}\leq p} (-1)^{\theta_1+\cdots+\theta_{k-1}}\sqrt{p-q+\theta_k}|\alpha(\Theta_{\theta_k=0})\alpha(\Theta_{\theta_k=1})| \\ &\quad \times \cos(\omega t - \varphi(\Theta_{\theta_k=0}) + \varphi(\Theta_{\theta_k=1})),\end{aligned} \tag{5.1}$$

where as in (4.12) $\Theta_{\theta_k=0}$ stands for the Θ-value in which θ_k is replaced by 0 (and similarly for $\Theta_{\theta_k=1}$). Observe that in (5.1), the contributions come in equal pairs; e.g. for $k=1$, the contribution coming from $\Theta=(0,\theta_2,\theta_3)$ is the same as that coming from $\Theta=(1,\theta_2,\theta_3)$, since $\sqrt{p-q+\theta_k}$ is independent of θ_k. Similarly, one finds:

$$\begin{aligned}\langle \hat{p}_k(t)\rangle_x &= (x, \hat{p}_k(t)x) = -\sum_{\theta_{123}\leq p} (-1)^{\theta_1+\cdots+\theta_{k-1}}\sqrt{p-q+\theta_k}|\alpha(\Theta_{\theta_k=0})\alpha(\Theta_{\theta_k=1})| \\ &\quad \times \sin(\omega t - \varphi(\Theta_{\theta_k=0}) + \varphi(\Theta_{\theta_k=1})).\end{aligned} \tag{5.2}$$

For instance, for our standard example (4.28), we find

$$\langle \hat{r}_1(t)\rangle_z = 0, \quad \langle \hat{r}_2(t)\rangle_z = 0, \quad \langle \hat{r}_3(t)\rangle_z = \sqrt{p}\cos(\omega t), \tag{5.3a}$$
$$\langle \hat{p}_1(t)\rangle_z = 0, \quad \langle \hat{p}_2(t)\rangle_z = 0, \quad \langle \hat{p}_3(t)\rangle_z = -\sqrt{p}\sin(\omega t). \tag{5.3b}$$

Note that each term in the right hand sides of (5.1)-(5.2) contains multiples

$$\alpha(\theta_1,\theta_2,\theta_3)\alpha(\tilde\theta_1,\tilde\theta_2,\tilde\theta_3) \quad \text{such that} \quad \theta_1+\theta_2+\theta_3 \neq \tilde\theta_1+\tilde\theta_2+\tilde\theta_3. \tag{5.4}$$

Therefore the right hand sides of (5.1)-(5.2) vanish if the system is in a stationary state ψ_q. The latter stems from the observation that in the stationary states, see (3.2), x is a linear combination of basis states $|p;\theta_1,\theta_2,\theta_3\rangle$ with fixed $\theta_1+\theta_2+\theta_3$, namely all non-zero coefficients $\alpha(\theta_1,\theta_2,\theta_3)$ in (4.23) have one and the same $q=\theta_1+\theta_2+\theta_3$. Thus we have

Conclusion 3 *The mean trajectories of the coordinates and momenta vanish if the system is in a stationary state ψ_q.*

In order to draw conclusions about the uncertainty of the coordinates and momenta, more precisely about their standard deviations, we also need the mean square deviations of $\hat{r}_k$ and $\hat{p}_k$, $k=1,2,3$. It follows from (4.5) that

$$\langle \hat{r}_k(t)^2\rangle_x = \langle \hat{p}_k(t)^2\rangle_x = \sum_{\theta_{123}\leq p} (p-q+\theta_k)|\alpha(\Theta)|^2. \tag{5.5}$$

Recall that the general definition of the standard deviation ΔX of an observable X in a state x is given by

$$\Delta X_x = \sqrt{\langle X^2\rangle_x - \langle X\rangle_x^2}. \tag{5.6}$$

So from the previous formulas one can write down the standard deviation of $\hat{r}_k(t)$ and $\hat{p}_k(t)$ in an arbitrary state x:

$$\begin{aligned}
\Delta\hat{r}_k(t)_x &= \Big[\sum_{\theta_{123}\le p}(p-q+\theta_k)|\alpha(\Theta)|^2 - \Big(\sum_{\theta_{123}\le p}(-1)^{\theta_1+\cdots+\theta_{k-1}}\sqrt{p-q+\theta_k} \\
&\quad\times|\alpha(\Theta_{\theta_k=0})\alpha(\Theta_{\theta_k=1})|\cos(\omega t-\varphi(\Theta_{\theta_k=0})+\varphi(\Theta_{\theta_k=1}))\Big)^2\Big]^{1/2}, && (5.7a)\\
\Delta\hat{p}_k(t)_x &= \Big[\sum_{\theta_{123}\le p}(p-q+\theta_k)|\alpha(\Theta)|^2 - \Big(\sum_{\theta_{123}\le p}(-1)^{\theta_1+\cdots+\theta_{k-1}}\sqrt{p-q+\theta_k} \\
&\quad\times|\alpha(\Theta_{\theta_k=0})\alpha(\Theta_{\theta_k=1})|\sin(\omega t-\varphi(\Theta_{\theta_k=0})+\varphi(\Theta_{\theta_k=1}))\Big)^2\Big]^{1/2}. && (5.7b)
\end{aligned}$$

Because of the double products in the expansion of the square, (5.7) cannot be simplified further for an arbitrary state vector x.

Let us just observe that in any one of the stationary states ψ_q the standard deviations become very simple and are time independent:

$$\Delta\hat{r}_j(t)_{\psi_0} = \Delta\hat{p}_j(t)_{\psi_0} = \sqrt{p}, \tag{5.8a}$$
$$\Delta\hat{r}_j(t)_{\psi_1} = \Delta\hat{p}_j(t)_{\psi_1} = \sqrt{p-1+|\alpha(\theta_j=1,\theta_k=\theta_l=0)|^2} \ge \sqrt{p-1}, \tag{5.8b}$$
$$\Delta\hat{r}_j(t)_{\psi_2} = \Delta\hat{p}_j(t)_{\psi_2} = \sqrt{p-1-|\alpha(\theta_j=0,\theta_k=\theta_l=1)|^2} \ge \sqrt{p-2}, \tag{5.8c}$$
$$\Delta\hat{r}_j(t)_{\psi_3} = \Delta\hat{p}_j(t)_{\psi_3} = \sqrt{p-2}, \tag{5.8d}$$

where $j\ne k\ne l\ne j\in\{1,2,3\}$.

Although formulas (5.7) look complicated, they are easy to apply. For instance, for our standard example (4.28), we find

$$\begin{aligned}
&\Delta\hat{r}_1(t)_z = \Delta\hat{p}_1(t)_z = \Delta\hat{r}_2(t)_z = \Delta\hat{p}_2(t)_z = \sqrt{\frac{2p-1}{2}},\\
&\Delta\hat{r}_3(t)_z = \sqrt{p}|\sin(\omega t)|, \quad \Delta\hat{p}_3(t)_z = \sqrt{p}|\cos(\omega t)|.
\end{aligned} \tag{5.9}$$

Equations (5.7) can be used for independent verification of some of the properties of the WQOs. Consider for instance the state $|2;1,1,0\rangle$. We know, see (4.21), that in this state the particle is polarized in the r_1r_2-plane, both $r_3=0$ and $p_3=0$. Equation (5.7) confirms this:

$$\Delta\hat{r}_3(t)_y = \Delta\hat{p}_3(t)_y = 0 \quad \text{in the state} \quad y = |2;1,1,0\rangle. \tag{5.10}$$

Let us note more generally that for any p and $k=1,2,3$ there exists a state x_k and a time t such that $\Delta\hat{r}_k(t)_{x_k}=0$ or $\Delta\hat{p}_k(t)_{x_k}=0$. For instance

$$\Delta\hat{r}_1(0)_{x_1} = 0 \quad \text{for} \quad x_1 = \frac{1}{\sqrt{2}}(|p;0,1,0\rangle + |p;1,1,0\rangle), \tag{5.11a}$$

$$\Delta\hat{r}_2(0)_{x_2} = 0 \quad \text{for} \quad x_2 = \frac{1}{\sqrt{2}}(|p;0,0,1\rangle + |p;0,1,1\rangle), \tag{5.11b}$$

$$\Delta\hat{r}_3(0)_{x_3} = 0 \quad \text{for} \quad x_3 = \frac{1}{\sqrt{2}}(|p;1,0,0\rangle + |p;1,0,1\rangle). \tag{5.11c}$$

As an immediate consequence we have

Conclusion 4 *The position and momentum operators of a WQO do not satisfy an uncertainty-like relation of the form*

$$\Delta\hat{r}_k(t)_x \Delta\hat{p}_k(t)_x \geq \gamma \tag{5.12}$$

for any $\gamma > 0$ holding simultaneously for all states x of the system at all times t (as is the case for a CQO and, more generally, for any canonical quantum system).

If however, x is any stationary state ψ_q, then in the typical case with $p > 2$, (5.8) yields

$$\Delta\hat{r}_k(t)_{\psi_q} = \Delta\hat{p}_k(t)_{\psi_q} \geq \sqrt{p-2} \tag{5.13}$$

for all $k = 1,2,3$. Therefore for any stationary state and any time

$$\Delta\hat{r}_i(t)_{\psi_q}\Delta\hat{r}_j(t)_{\psi_q} \geq p-2, \quad \Delta\hat{r}_i(t)_{\psi_q}\Delta\hat{p}_j(t)_{\psi_q} \geq p-2, \quad \Delta\hat{p}_i(t)_{\psi_q}\Delta\hat{p}_j(t)_{\psi_q} \geq p-2, \tag{5.14}$$

with $p - 2 > 0$.

Returning to the case of an arbitrary state $x \in W(p)$, uncertainty-like relations of the type (5.12) certainly will exist, but the uncertainty parameter γ may be zero. They can be derived from the general uncertainty relation [50]

$$\Delta\hat{F}(t)_x \Delta\hat{G}(t)_x \geq \frac{1}{2}\left| \langle\ [\hat{F}(t),\hat{G}(t)]\ \rangle_x \right|, \tag{5.15}$$

that applies to any two Hermitian operators $\hat{F}$ and $\hat{G}$ for any $x \in W(p)$.

Applying this in the case $\hat{F}(t) = \hat{r}_k(t)$ and $\hat{G}(t) = \hat{p}_k(t)$ and the arbitrary state x as defined in (4.23) gives

$$\Delta\hat{r}_k(t)_x \Delta\hat{p}_k(t)_x \geq \left| \sum_{\theta_{123}\leq p} (-1)^{\theta_k}(p-q+\theta_k)|\alpha(\Theta)|^2 \right|. \tag{5.16}$$

It should be noted that the sign factors $(-1)^{\theta_k}$ are such that cancellations may occur and may yield zero on the right hand side, as is the case, for example, if $k = 3$ and x is the state $x_3 = \frac{1}{\sqrt{2}}(|p;1,0,0\rangle + |p;1,0,1\rangle)$ introduced in (5.11c).

On the other hand if (5.16) is restricted to the stationary state ψ_0, for example, then it yields a relation very similar, when properly dimensionalized, to the Heisenberg uncertainty relation, namely:

$$\Delta\hat{R}_k(t)_{\psi_0}\Delta\mathcal{P}_k(t)_{\psi_0} \geq \frac{\hbar p}{2}. \tag{5.17}$$

Formulas of the type (5.16) in the case of $\Delta\hat{r}_k(t)_x\Delta\hat{r}_l(t)_x$, $\Delta\hat{r}_k(t)_x\Delta\hat{p}_l(t)_x$ and $\Delta\hat{p}_k(t)_x\Delta\hat{p}_l(t)_x$ with $k \neq l$ are however much more involved and will not be analyzed here systematically. We just give a typical example illustrating the impact of the non-commutative geometry:

$$\begin{aligned}
\Delta\hat{r}_1(t)_x\Delta\hat{r}_2(t)_x &\geq \frac{1}{2}\,|\langle\, [\hat{r}_1(t), \hat{r}_2(t)]\,\rangle_x| \\
&= |2\sqrt{p(p-1)}|\alpha(0,0,0)\alpha(1,1,0)|\sin(2\omega t - \phi(0,0,0) + \phi(1,1,0)) \\
&\quad +2\sqrt{(p-1)(p-2)}|\alpha(0,0,1)\alpha(1,1,1)|\sin(2\omega t - \phi(0,0,1) + \phi(1,1,1)) \\
&\quad +(2p-1)|\alpha(1,0,0)\alpha(0,1,0)|\sin(\phi(1,0,0) - \phi(0,1,0)) \\
&\quad +(2p-3)|\alpha(1,0,1)\alpha(0,1,1)|\sin(\phi(1,0,1) - \phi(0,1,1))\,| \,. \qquad (5.18)
\end{aligned}$$

6 Comparison with the canonical quantum oscillator

From the discussions so far it becomes clear that the WQOs are very different from the Bose canonical quantum oscillators (CQOs). Therefore it is somewhat of a surprise that one can establish a one-to-one correspondence between some mean trajectories of the 3D CQO and the $p=1$ mean trajectories of the WQO. This will be the main topic of the present section.

In dimensionless units, see (4.3), the coordinates $\bar{r}_k$ and momenta $\bar{p}_k$, $k=1,2,3$ of a 3D canonical oscillator (2.5) read:

$$\bar{r}_k(t) = b_k^+ e^{\,i\omega t} + b_k^-\, e^{-i\omega t}, \quad \bar{p}_k(t) = i\,(b_k^+ e^{\,i\omega t} -\, b_k^-\, e^{-i\omega t}). \qquad (6.1)$$

Let us consider first a simple example. As a Bose analogue of our standard state z we set

$$\bar{z} = \frac{1}{\sqrt{2}}|0,0,0) + \frac{1}{\sqrt{2}}|0,0,1). \qquad (6.2)$$

It is a simple computation to show that the mean trajectories of the coordinates and momenta in the state $\bar{z}$ of the Bose oscillator read:

$$\langle \bar{r}_1(t)\rangle_z = 0, \quad \langle \bar{r}_2(t)\rangle_z = 0, \quad \langle \bar{r}_3(t)\rangle_z = \cos(\omega t), \qquad (6.3a)$$

$$\langle \bar{p}_1(t)\rangle_z = 0, \quad \langle \bar{p}_2(t)\rangle_z = 0, \quad \langle \bar{p}_3(t)\rangle_z = -\sin(\omega t). \qquad (6.3b)$$

The above trajectories are the same as those of the WQO given in (5.3) provided in the latter that $p=1$. This was the first indication that some of the trajectories of the WQO are the same as those of the canonical Bose oscillator. The question is how far does this similarity go. In the next proposition we summarize the results which we are able to establish.

Proposition 3 *To each $p=1$ mean trajectory in the phase space of the Wigner quantum oscillator there corresponds an identical trajectory of the 3D Bose canonical quantum oscillator.*

Let $p=1$. By a straightforward computation one shows that the mean trajectory of the WQO in the state

$$x = \alpha(0,0,0)|1;0,0,0\rangle + \alpha(1,0,0)|1;1,0,0\rangle + \alpha(0,1,0)|1;0,1,0\rangle + \alpha(0,0,1)|1;0,0,1\rangle \quad (6.4)$$

is the same as the mean trajectory of the Bose oscillator in the state

$$x^* = \alpha(0,0,0)^*|0,0,0) + \alpha(1,0,0)^*|1,0,0) + \alpha(0,1,0)^*|0,1,0) + \alpha(0,0,1)^*|0,0,1). \quad (6.5)$$

The $*$ in the right hand side of (6.5) denotes complex conjugation.

Explicitly the mean trajectories corresponding to (6.4) and (6.5) read:

$$\langle \hat{r}_k(t)\rangle_x = \langle \bar{r}_k(t)\rangle_{x^*} = 2\ |\alpha(0,0,0)\alpha(0,0,0)_{\theta_k=1}|\ \cos\left(\omega t + \varphi(0,0,0)_{\theta_k=1} - \varphi(0,0,0)\right), \quad (6.6a)$$

$$\langle \hat{p}_k(t)\rangle_x = \langle \bar{p}_k(t)\rangle_{x^*} = -2\ |\alpha(0,0,0)\alpha(0,0,0)_{\theta_k=1}|\ \sin\left(\omega t + \varphi(0,0,0)_{\theta_k=1} - \varphi(0,0,0)\right), \quad (6.6b)$$

where $\alpha(0,0,0)_{\theta_k=1}$ denotes $\alpha(1,0,0)$, $\alpha(0,1,0)$, $\alpha(0,0,1)$ according as $k = 1,2,3$, respectively.

However, although the standard deviations of the coordinates and momenta of the WQO in a state x and of the CQO in a state x^* also look somewhat similar, they are in fact different:

$$\text{WQO}: \quad \Delta\hat{r}_k(t)_x = \Big[|\alpha(0,0,0)|^2 + |\alpha(0,0,0)_{\theta_k=1}|^2 \quad (6.7)$$
$$-4|\alpha(0,0,0)\alpha(0,0,0)_{\theta_k=1}|^2 \cos^2(\omega t - \varphi(0,0,0) + \varphi(0,0,0)_{\theta_k=1})\Big]^{1/2}, \quad k = 1,2,3.$$

$$\text{CQO}: \quad \Delta\bar{r}_k(t)_{x^*} = \Big[1 + 2|\alpha(0,0,0)_{\theta_k=1}|^2 \quad (6.8)$$
$$-4|\alpha(0,0,0)\alpha(0,0,0)_{\theta_k=1}|^2 \cos^2(\omega t - \varphi(0,0,0) + \varphi(0,0,0)_{\theta_k=1})\Big]^{1/2}, \quad k = 1,2,3.$$

Our standard state z provides a good illustration of the difference:

$$\text{WQO}: \quad \Delta\hat{r}_1(t)_z = \Delta\hat{r}_2(t)_z = \frac{1}{\sqrt{2}}, \quad \Delta\hat{r}_3(t)_z = |\sin(\omega t)|, \quad (6.9)$$

$$\text{CQO}: \quad \Delta\bar{r}_1(t)_{z^*} = \Delta\bar{r}_2(t)_{z^*} = 1, \quad \Delta\bar{r}_3(t)_{z^*} = (1 + \sin^2(\omega t))^{1/2}. \quad (6.10)$$

Let us go further and compare the standard deviations corresponding to the state $y = |1;1,0,0\rangle$ and its Bose "partner" $y^* = |1,0,0)$.

$$\text{WQO}: \quad \Delta\hat{r}_1(t)_y = \Delta\hat{p}_1(t)_y = 1, \quad \Delta\hat{r}_k(t)_y = \Delta\hat{p}_k(t)_y = 0, \quad k = 1,2, \quad (6.11)$$

$$\text{CQO}: \quad \Delta\bar{r}_1(t)_{y^*} = \Delta\bar{p}_1(t)_{y^*} = \sqrt{3}, \quad \Delta\bar{r}_k(t)_{y^*} = \Delta\bar{p}_k(t)_{y^*} = 1, \quad k = 1,2. \quad (6.12)$$

Equation (6.11) confirms that the oscillator in the state $|1;1,0,0\rangle$ "lives" on the r_1-axis. This is not the case for the CQO, either in the state $|1,0,0)$ or in any other state, since the right hand side of (6.8) never vanishes, as is implied also by the Heisenberg uncertainty relations.

We have compared also the sizes of the WQO and CQO corresponding to the basis states. For the CQO

$$\bar{r}(t)^2 = \sum_{k=1}^{3}\left((b_k^+)^2 e^{2i\omega t} + \{b_k^+, b_k^-\} + (b_k^-)^2 e^{-2i\omega t}\right) \quad (6.13)$$

is not an integral of motion. However for the average value of $\bar{r}^2$ in the state x^* we find

$$\langle \bar{r}(t)^2 \rangle_{x^*} = 5 - 2|\alpha(0,0,0)|^2. \tag{6.14}$$

Thus we have

$$\text{WQO}: \quad \langle \hat{r}(t)^2 \rangle_{|1;0,0,0\rangle} = 3, \quad \langle \hat{r}(t)^2 \rangle_{|1;1,0,0\rangle} = \langle \hat{r}(t)^2 \rangle_{|1;0,1,0\rangle} = \langle \hat{r}(t)^2 \rangle_{|1;0,0,1\rangle} = 1, \tag{6.15}$$

$$\text{CQO}: \quad \langle \bar{r}(t)^2 \rangle_{|0,0,0)} = 3, \quad \langle \bar{r}(t)^2 \rangle_{|1,0,0)} = \langle \bar{r}(t)^2 \rangle_{|0,1,0)} = \langle \bar{r}(t)^2 \rangle_{|0,0,1)} = 5. \tag{6.16}$$

Thus only the states $|1;0,0,0\rangle$ of the WQO and $|0,0,0)$ of the CQO have one and the same space dimensions. This is perhaps not surprising since only these states have one and the same energy $\epsilon = 3$ (in units of $\omega\hbar/2$). The energy of the other WQO states is 1, whereas for the other CQO states it is 5.

7 Concluding remarks

It is clear that while alternative, non-canonical solutions to the compatibility equations (2.4) between Hamilton's equations and the Heisenberg equations exist in our $sl(1|3)$ WQO model, they are in several very important respects quite different from the canonical solutions.

Firstly, each state space $W(p)$ of our one particle 3D WQO is finite-dimensional; 8-dimensional in the case of typical representations of $sl(1|3)$, and either 7 or 4-dimensional in the case of atypical representations.

Secondly, both the energy and the angular momentum are quantized, with equally spaced energy levels, all positive and with separation $\frac{1}{2}\hbar\omega$, and with the angular momentum restricted to be 0 or 1. Since there are only a finite number of energy levels, the energy is bounded. The degeneracies are always either 3 or 1. The lowest energy state is non-degenerate in all the typical cases, but degenerate in each of the atypical cases.

Thirdly, the spectrum of coordinates is also quantized, to the extent that in any stationary state measurements of the coordinates r_1, r_2 and r_3 give values consistent only with the particle being found at a finite number of possible sites, namely the various nests that we have identified. In the typical, $p > 2$ case, the number of possible nests is 64, while in the atypical cases it is 44 if $p = 2$ and only 14 if $p = 1$. In all cases the distance of the particle from the origin is bounded and may take on only the values $\sqrt{(\hbar/2m\omega)(3p-2q)}$ with $q \in \{0,1,2,3\}$ such that $q \leq p$.

Fourthly, not only is the mean trajectory of the particle in any stationary state zero, but there exist both typical and atypical states for which the standard deviation of some coordinate r_k, or some component of linear momentum p_k, is also zero. This implies that for the WQO there exists no uncertainty relation involving a non-zero uncertainty parameter γ that applies to all states at all times.

Fifthly, the atypical case is distinguished from the typical case in possessing stationary states of dimension lower than three, namely two-dimensional in the case $p = 2$ and one-dimensional in the case $p = 1$.

Many of these non-standard results are a consequence of the fact that the underlying geometry of this WQO model is non-commutative. This means that their interpretation must be undertaken carefully. In particular it should be stressed that it is not possible to specify precisely the position of the particle. Fortunately, the square operators $\hat{r}_1^2$, $\hat{r}_2^2$ and $\hat{r}_3^2$ not only mutually commute but also commute with the Hamiltonian. Their common eigenstates are the stationary states $|p;\Theta\rangle$, for which the eigenvalues of $\hat{r}_1^2$, $\hat{r}_2^2$ and $\hat{r}_3^2$ are simultaneously fixed to be either p, $p-1$ or $p-2$. Thus the spectrum of the measured values of each of the coordinates themselves, r_1, r_2 and r_3, is necessarily restricted to the set of values $\pm\sqrt{p}$, $\pm\sqrt{p-1}$ and $\pm\sqrt{p-2}$. Any measurement of a coordinate, r_1 say, results in one or other of the allowed positive or negative values of r_1 with, as we have shown, equal probability, leaving the signs of the other coordinates undetermined. Thus the particle in a stationary state $|p;\Theta\rangle$ has a certain probability of being within one or other of the relevant nests, but it is not to be thought of as localized in any particular nest.

In this article we have restricted ourselves to the relatively simple case of an $n=1$, single particle 3-dimensional WQO with a relatively low, 2^3-dimensional Fock space $W(p)$ associated with typical irreducible representations of $sl(1|3)$, and even lower dimensions for atypical representations. A very natural next step is to generalize the results to the case of an n-body 3-dimensional WQO as introduced in [2]. In such a case the dimension of the Fock space is 2^{3n}-dimensional for typical representations, and again lower dimensions for atypical representations. The relevant calculations are somewhat intricate and will be the subject of a separate article, in which the restriction from the Lie superalgebra $sl(1|3n)$ to the simple Lie algebra $so(3)$ of the rotation group plays a key role in the determination of the possible angular momentum states of our multiparticle system. It suffices to say at this stage that not only are the energy and angular momentum quantized and bounded but, in the corresponding stationary states of fixed energy and angular momentum, so are the single particle coordinates and components of linear momentum. As is to be expected the corresponding nest structure is more complicated and we have to contend with the relevant class A statistics, and account for the numbers of particles whose coordinates, when measured, can coincide with those of the nests, as well as more complicated patterns of degeneracy.

Further generalizations also come to mind. In particular the class of irreducible representations considered here are those specified by the parameter p. There exist other finite-dimensional irreducible representations of $sl(1|3)$, and more generally of $sl(1|3n)$, that are specified not just by a single positive integer p, but by a partition or equivalently a sequence of Kac-Dynkin indices [32]. These can be expected to provide other interesting models of the Wigner quantum oscillator in the one-particle case and, more particularly, in multi-particle cases. At the same time it would be interesting to explore in the same way other non-oscillator Wigner quantum systems. For examples of this kind see [7, 10, 13].

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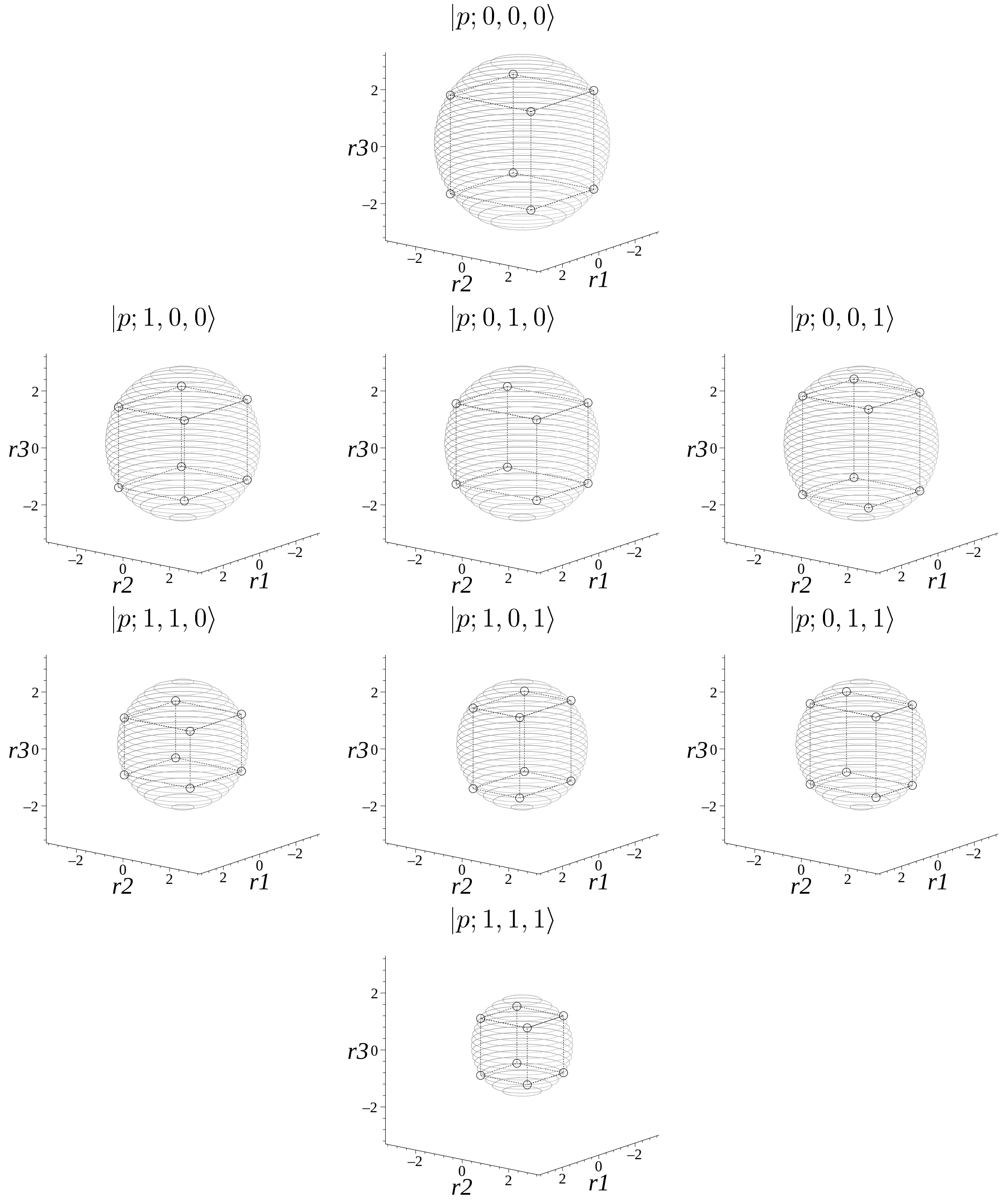

Figure 1. Identification of the results of all possible measurements of the coordinates of the particle for each of the stationary states $|p; \Theta\rangle$ of $W(p)$ with $p > 2$. The eight small circles on each sphere are the nests, the places where the oscillating particle can be spotted. The nests in the states of the first ($q = 0$), the second ($q = 1$), the third ($q = 2$) and the last line ($q = 3$) from the top, are on spheres with radii $\varrho_0 > \varrho_1 > \varrho_2 > \varrho_3$, see (3.8), and energies $E_0 > E_1 > E_2 > E_3$, (see 3.1), respectively.

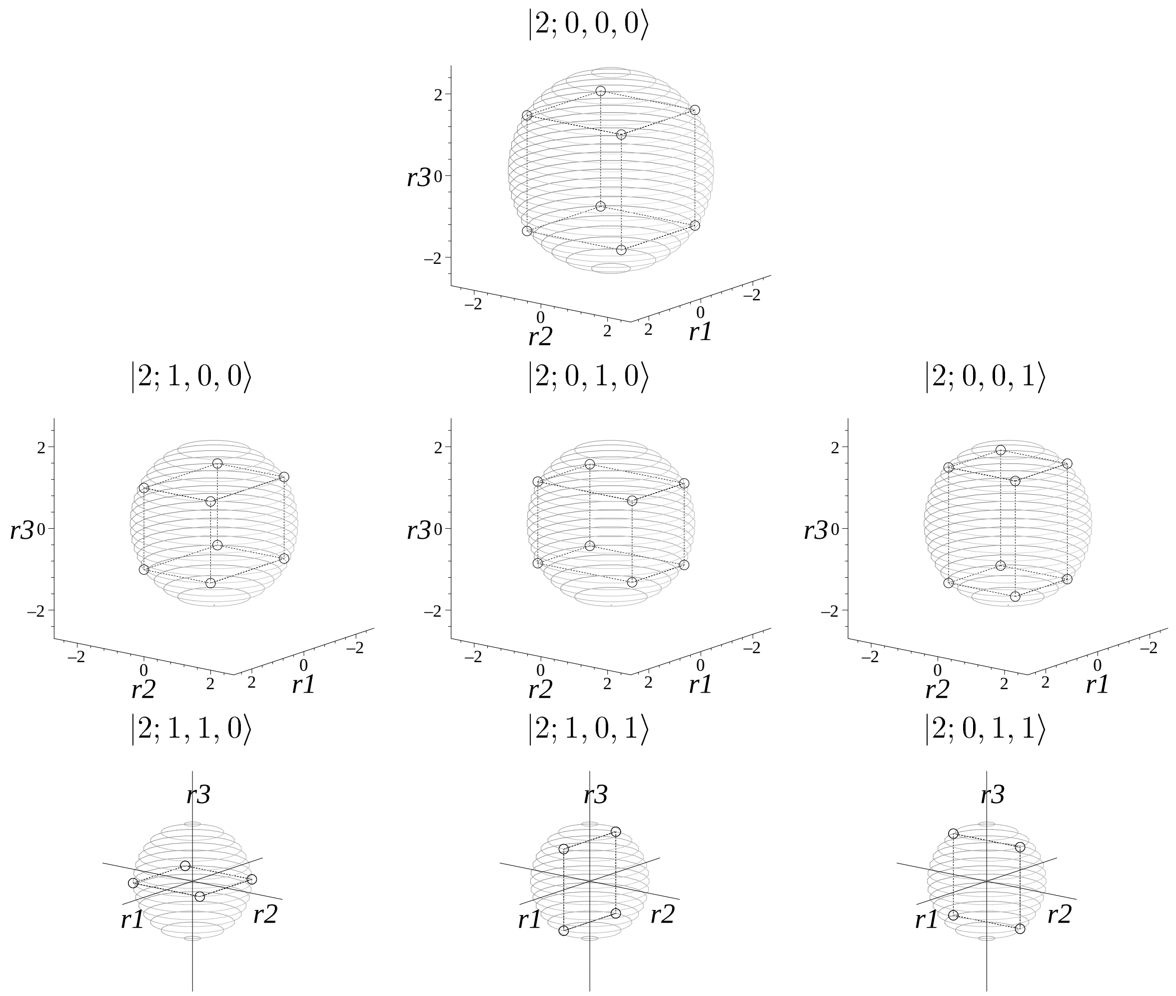

Figure 2. Identification of the results of all possible measurements of the coordinates of the particle for each of the stationary states $|p;\Theta\rangle$ of $W(p)$ with $p=2$. For $q=0$ or $q=1$ (the states on the first and the second line from the top), the coordinates of the particle correspond to one of the eight nests on the sphere, indicated by circles. For $q=2$ (the bottom line), there are three independent lowest energy states. For these states, there are only 4 possible nests on each sphere with radius $\sqrt{2}$. These 4 nests are in the r_1r_2-plane for $\Theta=(1,1,0)$, in the r_1r_3-plane for $\Theta=(1,0,1)$, and in the r_2r_3-plane for $\Theta=(0,1,1)$.

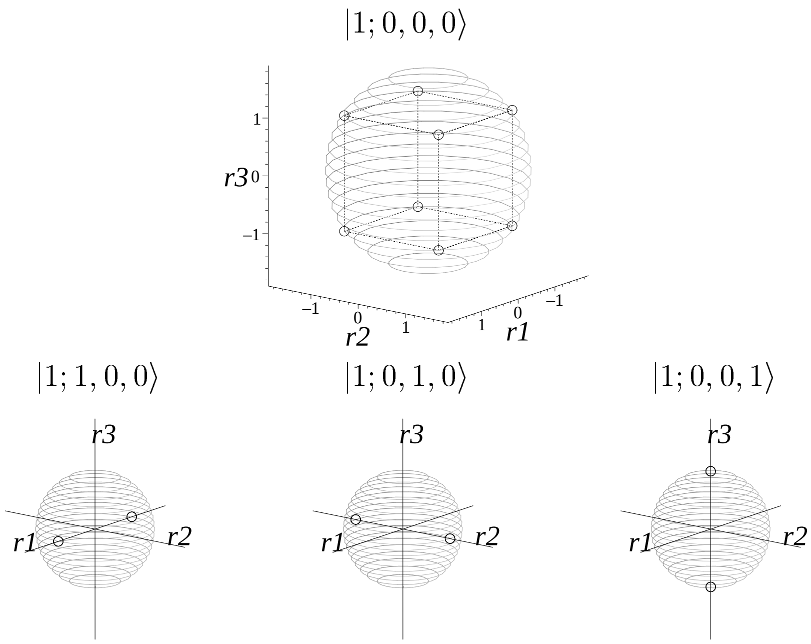

Figure 3. Identification of the results of all possible measurements of the coordinates of the particle for each of the stationary states $|p;\Theta\rangle$ of $W(p)$ with $p=1$. For $q=0$ (the top picture) the coordinates of the particle corresponds to one of the eight nests on the sphere, indicated by circles. For $q=1$ (the bottom line), there are three independent lowest energy states. For each such state the nests are at opposite poles on a sphere with radius 1.