Symmetry Groups of Bailey's Transformations for ${}_{10}\phi_9$-series

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Abstract

Although most of the symmetry groups or "invariance groups" associated with two term transformations between (basic) hypergeometric series have been studied and identified, this is not the case for the most general transformation formulae in the theory of basic hypergeometric series, namely Bailey's transformations for ${}_{10}\phi_9$-series. First, we show that the invariance group for both Bailey's two term transformations for terminating ${}_{10}\phi_9$-series and Bailey's four term transformations for non-terminating ${}_{10}\phi_9$-series (rewritten as a two term transformation of a so called Φ-series) is isomorphic to the Weyl group of type E_6. We continue our recent research concerning the group structure underlying three term transformations [10] and demonstrate that the group associated with a three term transformation between these Φ-series, each admitting Bailey's two term transformation, is the Weyl group of type E_7. We do this by giving a description of the root system of type E_7 that allows to find a transformation between equivalent three term identities in an easy way. A computation shows that there are five, essentially different, three term transformations between these Φ-series; we give an explicit form of each of these five transformations in an elegant way. To our knowledge only one of these transformations has appeared in the literature.

Keywords: Basic hypergeometric series, Bailey's transformations, Symmetry group, Root systems, E_6, E_7
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1 Introduction

This article deals with transformations between basic hypergeometric series, and we use the (standard) notation of [3] when working with such series. The *q-shifted factorial* is

$$(a;q)_0 = 1, \quad \text{and } (a;q)_n = \prod_{k=0}^{n-1}(1-a\,q^k), \text{ for } n = 1,2,\ldots,\infty,$$

and $(a_1,\ldots,a_m;q)_n = (a_1;q)_n\cdots(a_m;q)_n$. To ease notation, from Section 4 onwards, it will prove convenient to introduce the following shorthand notation, which is inspired by [3, Exercise 2.16]

$$S(a_1,\ldots,a_n) = (a_1,\ldots,a_n,q/a_1,\ldots,q/a_n;q)_\infty.$$

In this article, we only work with *very-well-poised* basic hypergeometric series of the form

$$\begin{aligned} &{}_{r+1}W_r(a_1;a_4,a_5,\ldots,a_{r+1};q,z) \\ &\quad = {}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_r \end{matrix};q,z\right) = \sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_k}{(q,b_1,b_2,\ldots,b_r;q)_k}z^k, \end{aligned} \tag{1}$$

where, by definition of a very-well-poised series, the following relations between the parameters hold

$$qa_1 = a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r \quad \text{and} \quad a_2 = qa_1^{1/2},\ a_3 = -qa_1^{1/2}.$$

A basic hypergeometric series is balanced if $b_1 \cdots b_r = qa_1 \cdots a_{r+1}$ and $z = q$; all encountered series will be balanced. In general, a hypergeometric series is *terminating* when one of its numerator parameters a_i equals q^{-n}, where n is a nonnegative integer. If this is not the case, then the series is non-terminating; we assume that the convergence conditions for non-terminating series are always satisfied.

A very-well-poised basic hypergeometric series has so called *trivial transformations*, i.e. one can freely permute the parameters a_4 up to a_{r+1} without changing the series. Some very-well-poised basic hypergeometric series, however, also satisfy non-trivial transformations. A well known example is Bailey's transformation formula for a terminating balanced very-well-poised ${}_{10}\phi_9$-series, see e.g. [3, Eq. 2.9.1] or [9, T10901]:

$$\begin{aligned} &{}_{10}W_9(a; q^{-n}, c, d, e, f, g, h; q, q) \\ &\quad = \frac{(aq, aq/fg, a^2q^2/cdef, a^2q^2/cdeg; q)_n}{(aq/f, aq/g, a^2q^2/cdefg, a^2q^2/cde; q)_n} \, {}_{10}W_9(\frac{a^2q}{cde}; q^{-n}, \frac{aq}{de}, \frac{aq}{ce}, \frac{aq}{cd}, f, g, h; q, q), \end{aligned} \tag{2}$$

where $a^3q^2 = q^{-n}cdefgh$. This last condition expresses the balancing requirement. It is easily verified that the series on the right hand side of this identity is balanced as well, and hence, this identity can be *iterated*. In this case, direct iteration would simply yield the identity transformation. One can however also include permutations of c up to h and then one gets an interesting group of parameter transformations denoted by H. This group will be studied here and identified as the Weyl group of type E_6. We point out that q^{-n} may *not* be included in the permutations as it plays a special role, ensuring the termination of the series. For similar examples of such symmetry groups of basic hypergeometric series transformations, see [11].

The arguments of the series on the right hand side of (2) are "linear" combinations of the arguments of the series on the left hand side, albeit in multiplicative form. Nevertheless, it is very easy to translate these linear combinations into matrix form. For the transformation at hand, one would write the following nine-dimensional matrix acting on a nine-dimensional vector in the following way

$$\begin{pmatrix} 2 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} a \\ q^{-n} \\ c \\ d \\ e \\ f \\ g \\ h \\ q \end{pmatrix} = \begin{pmatrix} a^2q/cde \\ q^{-n} \\ aq/de \\ aq/ce \\ aq/cd \\ f \\ g \\ h \\ q \end{pmatrix}. \tag{3}$$

One sees that this is ordinary matrix vector multiplication with addition replaced by multiplication and multiplication by exponentiation. Since we act on vectors from the left, composition of group transforms will be done right to left; in this way composition of group transforms corresponds to ordinary matrix multiplication.

In [10] we started our study of "invariance groups of three term transformations", and there it is clearly explained what we mean by this concept. For now, it suffices to say that for a certain linear combination, denoted by Φ, of two ${}_{10}W_9$-series there exists a known identity containing three of these Φ-series [4, Eq. 6.5]. It is important to remark that this Φ-series satisfies a two term identity that is structurally identical to (2), implying that the invariance group for Φ is H

as well. The arguments of each of the Φ-series in the three term identity are regarded as group element transforms of the "original" arguments, and these are added to the group H to form a bigger group G, which will be identified as the Weyl group of type E_7. In [4], a particular three term transformation between Φ-series is given. It also follows from the analysis in [4] that there exists a three term transformation for any set of three Φ-series from a set of 56 Φ-series. This set of 56 Φ-series is constructed in [4] as a set of solutions of a complicated second order difference equation. Although the construction in [4] is ingenious, it is not clear why the number of solutions is 56. Here, we show that this is related to the $56 = |E_7|/|E_6|$ cosets of E_6 in E_7. Between any three of the 56 cosets there exists a three term transformation. We show that there are five essentially different three term transformations, and we give these five transformations explicitly.

For completeness, we also introduce the notation for q-integrals [3]:

$$\int_a^b f(t), d_q(t) \equiv \int_0^b f(t)\, d_q(t) - \int_0^a f(t)\, d_q(t),$$

where

$$\int_0^a f(t)\, d_q(t) = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n.$$

The structure of this article is as follows. In Section 2, we study the relevant two term transformations, showing, by given an explicit description, that the invariance group of these transformations is the Weyl group of type E_6. In the next Section, we study the group structure of a three term transformation connecting three Φ-series. This is an entirely different concept from that of an invariance group of a two term transformation; in a previous article studying (different) three term transformations [10], it is clearly explained what is meant by this concept. We will show that of the 27720 three term transformations, there are only five that are essentially different from each other, meaning that these five cannot be obtained from each other using substitutions. All five can however be derived from one by using an elimination procedure. In Section 4, we give these five identities, paying special attention to get the coefficients in these identities as simple as possible.

Finally, we remark that although in principal the order of the numerator parameters in a very-well-poised basic hypergeometric series is immaterial (but for the first parameter), throughout this article, we consider the parameters to be fixed in the order given.

2 Bailey's Two Term Transformations

Transformation for terminating series. We repeat Bailey's transformation for terminating balanced very-well-poised ${}_{10}\phi_9$-series:

$$\begin{aligned}
&{}_{10}W_9(a; q^{-n}, c, d, e, f, g, h; q, q)\\
&\quad = \frac{(aq, aq/fg, a^2q^2/cdef, a^2q^2/cdeg; q)_n}{(aq/f, aq/g, a^2q^2/cdefg, a^2q^2/cde; q)_n}\, {}_{10}W_9(\frac{a^2q}{cde}; q^{-n}, \frac{aq}{de}, \frac{aq}{ce}, \frac{aq}{cd}, f, g, h; q, q),
\end{aligned} \tag{4}$$

where n is a nonnegative integer and the parameters satisfy the following restriction $a^3q^2 = q^{-n}cdefgh$. Besides this transformation, also permutations of the parameters c, d, e, f, g and h are allowed; or more correctly, permutations of the third up to the eighth argument of the ${}_{10}W_9$-series are allowed. We now introduce a rescaling of the ${}_{10}W_9$-series for which the transformation (4) can be written more naturally. Using the restriction between the parameters, one rewrites part of the factor in (4) as follows:

$$\frac{(aq/fg;q)_n}{(a^2q^2/cdefg;q)_n} = \frac{(q^2a^2/cdeh;q)_n}{(aq/h;q)_n}\left(\frac{cde}{aq}\right)^n.$$

This allows one to rewrite (4) as:

$$w(a;q^{-n};c,d,e,f,g,h) = w(\frac{a^2q}{cde};q^{-n};\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},f,g,h), \tag{5}$$

with

$$\begin{aligned} &w(a;q^{-n};c,d,e,f,g,h) \\ &\quad\equiv \frac{(aq/c,aq/d,aq/e,aq/f,aq/g,aq/h;q)_n}{(aq;q)_n a^n}\,{}_{10}W_9(a;q^{-n},c,d,e,f,g,h;q,q). \end{aligned} \tag{6}$$

In the notation of (6), we have used two semicolons to indicate that the first and second argument play a special role, while the function w is symmetric in its last six arguments.

Transformation for non-terminating series. Bailey's four term transformation formula is also known, see e.g. [3, Eq. 2.12.9]. The equation given there however is written in a rather non-symmetric way; to devise a more symmetric form of this identity, we start with the more compact way of writing this transformation formula using q-integrals [3, Eq. 2.12.10]:

$$\begin{aligned} &\int_a^b \frac{(qt/a,qt/b,t/\sqrt{a},-t/\sqrt{a},qt/c,qt/d,qt/e,qt/f,qt/g,qt/h;q)_\infty}{(t,bt/a,qt/\sqrt{a},-qt/\sqrt{a},ct/a,dt/a,et/a,ft/a,gt/a,ht/a;q)_\infty}\,d_qt \\ &\quad= \frac{a}{\lambda}\frac{(b/a,aq/b,\lambda c/a,\lambda d/a,\lambda e/a,bf/\lambda,bg/\lambda,bh/\lambda;q)_\infty}{(b/\lambda,\lambda q/b,c,d,e,bf/a,bg/a,bh/a;q)_\infty} \\ &\qquad\times\int_\lambda^b \frac{(qt/\lambda,qt/b,t/\sqrt{\lambda},-t/\sqrt{\lambda},aqt/c\lambda,aqt/d\lambda,aqt/e\lambda,qt/f,qt/g,qt/h;q)_\infty}{(t,bt/\lambda,qt/\sqrt{\lambda},-qt/\sqrt{\lambda},ct/a,dt/a,et/a,ft/\lambda,gt/\lambda,ht/\lambda;q)_\infty}\,d_qt, \end{aligned} \tag{7}$$

where $\lambda = qa^2/cde$ and $a^3q^2 = bcdefgh$. By redistributing the factors in front of the q-integrals (and adding some extra factors for symmetry reasons), it is clear that one should consider:

$$\begin{aligned} I(a;b;c,d,e,f,g,h) &\equiv \frac{1}{a}\frac{(c,d,e,f,g,h,bc/a,bd/a,be/a,bf/a,bg/a,bh/a;q)_\infty}{(b/a,aq/b;q)_\infty} \\ &\quad\times\int_a^b \frac{(qt/a,qt/b,t/\sqrt{a},-t/\sqrt{a},qt/c,qt/d,qt/e,qt/f,qt/g,qt/h;q)_\infty}{(t,bt/a,qt/\sqrt{a},-qt/\sqrt{a},ct/a,dt/a,et/a,ft/a,gt/a,ht/a;q)_\infty}\,d_qt. \end{aligned}$$

Using this notation, the identity (7) becomes:

$$I(a;b;c,d,e,f,g,h) = I(\frac{a^2q}{cde};b;\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},f,g,h), \tag{8}$$

with $a^3q^2 = bcdefgh$. Note that this identity, including the restriction, is structurally identical to the two term transformation of terminating ${}_{10}\phi_9$-series (5) (with b playing the role of q^{-n}).

Using the definition of the q-integral, one rewrites I as a difference of two very-well-poised ${}_{10}\phi_9$-series; after cancelling common factors on the left and right hand side of (8), one finds that:

$$\Phi(a;b;c,d,e,f,g,h) = \Phi(\frac{a^2q}{cde};b;\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},f,g,h), \tag{9a}$$

$$\text{where}\qquad a^3q^2 = bcdefgh, \tag{9b}$$

and

$$\begin{aligned}
&\Phi(a;b;c,d,e,f,g,h)\\
&\quad\equiv \frac{(aq/c,aq/d,aq/e,aq/f,aq/g,aq/h,bc/a,bd/a,be/a,bf/a,bg/a,bh/a;q)_\infty}{(b/a,aq;q)_\infty}\\
&\qquad\times {}_{10}W_9(a;b,c,d,e,f,g,h;q,q)\\
&\qquad\quad+\frac{(bq/c,bq/d,bq/e,bq/f,bq/g,bq/h,c,d,e,f,g,h;q)_\infty}{(a/b,b^2q/a,;q)_\infty}\\
&\qquad\qquad\times {}_{10}W_9(b^2/a;b,bc/a,bd/a,be/a,bf/a,bg/a,bh/a;q,q).
\end{aligned} \tag{9c}$$

One can perform on Φ the transformation (9a), together with permutations of the last six arguments. Repeated application of (9a) together with the mentioned permutations yields (as in the terminating case) an invariance group of order 51840. Given that both transformations are structurally identical, their invariance groups will be isomorphic. The study of this invariance group is our next subject. Note that although (9a), when expanded, actually comprises four ${}_{10}W_9$-series, we still consider it to be a two term transformation, as the natural object now is no longer a single (possibly rescaled) ${}_{10}W_9$-series, but the linear combination (9c) of two such series.

Invariance group of Bailey's two term transformations. To study the invariance group of Bailey's two term transformations, we consider six transformations (reflections) r_1 up to r_6. The first five of which simply correspond to swapping two adjacent arguments of (a,b,c,d,e,f,g,h,q):

$$r_1 \equiv c \leftrightarrow d, \quad r_2 \equiv d \leftrightarrow e, \quad r_3 \equiv e \leftrightarrow f, \quad r_4 \equiv f \leftrightarrow g, \quad r_5 \equiv g \leftrightarrow h.$$

More rigorously, one writes

$$r_1(a,b,c,d,e,f,g,h,q) = (a,b,d,c,e,f,g,h,q),$$

and analogously for r_2 up to r_5. The sixth transformation corresponds to (9a):

$$r_6(a,b,c,d,e,f,g,h,q) = (\frac{a^2q}{cde}, b, \frac{aq}{de}, \frac{aq}{ce}, \frac{aq}{cd}, f,g,h,q).$$

Using the GAP-program [2], one verifies quite easily that $|H| = |\langle r_1,r_2,r_3,r_4,r_5,r_6\rangle| = 51840$. We will now show that the group H is isomorphic to the Weyl group of type E_6. First, we shall give a description of this Weyl group, and since the Lie algebra E_6 plays no role here, the Weyl group itself will be denoted by E_6. For ease and familiarity of notation however, the description and especially the action of group elements on vectors of the relevant vector space will be given in the usual *additive* form, i.e. using ordinary matrix vector multiplication, and not the action used in (3). When connecting this description with the series transformations, we will "translate" these results – very straightforwardly – into multiplicative form.

In the literature one can find many descriptions of the root system of type E_6 [1, 5, 6]. For our purposes, it will be crucial to have a description of the Weyl group E_6 for which the symmetric subgroup S_6 is evident (note how the reflections r_1 up to r_5 generate the symmetric group S_6). In terms of root systems, this implies that the root system of E_6 should have a natural subsystem of type A_5.

To describe the appropriate root system and Weyl group of type E_6, consider the 8-dimensional real vector space $\mathbb{R}^8$ with orthonormal basis vectors ϵ_i ($i = 1,\ldots,8$), i.e. with inner product

$$\langle \epsilon_i, \epsilon_j\rangle = \delta_{ij}. \tag{10}$$

The roots of E_6 will be elements of the 6-dimensional subspace V of $\mathbb{R}^8$ consisting of those elements $\sum_{i=1}^{8} c_i\epsilon_i$ with $c_1 + c_2 = 0$ and $\sum_{i=3}^{8} c_i = 0$. A set of simple roots of E_6 is then given by the elements

$$\alpha_i = \epsilon_{i+2} - \epsilon_{i+3} \ (i = 1, \ldots, 5) \quad \text{and} \quad \alpha_6 = \frac{1}{2}(-\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8). \tag{11}$$

The corresponding Dynkin diagram is given in Figure 1(a); using the inner product (10) it is easy to check that the Cartan matrix of E_6 arises. Also the A_5 subsystem is evident. Note that all 72 non-zero roots (and their number) are given by

$$\pm(\epsilon_i - \epsilon_j), \qquad (1 \le i < j \le 2; \ \text{ or } 3 \le i < j \le 8); \qquad 32 \tag{12a}$$

$$\frac{1}{2}\Big(\sum_{i=1}^{8}(-1)^{a_i}\epsilon_i\Big), \qquad (a_i \in \{0,1\}; \ \sum_{i=1}^{2} a_i = 1 \text{ and } \sum_{i=3}^{8} a_i = 3). \qquad 40 \tag{12b}$$

So all coefficients in $\sum_{i=1}^{8}(-1)^{a_i}\epsilon_i$ are ± 1, and the condition is equivalent to saying that among the first two coefficients there is one $+1$ and one -1, and among the last six coefficients there are three $+1$'s and three -1's.

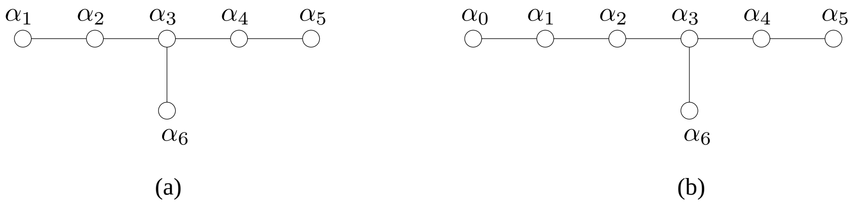

Figure 1: Dynkin diagrams of the root systems of type E_6 and E_7.

The Weyl group E_6 is now generated by the six simple reflections $\tilde{r}_i \equiv \tilde{r}_{\alpha_i}$, with $i \in \{1, \ldots, 6\}$, where

$$\tilde{r}_i(x) = x - 2\frac{\langle x, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i = x - \langle x, \alpha_i\rangle\alpha_i. \tag{13}$$

Herein, $x = \sum_{i=1}^{8} x_i\epsilon_i$, sometimes denoted by its coordinates $x = (x_1, x_2; \ x_3, x_4, x_5, x_6, x_7, x_8)$, and since each such reflection keeps both $x_1 + x_2$ and $\sum_{i=1}^{8} x_i$ fixed one can think of the Weyl group acting in the 6-dimensional hyperplane of $\mathbb{R}^8$ with

$$\ell(x) \equiv x_1 + x_2 = C, \quad \text{and} \quad |x| \equiv \sum_{i=1}^{8} x_i = C' \tag{14}$$

for some constants C and C'. Note that the subgroup S_6 of E_6, generated by $\tilde{r}_i$ with $1 \le i \le 5$, acts on x by permuting the coordinates x_j, with $j \in \{3, \ldots, 8\}$; more particularly $\tilde{r}_i$, with $i \in \{1, 2, 3, 4, 5\}$, acts on x by swapping the coordinates at positions $i+2$ and $i+3$. For further reference we give the action of $\tilde{r}_6$ here explicitly: $x' = \tilde{r}_6(x)$ with

$$\begin{aligned} x'_1 &= x_1 + y, \quad x'_2 = x_2 - y, \quad x'_3 = x_3 + y, \quad x'_4 = x_4 + y, \\ x'_5 &= x_5 + y, \quad x'_6 = x_6 - y, \quad x'_7 = x_7 - y, \quad x'_8 = x_8 - y, \end{aligned} \tag{15}$$

where y is a shorthand for $\frac{1}{4}(-x_1 + x_2 - x_3 - x_4 - x_5 + x_6 + x_7 + x_8)$.

The order of the Weyl group E_6 is 51840. It will be useful to describe the orbit of a general element x (that is, an element not on one of the reflection planes of the Weyl group). To list the 51840 images of x, it is sufficient to list the $51840/6! = 72$ S_6 orbits by their representing elements. Two of these orbits are represented by

$$(x_1, x_2;\ x_3, x_4, x_5, x_6, x_7, x_8) \text{ and } (x_2, x_1;\ x_3, x_4, x_5, x_6, x_7, x_8). \tag{16}$$

The next 20 S_6 orbits are directly represented by an element $\tilde{r}_\alpha(x)$, where α is one of the 40 roots given in (12b). More concretely, if $\alpha = \frac{1}{2}(\sum_{i=1}^{8}(-1)^{a_i}\epsilon_i)$, with the conditions given in (12b), then the components of $x' = \tilde{r}_\alpha(x)$ are given by

$$x'_i = \sum_{j=1}^{8}(\delta_{ij} - \frac{(-1)^{a_i+a_j}}{4})x_j, \quad \text{for } i \in \{1, \ldots, 8\}. \tag{17}$$

We only have 20 of these orbits since $\tilde{r}_\alpha(x) = \tilde{r}_{-\alpha}(x)$. One finds another set of 20 orbits by acting with $\tilde{r}_\alpha$ on the representative of the second orbit given in (16), or stated otherwise, one acts first with $\tilde{r}_{\epsilon_1-\epsilon_2}$ on x and then with $\tilde{r}_\alpha$ on the result. A representing element $x' = \tilde{r}_\alpha\tilde{r}_{\epsilon_1-\epsilon_2}(x)$ has the following coordinates:

$$x'_i = (\delta_{i1} - \frac{(-1)^{a_i+a_1}}{4})x_2 + (\delta_{i2} - \frac{(-1)^{a_i+a_2}}{4})x_1 + \sum_{j=3}^{8}(\delta_{ij} - \frac{(-1)^{a_i+a_j}}{4})x_j. \tag{18}$$

Finally, the remaining 30 S_6 orbits are found by reflections of the type $r_\alpha r_\beta(x)$, with both α and β of type (12b). These 30 orbits are determined by two indices (i, j) with $\{i, j\} \subset \{3, 4, 5, 6, 7, 8\}$. More, explicitly, let

$$\alpha = \frac{1}{2}(\sum_{p=1}^{8}(-1)^{a_p}\epsilon_p) \quad \text{and} \quad \beta = \frac{1}{2}(\sum_{p=1}^{8}(-1)^{b_p}\epsilon_p)$$

with the same conventions as in (12b), and where the coefficients of α and β overlap in the positions $\{1, 2, i, j\}$ and differ in the other positions, or more explicitly

$$a_1 = b_1 = a_j = b_j,\ a_2 = b_2 = a_i = b_i,\ a_k = 1 - b_k,\ a_l = 1 - b_l,\ a_m = 1 - b_m,\ a_n = 1 - b_n,$$

where $\{k, l, m, n\} = \{3, 4, 5, 6, 7, 8\} \setminus \{i, j\}$. A representing element x' has the following coordinates:

$$\begin{array}{ll}
x'_1 = -x_i + \frac{1}{2}(x_1 + x_2 + x_i + x_j) & x'_k = -x_k + \frac{1}{2}(x_k + x_l + x_m + x_n) \\
x'_2 = -x_j + \frac{1}{2}(x_1 + x_2 + x_i + x_j) & x'_l = -x_l + \frac{1}{2}(x_k + x_l + x_m + x_n) \\
x'_i = -x_1 + \frac{1}{2}(x_1 + x_2 + x_i + x_j) & x'_m = -x_m + \frac{1}{2}(x_k + x_l + x_m + x_n) \\
x'_j = -x_2 + \frac{1}{2}(x_1 + x_2 + x_i + x_j) & x'_n = -x_n + \frac{1}{2}(x_k + x_l + x_m + x_n).
\end{array} \tag{19}$$

Thus the E_6 orbit of x, of size 51840, consists of the 72 S_6 orbits (16), (17), (18) and (19) (each of size 720).

With this description of the Weyl group E_6, we can now prove our statements about the invariance group of Φ (and of w). Furthermore, the description of a general orbit under the action of E_6 will allow us to list all distinct forms of the Φ-series.

Lemma 1 *The invariance group $H = \langle r_1, r_2, r_3, r_4, r_5, r_6\rangle$ of both (5) and (9a) is isomorphic to the Weyl group of type E_6.*

Proof: Let x_1 up to x_8 be eight variables subject to the condition that their product equals one, i.e. $\prod_{i=1}^8 x_i = 1$, and let

$$\begin{aligned} &a = q^{1/2}x_1^2, \quad &b = q^{1/2}x_1x_2, \quad &c = q^{1/2}x_1x_3, \quad &d = q^{1/2}x_1x_4, \\ &e = q^{1/2}x_1x_5, \quad &f = q^{1/2}x_1x_6, \quad &g = q^{1/2}x_1x_7, \quad &h = q^{1/2}x_1x_8, \end{aligned} \tag{20}$$

which are invertible relations, satisfying the constraint (9b) between a up to h, due to the fact that $\prod_{i=1}^8 x_i = 1$. In terms of the variables x_i, the arguments on the right hand side of (9a) are

$$\begin{aligned} &\frac{a^2q}{cde} = q^{1/2}\frac{x_1}{x_3x_4x_5}, \quad &b = q^{1/2}x_1x_2, \quad &\frac{aq}{de} = q^{1/2}\frac{1}{x_4x_5}, \quad &\frac{aq}{ce} = q^{1/2}\frac{1}{x_3x_5}, \\ &\frac{aq}{cd} = q^{1/2}\frac{1}{x_3x_4}, \quad &f = q^{1/2}x_1x_6, \quad &g = q^{1/2}x_1x_7, \quad &h = q^{1/2}x_1x_8. \end{aligned} \tag{21}$$

It is easily verified that the transition from (20) to (21) is indeed realized by transformation (15), written in *multiplicative* form. It is even easier to see that swapping two adjacent parameters of (c, d, e, f, g, h) corresponds to swapping x_{i+2} and x_{i+3}, where i gives the position of the first parameter to be swapped. This completes the proof. □

Since we have used the fact that $\prod_{i=1}^8 x_i = 1$ in the proof of this lemma, from now on we assume that the constant C' appearing in (14) equals zero:

$$C' = 0. \tag{22}$$

We now introduce a new function $\tilde{\Phi}$ of eight arguments:

$$\tilde{\Phi}(x) \equiv \Phi(q^{1/2}x_1^2; q^{1/2}x_1x_2; q^{1/2}x_1x_3, q^{1/2}x_1x_4, q^{1/2}x_1x_5, q^{1/2}x_1x_6, q^{1/2}x_1x_7, q^{1/2}x_1x_8), \tag{23}$$

which is simply the Φ-function (9c), but rewritten using the realization (20).

Theorem 1 *Let x_1 up to x_8 be eight variables satisfying $\prod_{i=1}^8 x_i = 1$. The function $\tilde{\Phi}(x)$ is then invariant under the Weyl group of type E_6 acting multiplicatively on the variables $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$. Stated otherwise, it holds that*

$$\tilde{\Phi}(x) = \tilde{\Phi}\big(\tilde{h}(x)\big),$$

for each element $\tilde{h}$ of the group $H = \langle \tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5, \tilde{r}_6\rangle \cong E_6$.

The description of the 72 S_6 orbits of a general element x, given in (16)–(19), enables us to list all 72 distinct forms of the Φ-series (each form still having the 720 "trivial" permutation symmetries of the last 6 variables):

$$\begin{aligned} \Phi(a; b; c, d, e, f, g, h) &= \Phi(a; b; c, d, e, f, g, h) && 1 && \text{(24a)} \\ &= \Phi(\frac{b^2}{a}; b; \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}, \frac{bg}{a}, \frac{bh}{a}) && 1 && \text{(24b)} \\ &= \Phi(\frac{a^2q}{cde}; b; \frac{aq}{de}, \frac{aq}{ce}, \frac{aq}{cd}, f, g, h) && 20 && \text{(24c)} \\ &= \Phi(\frac{abq}{cde}; b; \frac{aq}{de}, \frac{aq}{ce}, \frac{aq}{cd}, \frac{bf}{a}, \frac{bg}{a}, \frac{bh}{a}) && 20 && \text{(24d)} \\ &= \Phi(\frac{bc}{d}; b; c, \frac{bc}{a}, \frac{aq}{de}, \frac{aq}{df}, \frac{aq}{dg}, \frac{aq}{dh}) && 30 && \text{(24e)} \end{aligned}$$

Here, the first two orbits, represented by (24a) and (24b), refer to the two orbits given in (16). The next 20 orbits, represented by (24c) correspond to a reflection over a root α, see (17); for this particular case the reflection $\tilde{r}_\alpha$ is determined by the fact that $a_2 = a_6 = a_7 = a_8$. The next 20 orbits, represented by (24d), are also determined by a root α and correspond to (18); in this particular case the same root α as before will do the trick. The last 30 orbits correspond to (19); in this particular case one has $a_2 = b_2 = a_3 = b_3$ and $a_1 = b_1 = a_4 = b_4$.

The fact that there are 20 different forms of type (24c), for example, can also be seen from the Φ-arguments: there is a form of this type for any set of 3 elements (like $\{c, d, e\}$) out of the six elements $\{c, d, e, f, g, h\}$, thus there are $\binom{6}{3} = 20$ such forms.

An important observation is that all 72 distinct forms of the Φ-series (or, disregarding the trivial permutation symmetries, all 51840 forms) all have the same second parameter b. When no confusion is possible, we shall sometimes refer to this set (or to one of its representatives) as Φ^b.

As a final remark regarding the two term transformations we finish by saying that equations (24) are also valid in the terminating case, provided one changes Φ to w and b to q^{-n}. In this case (24b) corresponds to a reversal of series.

3 Three Term Transformations

In [4] it is shown that a certain linear combination of two very-well-poised balanced ${}_{10}\phi_9$-series satisfies a three term transformation. This linear combination is the one given in (9c) but multiplied by the inverse of the coefficient in front of the series ${}_{10}W_9(a; b, c, d, e, f, g, h)$, so that the coefficient of this series equals 1. This immediately implies that Φ also satisfies a three term transformation. We will study the group structure underlying this three term transformation, allowing us to characterize the different types of three term transformations that exist between Φ-series. If we swap G and C in formula [4, Eq. 6.5] (and write it with lower case parameters), then it is of the following form:

$$C_1\,\Phi(a; c; b, d, e, f, g, h) + C_2\,\Phi(\frac{q}{a}; \frac{q}{h}; \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}, \frac{q}{g}) + C_3\,\Phi(\frac{c^2}{a}; \frac{bc}{a}; c, \frac{cd}{a}, \frac{ce}{a}, \frac{cf}{a}, \frac{cg}{a}, \frac{ch}{a}) = 0, \tag{25}$$

where $a^3q^2 = bcdefgh$ and where the coefficients C_1 and C_2 are infinite products, while the coefficient C_3 is a difference of two infinite products.

We now regard the arguments of these Φ-series as group element transforms of the “original” arguments of the Φ-series; e.g. we let

$$r_0(a, b, c, d, e, f, g, h, q) = (\frac{c^2}{a}, \frac{bc}{a}, c, \frac{cd}{a}, \frac{ce}{a}, \frac{cf}{a}, \frac{cg}{a}, \frac{ch}{a}, q). \tag{26}$$

If we add r_0 as a generator to the group H, and call the resulting group G, then one verifies, using GAP [2], that $|G| = |\langle r_0, r_1, r_2, r_3, r_4, r_5, r_6\rangle| = 2903040$, which is the order of the Weyl group E_7. The arguments of the first two Φ-series do not contribute anything new; when regarded as group transforms, they are already included in the group G. This is easily checked by noting that the group size does not increase when the corresponding group transforms are added; it will also be immediately clear once the group G has been studied.

The three term transformation (25) can be written in many different ways, keeping the coefficients C_1, C_2 and C_3 fixed. Indeed, on each of the series one can apply an element of the group H (isomorphic to E_6) without changing the identity. In this sense, formula (25) may be regarded not only as an identity connecting three Φ-series, but as an identity connecting three different sets of Φ-series, each of size 51840. We repeat here that applying an element of H on a Φ-series preserves its second argument, so following the short hand notation introduced at the end of the previous section, one can write (25) as:

$$C_1\,\Phi^{c} + C_2\,\Phi^{q/h} + C_3\,\Phi^{bc/a} = 0. \tag{27}$$

We would like to prove that the group involved with the three term transformation is indeed the Weyl group E_7, and for the reason just stated, we need a description of the group E_7 for which its subgroup E_6 is evident; in terms of root systems this means that the root system of E_7 should have that of E_6 as a natural subsystem. As in the case of E_6 we use an additive notation when describing the root system.

The roots of E_7 can be described in the same space $\mathbb{R}^8$ as the roots of E_6 with inner product (10); they will now be elements of the 7-dimensional subspace V' consisting of elements $\sum_{i=1}^{8} c_i\epsilon_i$ with $\sum_{i=1}^{8} c_i = 0$. A set of simple roots of E_7 consists of the six simple roots α_i, with $i \in \{1,\ldots,6\}$, of E_6 given in (11) plus the extra root

$$\alpha_0 = \epsilon_1 - \epsilon_3. \tag{28}$$

The corresponding Dynkin diagram can be found in Figure 1(b). By construction, the E_6 subsystem of E_7 is evident. The 126 non-zero roots of E_7 consist of

$$\pm(\epsilon_i - \epsilon_j), \qquad (1 \le i < j \le 8); \qquad 56 \tag{29a}$$

$$\frac{1}{2}\Big(\sum_{i=1}^{8}(-1)^{a_i}\epsilon_i\Big), \qquad \Big(a_i \in \{0,1\}; \quad \sum_{i=1}^{8} a_i = 4\Big). \qquad 70 \tag{29b}$$

Note that the 72 non-zero E_6 roots given in (12) are indeed part of the 126 non-zero roots of (29).

The Weyl group E_7, generated by the seven reflections $\tilde{r}_i$, with $0 \le i \le 6$, acts on elements $x = \sum_{i=1}^{8} x_i\epsilon_i$ of $\mathbb{R}^8$. The quantity $|x| = \sum_{i=1}^{8} x_i$ is invariant under the action of E_7, so it is a fixed constant C' as in (14). In accordance with the choice (22), we shall assume that $C' = 0$ or $\sum_{i=1}^{8} x_i = 0$. If we define the E_6-*level* of x by

$$\ell(x) = x_1 + x_2, \tag{30}$$

then the action of the E_6 subgroup will keep the level fixed, see (14), whereas the remaining E_7 reflections will change the level. The order of the E_7 Weyl group is 2903040. Since we have already described the E_6 orbit of a general x, it is now sufficient to list the $2903040/51840 = 56$ representing elements of the 56 E_6 orbits in the E_7 orbit of x. By performing the reflections $\tilde{r}_i$, with $0 \le i \le 5$, and $\tilde{r}_{\epsilon_1-\epsilon_2}$ repeatedly, one can see that there are 28 orbits of the following type:

$$u = (x_{i_1}, x_{i_2};\ x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}, x_{i_8}), \qquad \ell(u) = x_{i_1} + x_{i_2}, \tag{31}$$

where $(i_1, i_2, \ldots, i_8)$ is a permutation of $(1, 2, \ldots, 8)$ with $i_1 < i_2$ and $i_3 < i_4 < \cdots < i_8$. Hence the orbit is characterized by the two numbers $1 \le i_1 < i_2 \le 8$, or equivalently by its E_6-level $x_{i_1} + x_{i_2}$.

To find the remaining 28 orbits, we consider the three roots

$$\begin{aligned}\alpha &= \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4-\epsilon_5-\epsilon_6-\epsilon_7-\epsilon_8),\\ \beta &= \frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4+\epsilon_5+\epsilon_6-\epsilon_7-\epsilon_8), \quad \text{and}\\ \gamma &= \frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5-\epsilon_6+\epsilon_7+\epsilon_8).\end{aligned}$$

One verifies easily that

$$\tilde{r}_{\epsilon_7-\epsilon_8}\tilde{r}_{\epsilon_5-\epsilon_6}\tilde{r}_{\epsilon_3-\epsilon_4}\tilde{r}_{\epsilon_1-\epsilon_2}\tilde{r}_{\gamma}\tilde{r}_{\beta}\tilde{r}_{\alpha}(x) = (-x_1,-x_2;-x_3,-x_4,-x_5,-x_6,-x_7,-x_8). \tag{32}$$

In view of the first 28 orbits, it it clear that the remaining 28 orbits are of the type

$$u = (-x_{i_1},-x_{i_2};\ -x_{i_3},-x_{i_4},-x_{i_5},-x_{i_6},-x_{i_7},-x_{i_8}), \quad \text{with} \quad \ell(u) = -x_{i_1}-x_{i_2}, \tag{33}$$

where again $(i_1,i_2,\ldots,i_8)$ is a permutation of $(1,2,\ldots,8)$ with $i_1<i_2$ and $i_3<i_4<\cdots<i_8$. Hence, also these orbits are characterized by two numbers $1\le i_1<i_2\le 8$, or equivalently by their E_6-level $-x_{i_1}-x_{i_2}$.

Thus the E_7 orbit of x, of size 2903040, consist of 56 E_6 orbits (each of size 51840); observe that each such E_6 orbit is uniquely characterized by its level.

The following lemma is now very easy to prove.

Lemma 2 *The group $G=\langle r_0,r_1,r_2,r_3,r_4,r_5,r_6\rangle$ associated with the three term transformation (25) is isomorphic to the Weyl group E_7.*

Proof: We return to the realization (20) already used in proving that H is isomorphic to the Weyl group E_6. Given the description of E_7 above, all we need to do is verify that r_0 given in (26) indeed corresponds to $\tilde{r}_0$ with $\tilde{r}_0(x)=(x_3,x_2,x_1,x_4,x_5,x_6,x_7,x_8)$, which is trivial. □

We have seen that each of the 56 E_6 orbits of a general element x is characterized by two numbers $1\le i_1<i_2\le 8$, and that there are 28 orbits of type (31) – which we shall label by (i_1,i_2) – corresponding to the E_6-level $x_{i_1}+x_{i_2}$, and 28 of type (33) – which we shall label by $(i_1,i_2)^*$ – corresponding to the E_6-level $-x_{i_1}-x_{i_2}$. The description of the 56 E_6 orbits can now be used to list the corresponding 56 sets of Φ-series. In this process, we can use the realization (20), and work as before with the multiplicative form of the transformations. As an example, let us identify the sets corresponding to the three terms of (25). The first series, $\Phi(a;c;b,d,e,f,g,h)$, can be written as $\tilde{\Phi}\big(\tilde{r}_{\epsilon_2-\epsilon_3}(x)\big)=\tilde{\Phi}(x_1;x_3;x_2,x_4,x_5,x_6,x_7,x_8)$ in this correspondence. Hence, it is labelled by $(1,3)$. Note that it is also characterized by its second parameter $c=q^{1/2}x_1x_3$.

The second series, $\Phi(q/a;q/h;q/b,q/c,q/d,q/e,q/f,q/g)$, can be written as

$$\tilde{\Phi}(x_1^{-1};x_8^{-1};x_2^{-1},x_3^{-1},x_4^{-1},x_5^{-1},x_6^{-1},x_7^{-1})$$

in the correspondence governed by (20) and (23). Hence, it is labelled by $(1,8)^*$. Note that the second parameter of this Φ-series is $q/h=q^{1/2}/x_1x_8$.

The third series, $\Phi(c^2/a;bc/a;c,cd/a,ce/a,cf/a,cg/a,ch/a)$, has just been identified in the proof of Lemma 2 as $\tilde{\Phi}(x_3;x_2;x_1,x_4,x_5,x_6,x_7,x_8)$ under the correspondence (20). Thus it is labelled by $(2,3)$. The second parameter of Φ is $bc/a=q^{1/2}x_2x_3$, so again there is an obvious relation between the characterizing parameter of Φ and the corresponding orbit label.

Thus we see that (25) connects three orbits (in this context thought of as sets of Φ-series, each of size 51840) determined by $(1,3)$, $(1,8)^*$ and $(2,3)$. Equation (27) could now also be rewritten as

$$C_1\,\tilde{\Phi}_{(1,3)} + C_2\,\tilde{\Phi}_{(1,8)^*} + C_3\,\tilde{\Phi}_{(2,3)} = 0.$$

The relation with the previously introduced short hand notation is clear:

$$\tilde{\Phi}_{(i,j)} \leftrightarrow \Phi^{q^{1/2}x_ix_j} \quad \text{and} \quad \tilde{\Phi}_{(i,j)^*} \leftrightarrow \Phi^{q^{1/2}/x_ix_j},$$

where the superscripts should be rewritten in terms of a up to h using (20). In this notation, the 56 series are denoted by Φ^u with $u \in U$. Herein, $U = U_1 \cup U_2$, where

$$U_1 = \{\frac{ab}{a}, \frac{ac}{a}, \ldots, \frac{gh}{a}\} \quad \text{and} \quad U_2 = \{\frac{aq}{ab}, \frac{aq}{ac}, \ldots, \frac{aq}{gh}\},$$

both sets containing 28 elements.

The 56 series considered here correspond precisely with a set of 56 solutions of a second order difference equation determined in [4]. It follows that there exists a three term transformation connecting any three of the 56 series. Since $\binom{56}{3} = 27720$, there are thus 27720 three term transformations between Φ-series. Of course, many three term transformations are equivalent with (25), since one can act on it with an arbitrary element of G to obtain a different looking, but equivalent, identity. For instance, applying the element of G corresponding to the transformation

$$a \to \frac{acq}{bdh},\ b \to \frac{aq}{dh},\ c \to c,\ d \to \frac{aq}{bh},\ e \to \frac{ce}{a},\ f \to \frac{cf}{a},\ g \to \frac{cg}{a},\ h \to \frac{aq}{d}, \tag{34}$$

yields a relation between the series Φ^b, Φ^c and $\Phi^{bd/a}$, or stated otherwise between $\tilde{\Phi}_{(1,2)}$, $\tilde{\Phi}_{(1,3)}$ and $\tilde{\Phi}_{(2,4)}$ respectively. It is however impossible to obtain the identity connecting the series Φ^b, Φ^c and $\Phi^{bc/a}$ by acting on (25) with an element of E_7. The question thus arises how many "prototype" relations one needs so that each of the 27720 relations is connected with such a prototype relation through the action with an element of E_7. To put it differently, we would like to determine the number of E_7 orbits when acting on three term identities.

Let c_1, c_2 and c_3 denote three different elements from $\{(i,j),(i,j)^*\}$ with $1 \le i < j \le 8$, and let X denote the set of all sets of the form $\{c_1,c_2,c_3\}$. The E_7 orbits when acting on X can be computed using GAP. In Appendix I we describe an elegant way of achieving this. One finds that there are five orbits, and thus one needs five prototype relations. In Table 1 we summarize the sizes of these orbits, together with a list that enables to determine for any $\{c_1,c_2,c_3\}$ the orbit it belongs to. In Table 1, i, j, k, l, m and n stand for different numbers between 1 and 8. With each orbit the patterns determining the three term identities belonging to that orbit are given. These patterns are given in two columns, where the pattern in the right column is the complement (obtained by replacing each x_i by $1/x_i$) of the pattern in the left column (and vice versa). The numbers given in the rightmost column are for one pattern.

Using this table one sees that (25) belongs to the second orbit since $\{(1,3),(1,8)^*,(2,3)\}$ matches $\{(i,j),(i,k),(j,l)^*\}$ by setting $i=3$, $j=1$, $k=2$ and $l=8$. Likewise, $\{(1,2),(1,3),(2,4)\}$ matches $\{(i,j),(i,k),(j,l)\}$ and hence also belongs the the second orbit (an already known fact since we have explicitly given a transformation between the two identities). On the other hand, the identity connecting the series Φ^b, Φ^c and $\Phi^{bc/a}$ belongs to the first orbit since $\{(1,2),(1,3),(2,3)\}$ matches the pattern $\{(i,j),(i,k),(j,k)\}$; a confirmation of the fact that this identity cannot be reached by transforming (25).

Orbit 1 (size 4032)	$(i,j),(i,k),(i,l)$	$(i,j)^*,(i,k)^*,(i,l)^*$	$8\times\binom{7}{3}=280$
	$(i,j),(i,k),(j,k)$	$(i,j)^*,(i,k)^*,(j,k)^*$	$\binom{8}{3}=56$
	$(i,j),(i,k),(l,m)^*$	$(i,j)^*,(i,k)^*,(l,m)$	$\binom{8}{2}\times 6\times\binom{5}{2}=1680$
Orbit 2 (size 7560)	$(i,j),(i,k),(j,l)$	$(i,j)^*,(i,k)^*,(j,l)^*$	$\binom{8}{2}\times 6\times 5=840$
	$(i,j),(i,k),(j,l)^*$	$(i,j)^*,(i,k)^*,(j,l)$	$8\times 7\times 6\times 5=1680$
	$(i,j),(k,l),(m,n)^*$	$(i,j)^*,(k,l)^*,(m,n)$	$\binom{8}{2}\times\binom{6}{2}\times\binom{4}{2}/2=1260$
Orbit 3 (size 12096)	$(i,j),(j,k),(l,m)$	$(i,j)^*,(j,k)^*,(l,m)^*$	$8\times\binom{7}{2}\times\binom{5}{2}=1680$
	$(i,j),(i,k),(j,k)^*$	$(i,j)^*,(i,k)^*,(j,k)$	$8\times\binom{7}{2}=168$
	$(i,j),(j,k),(j,l)^*$	$(i,j)^*,(j,k)^*,(j,l)$	$8\times 7\times\binom{6}{2}=840$
	$(i,j),(k,l),(i,m)^*$	$(i,j)^*,(k,l)^*,(i,m)$	$8\times 7\times 6\times\binom{5}{2}=3360$
Orbit 4 (size 2520)	$(i,j),(k,l),(m,n)$	$(i,j)^*,(k,l)^*,(m,n)^*$	$\binom{8}{2}\times\binom{6}{2}\times\binom{4}{2}/3!=420$
	$(i,j),(k,l),(i,k)^*$	$(i,j)^*,(k,l)^*,(i,k)$	$\binom{8}{2}\times 6\times 5=840$
Orbit 5 (size 1512)	$(i,j),(i,k),(i,j)^*$	$(i,j)^*,(i,k)^*,(i,j)$	$8\times 7\times 6=336$
	$(i,j),(k,l),(i,j)^*$	$(i,j)^*,(k,l)^*,(i,j)$	$\binom{8}{2}\times\binom{6}{2}=420$

Table 1: Characterization of the five orbits.

In the next section, we will give an identity for each of the five orbits given in Table 1. It is however, also interesting to know how to move around within each of the orbits. This is, one would like to be able to find transformations as (34) without relying on a computer program. To this end, we investigate how a sum of two x-variables, e.g. $x_i + x_j$, transforms when acted upon with an element of E_7, and more specifically with an element of the form $\tilde{r}_\alpha$, where α is a non-zero root of type (29b) of E_7. This is, we investigate how the E_6-level transforms under reflections of the form $\tilde{r}_\alpha$. Using (17) it is easy to show that $x_i + x_j$ remains invariant under $\tilde{r}_\alpha$ whenever $\{a_i, a_j\} = \{0, 1\}$. On the other hand, whenever $a_i = a_j$, then $x'_i + x'_j = -(x_k + x_l)$ where k and l are such that $a_i = a_j = a_k = a_l$.

It is clear that within each pattern, one can simply permute the indices to go from one identity to another; moreover using (32) one can also go from a pattern to its complement pattern on the other side of the table. In Table 2 we indicate how to move around within each orbit using reflections through roots α of type (29b), and, in some cases, the reflection (32). In this table, the primed indices such as (i', j') in the top rows are independent of the indices in the columns. They are included only to indicate the goal pattern; the source pattern is given in the first column. The subscripts used inside the table, e.g. a_i, refer to the indices in the first column. This table has a twofold purpose. First, it *proves* that the patterns given in Table 1 are indeed divided over the orbits as is given there. (The calculation given in Appendix I does not show this.) Secondly, it gives a means of transforming identities without having to rely on a computer program.

As an example, let us show how to find (34). We start with an identity connecting $\tilde{\Phi}_{(1,3)}$, $\tilde{\Phi}_{(2,3)}$ and $\tilde{\Phi}_{(1,8)^*}$, and we would like to find an identity connecting $\tilde{\Phi}_{(1,2)}$, $\tilde{\Phi}_{(1,3)}$ and $\tilde{\Phi}_{(2,4)}$. In a first step, we apply $\tilde{r}_\alpha$ where $a_1 = a_2 = a_4 = a_8$; this gives an identity connecting $\tilde{\Phi}_{(1,3)}$, $\tilde{\Phi}_{(2,3)}$ and $\tilde{\Phi}_{(2,4)}$. A simple observation now shows that one only has to swap x_1 and x_3 to reach the

Orbit 1	$(i',j'),(i',k'),(i',l')$	$(i',j'),(i',k'),(j',k')$	$(i',j'),(i',k'),(l',m')^*$	
$(i,j),(i,k),(i,l)$		$a_i=a_j=a_k=a_l,\ x\to -x$	$a_i=a_l=a_m=a_n$	
$(i,j),(i,k),(j,k)$	$a_i=a_j=a_k=a_l,\ x\to -x$		$a_j=a_k=a_m=a_n$	
$(i,j),(i,k),(l,m)^*$	$a_i=a_l=a_m=a_n$	$a_j=a_k=a_l=a_m$		
Orbit 2	$(i',j'),(i',k'),(j',l')$	$(i',j'),(i',k'),(j',l')^*$	$(i',j'),(k',l'),(m',n')^*$	
$(i,j),(i,k),(j,l)$		$a_j=a_l=a_k=a_m$	$a_i=a_j=a_m=a_n$	
$(i,j),(i,k),(j,l)^*$	$a_j=a_k=a_l=a_m$		$a_i=a_k=a_m=a_n$	
$(i,j),(k,l),(m,n)^*$	$a_i=a_k=a_m=a_n$	$a_i=a_j=a_k=a_p$		
Orbit 3	$(i',j'),(j',k'),(l',m')$	$(i',j'),(i',k'),(j',k')^*$	$(i',j'),(j',k'),(j',l')^*$	$(i',j'),(k',l'),(i',m')^*$
$(i,j),(j,k),(l,m)$		$a_i=a_k=a_l=a_m$	$a_j=a_l=a_m=a_n$	$a_j=a_k=a_m=a_n$
$(i,j),(i,k),(j,k)^*$	$a_j=a_k=a_l=a_m$		$a_i=a_j=a_k=a_m,\ x\to -x$	$a_i=a_k=a_m=a_n,\ x\to -x$
$(i,j),(j,k),(j,l)^*$	$a_j=a_l=a_m=a_n$	$a_i=a_j=a_k=a_l,\ x\to -x$		$a_i=a_j=a_l=a_m$
$(i,j),(k,l),(i,m)^*$	$a_i=a_k=a_m=a_n$	$a_j=a_k=a_l=a_m, x\to -x$	$a_i=a_j=a_k=a_m$	
Orbit 4	$(i',j'),(k',l'),(m',n')$	$(i',j'),(k',l'),(i',k')^*$		
$(i,j),(k,l),(m,n)$		$a_i=a_l=a_m=a_n$		
$(i,j),(k,l),(i,k)^*$	$a_i=a_k=a_m=a_n$			
Orbit 5	$(i',j'),(i',k'),(i',j')^*$	$(i',j'),(k',l'),(i',j')^*$		
$(i,j),(i,k),(i,j)^*$		$a_i=a_j=a_l=a_m$		
$(i,j),(k,l),(i,j)^*$	$a_i=a_j=a_k=a_m$			

Table 2: Transitions between the patterns in the orbits. We use reflections through a root α of type (29b), sometimes followed by (32).

desired identity. This means once has to perform the following substitution

$$x_1' = x_3 + y,\ x_3' = x_1 - y,\ x_i' = x_i + y,\ i \in \{2,4,8\},\ x_i' = x_i - y,\ i \in \{5,6,7\},$$

with $y = \frac{1}{4}(x_1 - x_2 - x_3 - x_4 + x_5 + x_6 + x_7 - x_8)$. Translating this into a transformation for a up to h in multiplicative form, using (20), gives exactly the transformation (34).

We summarize the main results of our analysis:

Theorem 2 *Let $G = E_7$ and $H = E_6$. The series $\tilde{\Phi}(x)$ satisfies $\tilde{\Phi}\big(\tilde{h}(x)\big) = \tilde{\Phi}(x)$ for each $\tilde{h} \in H$. The 2903040 series $\tilde{\Phi}\big(\tilde{g}(x)\big)$, with $\tilde{g} \in G$, can be divided in 56 sets corresponding to the 56 cosets of E_6 in E_7. Of these sets, 28 are represented by $\tilde{\Phi}_{(i_1,i_2)} = \tilde{\Phi}(x_{i_1}; x_{i_2}; x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}, x_{i_8})$, and 28 by $\tilde{\Phi}_{(i_1,i_2)^*} = \tilde{\Phi}(x_{i_1}^{-1}; x_{i_2}^{-1}; x_{i_3}^{-1}, x_{i_4}^{-1}, x_{i_5}^{-1}, x_{i_6}^{-1}, x_{i_7}^{-1}, x_{i_8}^{-1})$, where $1 \le i_1 < i_2 \le 8$; (i_1,i_2) and $(i_1,i_2)^*$ are referred to as the orbit labels of the (set of) Φ-series. For any three distinct orbit labels, there exists a relation (three term transformation) between the corresponding Φ-series. There are five different types of three term transformations, according to the combinations of orbit labels: these five types are summarized in Table 1. The action of an element $\tilde{g} \in G$ on a three term transformation of type K, with $K \in \{1,2,3,4,5\}$, yields another three term transformation of the same type. The element $\tilde{g}$ turning one particular three term transformation of type K into another given one of type K can be determined from Table 2.*

4 Prototypes of the transformations

It is one thing to know the number of orbits and their sizes, it is another to effectively know the coefficients of the transformation formulae. It is the purpose of this section to construct one three term transformation for each of the five different types K, with $K \in \{1,2,3,4,5\}$. These five three term transformations shall be referred to as the prototypes: all 27720 three term transformations can be deduced from these prototypes by Theorem 2. To derive these identities explicitly, we (mainly) use the method that was already employed in [4] to derive formula (25). They show that certain multiples of (9c) (subject to the replacement $(g,h) \to (gq^n, hq^{-n})$) are solutions of a second order difference equation in n, i.e. a difference equation of the form

$$X_{n+1} - \alpha_n X_n + \beta_n X_{n-1} = 0, \tag{35}$$

where α_n and β_n are complicated expressions given in [4, Eq. 3.2]. Since we are dealing with a second order difference equation, any three solutions of this difference equation are connected by a three term identity in which the coefficients are independent of n. In some cases, letting n tend to infinity yields a three term relation connecting three very-well-poised ${}_8W_7$-series, for which the three term transformation is known. Identification of the corresponding coefficients then gives the desired transformation formula.

Since one is considering solutions of the difference equation (35), one has to introduce the parameter n (a nonnegative integer) into the set of parameters a up to h, and this is done by replacing g and h respectively by gq^n and hq^{-n}; this leaves the condition (9b) intact. Using the notation of [4], the following series is a solution of the difference equation at hand:

$$\begin{aligned} X_n^{bgq^n/a} &= \Big(\frac{gh}{a}\Big)^n \frac{1}{(bgq^n/a, cgq^n/a, dgq^n/a, egq^n/a, fgq^n/a, gq^n, bhq^{-n}/a, b, q^{n+1}/h; q)_\infty} \\ &\times \frac{(gq^{2n}/h, bq^{-n}/g, gq^{n+1}/b; q)_\infty}{(bc/a, bd/a, be/a, bf/a, aq^{n+1}/bh, aq^{n+1}/ch, aq^{n+1}/dh, aq^{n+1}/eh, aq^{n+1}/fh)_\infty} \\ &\times \Phi\Big(\frac{q^{2n}g^2}{a}; \frac{bgq^n}{a}; \frac{cgq^n}{a}, \frac{dgq^n}{a}, \frac{egq^n}{a}, \frac{fgq^n}{a}, gq^n, \frac{gh}{a}\Big). \end{aligned} \tag{36}$$

The superscript indicates that bgq^n/a plays a special role: it is the characterizing argument (i.e. the second argument) of the Φ-series. Quite general, each of the 56 solutions X_n^u of (35) constructed in [4] is proportional to the Φ-series Φ^u under the appropriate replacement:

$$X_n^u \sim \Phi^u\big|_{(g,h)\to(gq^n,hq^{-n})},$$

with $u \in U$.

It can be seen that swapping the parameter bgq^n/a in (36) with one of the last six arguments of the Φ-series also gives solutions to the difference equation. More in particular, the series $X_n^{cgq^n/a}$ and $X_n^{dgq^n/a}$ are solutions as well. It is these three series that we assume to be connected in the following way:

$$P\,X_n^{bgq^n/a} + Q\,X_n^{cgq^n/a} + R\,X_n^{dgq^n/a} = 0, \tag{37}$$

where, as stated before, P, Q and R are independent of n. The large n asymptotics of $X_n^{bgq^n/a}$ is fairly easy to determine and one finds that

$$\begin{aligned}&\lim_{n\to\infty} X_n^{bgq^n/a}\\ &\quad= \frac{(gh/a, bq/c, bq/d, bq/e, bq/f, bq/a, b/g, gq/b; q)_\infty}{(b^2q/a, bc/a, bd/a, be/a, bf/a, b, bh/a, aq/bh; q)_\infty}\,{}_8W_7(\frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}; q, \frac{gh}{a}),\end{aligned}$$

provided that $|gh/a| < 1$. The asymptotics for the other two series are obtained by interchanging the role of b and c, respectively of b and d. One sees that the limiting relation of (37) yields a three term relation between three ${}_8W_7$-series. The relation between the three ${}_8W_7$-series at hand is known, and it reads:

$$\begin{aligned}&(\frac{b}{c}, \frac{cq}{b}, \frac{c}{d}, \frac{dq}{c}, \frac{cd}{a}, \frac{q}{b}, \frac{aq}{be}, \frac{aq}{bf}, \frac{bq}{a}, \frac{bq}{c}, \frac{bq}{d}, \frac{bq}{e}, \frac{bq}{f}, \frac{c^2q}{a}, \frac{d^2q}{a}; q)_\infty\,{}_8W_7(\frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}; q, \frac{gh}{a})\\
&+ (\frac{c}{b}, \frac{bq}{c}, \frac{b}{d}, \frac{dq}{b}, \frac{bd}{a}, \frac{q}{c}, \frac{aq}{ce}, \frac{aq}{cf}, \frac{cq}{a}, \frac{cq}{b}, \frac{cq}{d}, \frac{cq}{e}, \frac{cq}{f}, \frac{b^2q}{a}, \frac{d^2q}{a}; q)_\infty\,{}_8W_7(\frac{c^2}{a}; c, \frac{bc}{a}, \frac{cd}{a}, \frac{ce}{a}, \frac{cf}{a}; q, \frac{gh}{a})\\
&- (\frac{b}{c}, \frac{cq}{b}, \frac{c}{b}, \frac{bq}{c}, \frac{bc}{a}, \frac{q}{d}, \frac{aq}{de}, \frac{aq}{df}, \frac{dq}{a}, \frac{dq}{b}, \frac{dq}{c}, \frac{dq}{e}, \frac{dq}{f}, \frac{b^2q}{a}, \frac{c^2q}{a}; q)_\infty\,{}_8W_7(\frac{d^2}{a}; d, \frac{bd}{a}, \frac{cd}{a}, \frac{de}{a}, \frac{df}{a}; q, \frac{gh}{a}) = 0.\end{aligned}$$

This identity is equivalent with identity [10, Theorem 3] with $k = 1$, $l = 2$ and $m = 3$. Comparison of coefficients first yields an identity connecting the three X_n-series; rewriting this identity in term of the Φ-series (and replacing gq^n and hq^{-n} back by g and h) results in the following identity:

$$\begin{aligned}&(\frac{b}{c}, \frac{cq}{b}, \frac{c}{d}, \frac{dq}{c}, \frac{q}{b}, \frac{aq}{be}, \frac{aq}{bf}, \frac{aq}{bh}, \frac{cd}{a}; q)_\infty\,\Phi(\frac{g^2}{a}; \frac{bg}{a}; \frac{cg}{a}, \frac{dg}{a}, \frac{eg}{a}, \frac{fg}{a}, \frac{hg}{a}, g)\\
&\quad+ (\frac{c}{b}, \frac{bq}{c}, \frac{b}{d}, \frac{dq}{b}, \frac{q}{c}, \frac{aq}{ce}, \frac{aq}{cf}, \frac{aq}{ch}, \frac{bd}{a}; q)_\infty\,\Phi(\frac{g^2}{a}; \frac{cg}{a}; \frac{bg}{a}, \frac{dg}{a}, \frac{eg}{a}, \frac{fg}{a}, \frac{hg}{a}, g)\\
&\quad\quad- (\frac{c}{b}, \frac{bq}{c}, \frac{b}{c}, \frac{cq}{b}, \frac{q}{d}, \frac{aq}{de}, \frac{aq}{df}, \frac{aq}{dh}, \frac{bc}{a}; q)_\infty\,\Phi(\frac{g^2}{a}; \frac{dg}{a}; \frac{cg}{a}, \frac{bg}{a}, \frac{eg}{a}, \frac{fg}{a}, \frac{hg}{a}, g) = 0.\end{aligned} \tag{38}$$

This identity thus connects the three series $\tilde\Phi_{(2,7)}$, $\tilde\Phi_{(3,7)}$ and $\tilde\Phi_{(4,7)}$; this is thus an identity corresponding to the first pattern of the first orbit (there are 280 identities of this sort). To transform it to an identity connecting $\tilde\Phi_{(1,2)}$, $\tilde\Phi_{(1,3)}$ and $\tilde\Phi_{(1,4)}$ all one needs to do is swap x_1 and

x_7 in the realization (20). This is, one replaces a by $q^{1/2}x_7^2 = g^2/a$, and b by $q^{1/2}x_2x_7 = bg/a$, etc.; the complete transformation is given by:

$$a \to g^2/a,\ b \to bg/a,\ c \to cg/a,\ d \to dg/a,\ e \to eg/a,\ f \to fg/a,\ g \to g,\ h \to hg/a.$$

Applying this transformation on (38) gives, after some trivial manipulations on the coefficients:

$$\begin{aligned} & bd\,(\frac{cd}{a}, \frac{c}{d}, \frac{dq}{c}, \frac{aq}{be}, \frac{aq}{bf}, \frac{aq}{bg}, \frac{aq}{bh}; q)_\infty\, \Phi(a; b; c, d, e, f, g, h) \\ & + bc\,(\frac{bd}{a}, \frac{d}{b}, \frac{bq}{d}, \frac{aq}{ce}, \frac{aq}{cf}, \frac{aq}{cg}, \frac{aq}{ch}; q)_\infty\, \Phi(a; c; d, b, e, f, g, h) \\ & + cd\,(\frac{bc}{a}, \frac{b}{c}, \frac{cq}{b}, \frac{aq}{de}, \frac{aq}{df}, \frac{aq}{dg}, \frac{aq}{dh}; q)_\infty\, \Phi(a; d; b, c, e, f, g, h) = 0. \end{aligned} \tag{39}$$

Each of the three Φ-series in this identity is symmetric in $\{e, f, g, h\}$, and one sees that the three coefficients obey this symmetry in a way that is clear by simple inspection. Furthermore, one sees that the second and third coefficient are simply cyclic permutations of (b, c, d) of the first coefficient. This is thus a very elegant way to represent this identity.

One can also write this identity completely in terms of the variables x; this is one writes:

$$\alpha(x)\,\tilde\Phi(x) + \beta(x)\,\tilde\Phi\big(\tilde r_{\epsilon_2-\epsilon_3}\tilde r_{\epsilon_2-\epsilon_4}(x)\big) + \gamma(x)\,\tilde\Phi\big(\tilde r_{\epsilon_2-\epsilon_4}\tilde r_{\epsilon_2-\epsilon_3}(x)\big) = 0, \tag{40}$$

with

$$\begin{aligned} \alpha(x) &= x_2x_4\,(q^{1/2}x_3x_4, \frac{x_3}{x_4}, \frac{qx_4}{x_3}, \frac{q^{1/2}}{x_2x_5}, \frac{q^{1/2}}{x_2x_6}, \frac{q^{1/2}}{x_2x_7}, \frac{q^{1/2}}{x_2x_8}; q)_\infty, \\ \beta(x) &= \alpha\big(\tilde r_{\epsilon_2-\epsilon_3}\tilde r_{\epsilon_2-\epsilon_4}(x)\big) = \alpha(x_1, x_3, x_4, x_2, x_5, x_6, x_7, x_8), \\ \gamma(x) &= \alpha\big(\tilde r_{\epsilon_2-\epsilon_4}\tilde r_{\epsilon_2-\epsilon_3}(x)\big) = \alpha(x_1, x_4, x_2, x_3, x_5, x_6, x_7, x_8). \end{aligned}$$

To transform (39) to an identity connecting the series Φ^b, Φ^c and $\Phi^{bc/a}$ or, stated otherwise the series $\tilde\Phi_{(1,2)}$, $\tilde\Phi_{(1,3)}$ and $\tilde\Phi_{(2,3)}$, we can follow the procedure outlined in Table 2. In multiplicative notation this means one has to perform the following substitution in (40):

$$\begin{aligned} & x_1' = y/x_4, \quad && x_2' = y/x_2, \quad && x_3' = y/x_3, \quad && x_4' = y/x_1 \\ & x_5' = 1/x_5y, \quad && x_6' = 1/x_6y, \quad && x_7' = 1/x_7y, \quad && x_8' = 1/x_8y, \end{aligned}$$

with $y = (x_1x_2x_3x_4)^{1/2} = (x_5x_6x_7x_8)^{-1/2}$. The coefficient β for instance is then transformed into:

$$\begin{aligned} & \alpha(y/x_4, y/x_3, y/x_1, y/x_2, 1/x_5y, 1/x_6y, 1/x_7y, 1/x_8y) \\ & \quad = x_1x_4\,(q^{1/2}x_3x_4, \frac{x_2}{x_1}, \frac{qx_1}{x_2}, q^{1/2}x_3x_5, q^{1/2}x_3x_6, q^{1/2}x_3x_7, q^{1/2}x_3x_8; q)_\infty. \end{aligned}$$

Doing this for the three coefficients and the three series, and translating everything back to the parameters a up to h yields, with a minor rewriting of the coefficients:

$$\begin{aligned} & b\,(\frac{a}{b}, \frac{bq}{a}, \frac{cd}{a}, \frac{ce}{a}, \frac{cf}{a}, \frac{cg}{a}, \frac{ch}{a}; q)_\infty\, \Phi(a; b; c, d, e, f, g, h) \\ & - c\,(\frac{a}{c}, \frac{cq}{a}, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}, \frac{bg}{a}, \frac{bh}{a}; q)_\infty\, \Phi(a; c; b, d, e, f, g, h) \\ & + c\,(\frac{b}{c}, \frac{cq}{b}, d, e, f, g, h; q)_\infty\, \Phi(\frac{b^2}{a}; \frac{bc}{a}; b, \frac{bd}{a}, \frac{be}{a}, \frac{bf}{a}, \frac{bg}{a}, \frac{bh}{a}) = 0. \end{aligned} \tag{41}$$

Thus, identities (38), (39) and (41) are all three term transformations of the first type, transferable into each other by the action of an element $g \in G$.

We now give a relation belonging to the second orbit, satisfying the first pattern. Applying the relevant transformation on (25) gives, after some (non-trivial) manipulations the following identity, symmetric in $\{e,f,g,h\}$:

$$\begin{aligned} &\frac{ab}{cd}\,(\frac{cd}{a};q)_\infty \big(S(e,f,g,h,\frac{c}{b},\frac{d}{c}) - S(\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a},\frac{d}{a},\frac{a}{b})\big)\,\Phi(a;b;c,d,e,f,g,h) \\ &\quad - (\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{cg},\frac{aq}{ch},\frac{bd}{a},\frac{a}{c},\frac{cq}{a},\frac{a}{d},\frac{dq}{a};q)_\infty\,\Phi(a;c;b,d,e,f,g,h) \\ &\quad\quad + (e,f,g,h,\frac{aq}{de},\frac{aq}{df},\frac{aq}{dg},\frac{aq}{dh},c,\frac{a}{c},\frac{cq}{a},\frac{b}{c},\frac{cq}{b};q)_\infty\,\Phi(\frac{b^2}{a};\frac{bd}{a};b,\frac{bc}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a}) = 0. \end{aligned} \tag{42}$$

An easier way of finding the same transformation, is eliminating the series Φ^d from two identities, both belonging to the first orbit. One of these connects the series Φ^b, Φ^c and Φ^d, see (39), while the second one connects the series Φ^b, Φ^d and $\Phi^{bd/a}$, see (41) with c and d interchanged.

In order to find a relation belonging to the third orbit, we start with the three series $X_n^{gh/a}$, $X_n^{gq^n}$ and $X_n^{q^{n+1}/h}$ from the article [4]. Using the same limit process one arrives at an identity connecting three ${}_8W_7$-series, namely, with the notation of [10], w_0, w_{0^*} and w_5; identification of coefficients yields a relation between $\Phi^{gh/a}$, Φ^g and $\Phi^{q/h}$. This identity clearly belongs to the third orbit, as it satisfies the second pattern of this orbit. The simplification of this identity is however not trivial, and as an example of how to manipulate the coefficients in these identities we included an appendix showing the various steps required in the simplification process; these can be found in Appendix II. In the end one arrives at:

$$\begin{aligned} &\big(c^2\,S(\frac{d}{c},\frac{e}{c},\frac{f}{c},\frac{g}{c},\frac{h}{c},b) - a\,S(\frac{cd}{a},\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a},\frac{bc^2}{a})\big) \\ &\quad\times(\frac{aq}{bd},\frac{aq}{be},\frac{aq}{bf},\frac{aq}{bg},\frac{aq}{bh},\frac{b^2}{a},\frac{aq}{b^2};q)_\infty\,\Phi(a;b;c,d,e,f,g,h) \\ &+\big(a\,S(\frac{bd}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a},\frac{b^2c}{a}) - b^2\,S(\frac{d}{b},\frac{e}{b},\frac{f}{b},\frac{g}{b},\frac{h}{b},c)\big) \\ &\quad\times(\frac{aq}{cd},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{cg},\frac{aq}{ch},\frac{c^2}{a},\frac{aq}{c^2};q)_\infty\,\Phi(a;c;b,d,e,f,g,h) \\ &-\frac{a^2q}{bc^2}\,(\frac{aq}{de},\frac{aq}{df},\frac{aq}{dg},\frac{aq}{dh},\frac{aq}{ef},\frac{aq}{eg},\frac{aq}{eh},\frac{aq}{fg},\frac{aq}{fh},\frac{aq}{gh},\frac{bc}{a},b,c,\frac{b^2}{a},\frac{aq}{b^2},\frac{c^2}{a},\frac{aq}{c^2},\frac{c}{b},\frac{bq}{c};q)_\infty \\ &\quad\times\Phi(\frac{aq}{b^2};\frac{aq}{bc};\frac{q}{b},\frac{aq}{bd},\frac{aq}{be},\frac{aq}{bf},\frac{aq}{bg},\frac{aq}{bh}) = 0. \end{aligned} \tag{43}$$

Notice the symmetry in $\{d,e,f,g,h\}$ and how swapping b and c changes the sign of the last coefficient, due to the presence of the factor $(c/b, bq/c;q)_\infty$, whilst interchanging the first two coefficients.

As a remark, we note that we could also have derived this identity be eliminating Φ^d from the two identities connecting Φ^b, Φ^c, Φ^d and Φ^b, Φ^d, $\Phi^{aq/bd}$ respectively. The first one belongs to the first orbit, the second one to the second orbit. There would still remain some serious simplification to do, as this yields an identity where the coefficient of Φ^b is a sum of three products. The point is, however, that we could have found identity (43) by a double elimination procedure starting from the first orbit.

For the fourth orbit, one can once more use a limiting process, this time starting from the three series X_n^f, $X_n^{gh/a}$ and $X_n^{aq^{n+1}/fh}$. Again, after some considerable simplification resembling the calculations given in Appendix II, one arrives at the following identity, symmetric in $\{e,f,g,h\}$:

$$\begin{aligned}
&\frac{1}{a}\big(b\,S(\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a},d,\frac{c^2d}{a}) - cd\,S(\frac{cde}{a},\frac{cdf}{a},\frac{cdg}{a},\frac{cdh}{a},\frac{bc}{a},\frac{b}{c})\big)\\
&\qquad\times(\frac{a}{b},\frac{bq}{a},a,\frac{q}{a},\frac{q}{b},\frac{de}{a},\frac{df}{a},\frac{dg}{a},\frac{dh}{a},\frac{q}{e},\frac{q}{f},\frac{q}{g},\frac{q}{h};q)_\infty\,\Phi(a;b;c,d,e,f,g,h)\\
&\quad+\big(S(e,f,g,h,d,bc) - a\,S(\frac{e}{a},\frac{f}{a},\frac{g}{a},\frac{h}{a},\frac{bc}{a},\frac{d}{a})\big)\\
&\qquad\times(\frac{cd}{b},\frac{bq}{cd},\frac{b}{a},\frac{aq}{b},\frac{aq}{cd},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{cg},\frac{aq}{ch};q)_\infty\,\Phi(\frac{c^2}{a};\frac{cd}{a};c,\frac{bc}{a},\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a})\\
&\quad+\frac{q}{cd}\big(S(\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a},\frac{d}{a},\frac{a}{c}) - S(e,f,g,h,\frac{b}{c},\frac{d}{b})\big)\\
&\qquad\times(a,\frac{q}{a},\frac{cd}{b},\frac{bq}{cd},c,d,\frac{bc}{a},\frac{aq}{ef},\frac{aq}{eg},\frac{aq}{eh},\frac{aq}{fg},\frac{aq}{fh},\frac{aq}{gh};q)_\infty\,\Phi(\frac{q}{a};\frac{q}{c};\frac{q}{b},\frac{q}{d},\frac{q}{e},\frac{q}{f},\frac{q}{g},\frac{q}{h}) = 0.
\end{aligned}\tag{44}$$

In this case, one could obtain this identity be an elimination procedure using two identities from the second orbit.

An element of the fifth orbit, finally, cannot be reached using the limit process which was used for the previous four orbits. In this case, we have to rely on an elimination procedure. This is, we eliminate Φ^c from the identities (41) and (43); this will result in an identity connecting the series Φ^b, $\Phi^{bc/a}$ and $\Phi^{aq/bc}$. From the structure of the identities (41) and (43) it is clear that at first the coefficient of Φ^b will involve a sum of four S-products, that the coefficient of $\Phi^{bc/a}$ will involve a sum of two S-products, while the coefficient of $\Phi^{aq/bc}$ will simply be an infinite product. Since we started with two identities symmetric in $\{d,e,f,g,h\}$ the resulting identity too will have this desirable property. One obvious simplification can be made using (46), after multiplying all coefficients by $S(a/bc)$:

$$S(\frac{c^2}{a},\frac{a}{b},\frac{b^2c}{a},\frac{a}{bc}) = \frac{c}{b}S(\frac{bc^2}{a},\frac{a}{c},\frac{b^2}{a},\frac{a}{bc}) - \frac{c^2}{a}S(\frac{a^2}{b^2c^2},\frac{b}{c},b,c,).$$

This allows to reduce the sum of four S-products to a sum of three S-products. The resulting identity is:

$$\begin{aligned}
&\big(c^3\,S(\frac{bd}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a},\frac{d}{c},\frac{e}{c},\frac{f}{c},\frac{g}{c},\frac{h}{c},\frac{b^2}{a},\frac{a}{c},\frac{bc}{a},b) - b^3\,S(\frac{cd}{a},\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a},\frac{d}{b},\frac{e}{b},\frac{f}{b},\frac{g}{b},\frac{h}{b},\frac{c^2}{a},\frac{a}{b},\frac{bc}{a},c)\\
&\quad - ac\,S(\frac{bd}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a},\frac{cd}{a},\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a},\frac{b}{c},b,c,\frac{b^2c^2}{a^2})\big)\,\Phi(a;b;c,d,e,f,g,h)\\
&\quad+c\,\big(a\,S(\frac{bd}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a},\frac{b^2c}{a}) - b^2\,S(\frac{d}{b},\frac{e}{b},\frac{f}{b},\frac{g}{b},\frac{h}{b},c)\big)\\
&\qquad\times(\frac{aq}{cd},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{cg},\frac{aq}{ch},d,e,f,g,h,\frac{c^2}{a},\frac{aq}{c^2},\frac{b}{c},\frac{cq}{b},\frac{bc}{a},\frac{aq}{bc};q)_\infty\,\Phi(\frac{b^2}{a};\frac{bc}{a};b,\frac{bd}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a})\\
&\quad-\frac{a^2q}{bc}(\frac{aq}{de},\frac{aq}{df},\frac{aq}{dg},\frac{aq}{dh},\frac{aq}{ef},\frac{aq}{eg},\frac{aq}{eh},\frac{aq}{fg},\frac{aq}{fh},\frac{aq}{gh},\frac{bd}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a};q)_\infty\\
&\qquad\times(\frac{b^2}{a},\frac{aq}{b^2},\frac{c^2}{a},\frac{aq}{c^2},\frac{c}{b},\frac{bq}{c},\frac{a}{c},\frac{cq}{a},\frac{bc}{a},\frac{aq}{bc},b,c,\frac{bc}{a};q)_\infty\,\Phi(\frac{aq}{b^2};\frac{aq}{bc};\frac{q}{b},\frac{aq}{bd},\frac{aq}{be},\frac{aq}{bf},\frac{aq}{bg},\frac{aq}{bh}) = 0.
\end{aligned}\tag{45}$$

Theorem 3 *Following the notation and description of Theorem 2, all 27720 three term transformations between the 56 different* Φ*-series are equivalent to one of the five prototypes (39), (42), (43), (44) or (45).*

5 Conclusion

In this article, we have studied Bailey's transformations for both terminating and non-terminating series. We have shown, by given an explicit description of the root system E_6, that the invariance group for the two term transformations for both the terminating balanced very-well-poised ${}_{10}\phi_9$ and the non-terminating Φ-series is the Weyl group of type E_6, a fact maybe known to some in the field but, to our knowledge, never written down explicitly.

Secondly, we have studied the group structure underlying the three term transformations for Φ-series, which we have unravelled. The group involved is the Weyl group of type E_7, again shown using an explicit description of the appropriate root system. An identity involving three Φ-series can be seen as connecting three sets (each of size 51840) of these series; they can be seen as E_6 cosets in E_7. Between any three of these 56 cosets there exists a three term identity, and we have shown that one needs five prototypes in order to describe each of the $\binom{56}{3} = 27720$ relationships. We have given an example of each of these prototypes, and we have given a table indicating how to obtain all other three term transformations.

The explicit identification of the symmetry groups for the two term transformations and for the three term transformations is of importance in itself. A significant consequence is the unravelling of the five different types of three term transformations. Our study of the symmetry groups and the three term transformations should also be of interest in the construction of hypergeometric solutions to the q-Painlevé equations, where similar affine Weyl group symmetries have been encountered [8, 7].

I Calculation of the orbits using GAP

We know that there are 56 cosets of E_6 in E_7, and our goal is to find out how many different orbits there are when E_7 acts on sets consisting of three cosets each; the number of such sets is $\binom{56}{3} = 27720$. To this end, we will construct a permutation group $P_3 \subseteq S_{27720}$, and we will use GAP to determine the different orbits when P_3 acts pointwise on the set $\{1, 2, \ldots, 27720\}$; the underlying idea being that each set $\{c_i, c_j, c_k\}$ consisting of three cosets is mapped onto an element of $\{1, 2, \ldots, 27720\}$. As an intermediate step however we first construct a permutation group $P_2 \subseteq S_{56}$ that acts naturally on each of the 56 cosets, i.e. each cosets is assigned a number between 1 and 56 and the group P_2 then sends i to j if and only if the i-th coset is mapped onto the j-th coset using the natural action of E_7 on cosets. The GAP-code implementing this strategy is the following:

```
E7 := Group(m0,m1,m2,m3,m4,m5,m6);
E6 := Group(m1,m2,m3,m4,m5,m6);
rc := RightCosets(E7,E6);;
P2 := Action(E7, rc, OnRight);;
comb := Combinations([1..56],3);;
P3 := Action(P2,comb,OnSets);;
orb := Orbits(P3,[1..27720],OnPoints);;
```

Herein, m_0 up to m_6 are eight-dimensional matrices corresponding to the reflections $\tilde{r}_0$ up to $\tilde{r}_6$. Examining the object `orb` yields that there are indeed five orbits, and that their sizes are as given in Table 1. This is shown here:

```
gap> Size(orb);
5
gap> [Size(orb[1]), Size(orb[2]), Size(orb[3]), Size(orb[4]), Size(orb[5])];
[ 4032, 7560, 12096, 1512, 2520 ]
```

One sees that the order of the orbits given here is different from the order in Table 1; we have chosen to sort the orbits in order of increasing complexity of their identities.

II Calculation of an element of the third orbit

During the course of the manipulations, we will need some product identities; these are known and are already given in [3, Exercises 5.21 and 5.22], albeit in a different form, less suitable for direct application of the identities. The identities are respectively:

$$S(a^2,b^2,c^2,d^2)=S(\frac{bcd}{a},\frac{acd}{b},\frac{abd}{c},\frac{abc}{d})-a^2\,S(\frac{cd}{ab},\frac{bd}{ac},\frac{bc}{ad},abcd). \tag{46}$$

and

$$\begin{aligned}S(a^3,b^3,c^3,d^3,e^3,f^3,\frac{f^3}{e^3})&=S(\frac{bcdf^2}{a^2e},\frac{acdf^2}{b^2e},\frac{abdf^2}{c^2e},\frac{abcf^2}{d^2e},\frac{abcd}{ef},e^3,\frac{e^2abcd}{f})\\&-\frac{f^3}{e^3}\,S(\frac{bcde^2}{a^2f},\frac{acde^2}{b^2f},\frac{abde^2}{c^2f},\frac{abce^2}{d^2f},\frac{abcd}{ef},f^3,\frac{f^2abcd}{e})\\&+\frac{abcd}{ef}\,S(\frac{a^2ef}{bcd},\frac{b^2ef}{acd},\frac{c^2ef}{abd},\frac{d^2ef}{abc},\frac{f^3}{e^3},\frac{abcde^2}{f},\frac{abcdf^2}{e}).\end{aligned} \tag{47}$$

We will also use the following trivial identities very frequently:

$$S(a)=S(q/a),\quad \text{and}\quad S(a)=-a\,S(aq).$$

To find a three term identity belonging to the third orbit, we start with the three series $X_n^{gh/a}$, $X_n^{q^ng}$ and $X_n^{q^{n+1}/h}$. Using the described limit process one arrives at an identity connecting w_0, w_{0^*} and w_5 [10]; identification of coefficients yields:

$$\begin{aligned}&\big(S(b,d,e,\frac{cf}{a},\frac{cgh}{aq},\frac{fgh}{a})-S(\frac{bd}{a},\frac{be}{a},\frac{de}{a},\frac{a}{f},\frac{c}{a},\frac{gh}{a})\big)\\&\quad\times(\frac{aq}{bh},\frac{aq}{ch},\frac{aq}{dh},\frac{aq}{eh},\frac{aq}{fh},f,\frac{f}{a},\frac{aq}{f};q)_\infty\,\Phi(\frac{g^2}{a};\frac{gh}{a};\frac{bg}{a},\frac{cg}{a},\frac{dg}{a},\frac{eg}{a},\frac{fg}{a},g)\\&+\big(S(\frac{bde}{a},\frac{bf}{a},\frac{df}{a},\frac{ef}{a},\frac{fgh}{a},\frac{cf}{a},\frac{a}{h},h)-S(\frac{fh}{a},\frac{f}{h},b,d,e,\frac{cf}{a},\frac{cgh}{aq},\frac{fgh}{a})\\&\quad+S(\frac{fh}{a},\frac{f}{h},\frac{bd}{a},\frac{be}{a},\frac{de}{a},\frac{a}{f},\frac{c}{a},\frac{gh}{a})\big)(\frac{q}{b},\frac{q}{c},\frac{q}{d},\frac{q}{e};q)_\infty\,\Phi(a;g;b,c,d,e,f,h)\\&+\frac{aq}{fgh}(f,g,h,\frac{gh}{a},\frac{aq}{bc},\frac{aq}{bd},\frac{aq}{be},\frac{aq}{bf},\frac{aq}{cd},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{de},\frac{aq}{df},\frac{aq}{ef},\frac{f}{a},\frac{aq}{f},\frac{bde}{a},\frac{aq}{bde},\frac{a}{h},\frac{hq}{a};q)_\infty\\&\quad\times\Phi(\frac{q}{a};\frac{q}{h};\frac{q}{b},\frac{q}{c},\frac{q}{d},\frac{q}{e},\frac{q}{f},\frac{q}{g})=0.\end{aligned} \tag{48}$$

We will now rewrite identity (48) as a relation between Φ^b, Φ^c and $\Phi^{aq/bc}$, where we keep a twofold goal in mind; firstly we would like to reduce the sum of three S-products in the coefficient of Φ^c to a sum (or difference) of two S-products, and secondly, we would like that the identity shows symmetry in $\{d,e,f,g,h\}$ in a trivial way. After applying the transformation:

$$a \to \frac{c^2}{a},\ b \to \frac{cd}{a},\ c \to \frac{ce}{a},\ d \to \frac{cf}{a},\ e \to \frac{cg}{a},\ f \to \frac{ch}{a},\ g \to c,\ h \to \frac{bc}{a},$$

we will roughly perform the following steps:

- Use a double application of (46) on both the coefficients of Φ^b and Φ^c.
- Apply (47) on the resulting coefficient of Φ^c, which will then allow to use (46) once again. This will realize the requested simplification.
- Apply (47) once more on the (now simplified) coefficient of Φ^c which will establish the required symmetry in $\{d,e,f,g,h\}$.

The coefficient of Φ^b is

$$\Big(S(\frac{cd}{a},\frac{cf}{a},\frac{cg}{a},\frac{bce}{aq},\frac{bch}{a},\frac{eh}{a}) - S(\frac{fg}{a},\frac{df}{a},\frac{dg}{a},\frac{c}{h},b,\frac{e}{c})\Big)(\frac{aq}{bd},\frac{aq}{be},\frac{aq}{bf},\frac{aq}{bg},\frac{aq}{bh},\frac{ch}{a},\frac{h}{c},\frac{cq}{h};q)_\infty, \quad (49)$$

while those of Φ^c and $\Phi^{aq/bc}$ are:

$$\begin{aligned}\Big(\big(S(\frac{fg}{a},\frac{df}{a},\frac{dg}{a},\frac{c}{h},b,\frac{e}{c}) - S(\frac{cd}{a},\frac{cf}{a},\frac{cg}{a},\frac{bce}{aq},\frac{bch}{a},\frac{eh}{a})\big)S(\frac{h}{b},\frac{bh}{a})\\ + S(\frac{beh}{aq},\frac{dh}{a},\frac{eh}{a},\frac{fh}{a},\frac{gh}{a},\frac{bch}{a},\frac{c}{b},\frac{bc}{a})\Big)(\frac{aq}{cd},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{cg};q)_\infty,\end{aligned} \quad (50)$$

respectively

$$\frac{aq}{bch}(\frac{ch}{a},\frac{h}{c},\frac{cq}{h},\frac{beh}{aq},\frac{aq^2}{beh},\frac{bc}{a},b,c,\frac{aq}{de},\frac{aq}{df},\frac{aq}{dg},\frac{aq}{dh},\frac{aq}{ef},\frac{aq}{eg},\frac{aq}{eh},\frac{aq}{fg},\frac{aq}{fh},\frac{aq}{gh},\frac{c}{b},\frac{bq}{c};q)_\infty \quad (51)$$

One sees that coefficient (51) is symmetric in $\{d,e,f,g,h\}$ but for the factors $(ch/a;q)_\infty$, $S(h/c)$ and $S(beh/aq)$. Furthermore, one notices that the first two of these factors also appear in (49). It is obvious that (50) lacks the product $(aq/ch;q)_\infty$ to make its common factor symmetric. Our first goal is now to rewrite the difference in (49) so that it contains the factor $S(beh/aq)$. In a later stage we would like to extract $S(ch/a,h/c)$ from (50) as well.

Multiply all coefficients by $S(a/c^2)$ and apply the following double application of (46) twice, i.e. on both the coefficient of Φ^b and Φ^c:

$$\begin{aligned}&S(\frac{fg}{a},\frac{df}{a},\frac{dg}{a},\frac{a}{c^2},\frac{c}{h},b,\frac{e}{c}) = -\frac{c}{h}S(\frac{d}{c},\frac{e}{c},\frac{f}{c},\frac{g}{c},\frac{h}{c},b,\frac{beh}{aq}) - \frac{a}{c^2}S(\frac{cd}{a},\frac{cf}{a},\frac{cg}{a},\frac{dfg}{ac},\frac{c}{h},b,\frac{e}{c})\\ &= -\frac{c}{h}S(\frac{d}{c},\frac{e}{c},\frac{f}{c},\frac{g}{c},\frac{h}{c},b,\frac{beh}{aq}) + S(\frac{cd}{a},\frac{cf}{a},\frac{cg}{a},\frac{a}{c^2},\frac{eh}{a},\frac{bch}{a},\frac{bce}{aq}) + \frac{a}{ch}S(\frac{cd}{a},\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a},\frac{bc^2}{a},\frac{beh}{aq}).\end{aligned}$$

First, we applied (46) on $S(fg/a,df/a,dg/a,a/c^2)$ and then on $S(dfg/ac,c/h,b,e/c)$. After this a factor $S(beh/aq)$ can be cancelled out from each of the three coefficients, and the coefficient of Φ^c is:

$$\begin{aligned}\Big(\big(\frac{a}{ch}S(\frac{cd}{a},\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a},\frac{bc^2}{a}) - \frac{c}{h}S(\frac{d}{c},\frac{e}{c},\frac{f}{c},\frac{g}{c},\frac{h}{c},b)\big)S(\frac{h}{b},\frac{bh}{a})\\ + S(\frac{dh}{a},\frac{eh}{a},\frac{fh}{a},\frac{gh}{a},\frac{bch}{a},\frac{c}{b},\frac{bc}{a},\frac{a}{c^2})\Big)(\frac{aq}{cd},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{cg};q)_\infty.\end{aligned}$$

Multiply all coefficients by $S(bc/h)$[1] and apply (47) on the coefficient of Φ^c:

$$S(\frac{d}{cq},\frac{e}{c},\frac{f}{c},\frac{g}{c},\frac{h}{c},b,\frac{bc}{h}) = -\frac{bcd}{aq}S(\frac{bcd}{a},\frac{bce}{a},\frac{bcf}{a},\frac{bcg}{a},\frac{a}{ch},\frac{h}{c},\frac{a}{c^2}) - \frac{d}{cq}S(\frac{dh}{a},\frac{eh}{a},\frac{fh}{a},\frac{gh}{a},b,\frac{a}{c^2},\frac{bc^2}{a}) - \frac{ad}{c^3q}S(\frac{cd}{a},\frac{ce}{a},\frac{cf}{a},\frac{cg}{a},\frac{ch}{a},\frac{bc}{h},\frac{bc^2}{a}).$$

At this point the coefficient of Φ^c is:

$$(\frac{aq}{cd},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{cg},\frac{a}{c^2},\frac{c^2q}{a};q)_\infty \times \Big(-\frac{bc^3}{ah}S(\frac{bcd}{a},\frac{bce}{a},\frac{bcf}{a},\frac{bcg}{a},\frac{a}{ch},\frac{h}{c},\frac{h}{b},\frac{bh}{a}) + S(\frac{dh}{a},\frac{eh}{a},\frac{fh}{a},\frac{gh}{a})\big(S(\frac{bch}{a},\frac{c}{b},\frac{bc}{a},\frac{bc}{h}) - \frac{c}{h}S(b,\frac{bc^2}{a},\frac{h}{b},\frac{bh}{a})\big)\Big).$$

It is now clear that one has to apply (46) again:

$$S(\frac{bch}{a},\frac{c}{b},\frac{bc}{a},\frac{bc}{h}) = \frac{c^2}{a}\,S(\frac{a}{ch},c,\frac{b^2c}{a},\frac{h}{c}) + \frac{c}{h}\,S(b,\frac{bc^2}{a},\frac{h}{b},\frac{bh}{a})$$

in order to reduce the sum of three S-products. At this point $S(a/ch,h/c)$ can be factored out of the difference; all three coefficients now have $(h/c,qc/h,ch/a;q)_\infty$ as a factor, and the coefficient at hand becomes:

$$(-\frac{bc^3}{ah}S(\frac{bcd}{a},\frac{bce}{a},\frac{bcf}{a},\frac{bcg}{a},\frac{h}{b},\frac{bh}{a}) + \frac{c^2}{a}S(\frac{dh}{a},\frac{eh}{a},\frac{fh}{a},\frac{gh}{a},c,\frac{b^2c}{a})) \times (\frac{aq}{cd},\frac{aq}{ce},\frac{aq}{cf},\frac{aq}{cg},\frac{aq}{ch},\frac{a}{c^2},\frac{c^2q}{a};q)_\infty.$$

Finally, we apply (47) once again, after multiplying each coefficient by $S(b^2/a)$, to make the symmetry in $\{d,e,f,g,h\}$ clear:

$$S(\frac{dhq}{a},\frac{eh}{a},\frac{fh}{a},\frac{gh}{a},c,\frac{b^2c}{a},\frac{b^2}{a}) = \frac{b^2}{dh}\,S(\frac{d}{b},\frac{e}{b},\frac{f}{b},\frac{g}{b},\frac{h}{b},c,\frac{bc}{h}) - \frac{a}{dh}\,S(\frac{bd}{a},\frac{be}{a},\frac{bf}{a},\frac{bg}{a},\frac{bh}{a},\frac{b^2c}{a},\frac{bc}{h}) - \frac{abc}{dh^2}\,S(\frac{bcd}{a},\frac{bce}{a},\frac{bcf}{a},\frac{bcg}{a},\frac{h}{b},\frac{bh}{a},\frac{b^2}{a}).$$

After some final trivial simplifications the identity (48) finally becomes the already given identity (43).

References

[1] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines.* Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[2] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2005. (`http://www.gap-system.org`).

[1] It may seem strange to introduce a non-symmetric factor, but over the course of later manipulations, it will disappear again.

[3] G. Gasper and M. Rahman. *Basic Hypergeometric Series*, volume 35 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2004.

[4] D.P. Gupta and D.R. Masson. Contiguous relations, continued fractions and orthogonality. *Trans. Amer. Math. Soc.*, **350**(2):769–808, 1998.

[5] J.E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[6] N. Jacobson. *Lie algebras.* Dover Publications Inc., New York, 1979. Republication of the 1962 original.

[7] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, and Y. Yamada. Hypergeometric solutions to the q-Painlevé equations. *Int. Math. Res. Not.*, **2004**(47):2497–2521, 2004.

[8] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, and Y. Yamada. Construction of hypergeometric solutions to the q-Painlevé equations. *Int. Math. Res. Not.*, **2005**(24):1441–1463, 2005.

[9] C. Krattenthaler. HYP and HYPQ: Mathematica packages for the manipulation of binomial sums and hypergeometric series, respectively q-binomial sums and basic hypergeometric series. *J. Symbolic Comput.*, **20**(5–6):737–744, 1995.

[10] S. Lievens and J. Van der Jeugt. Invariance groups of three term transformations for basic hypergeometric series. *J. Comput. Appl. Math.*, 2006, in press

[11] J. Van der Jeugt and K. Srinivasa Rao. Invariance groups of transformations of basic hypergeometric series. *J. Math. Phys.*, **40**(12):6692–6700, 1999.