The semiconfined harmonic oscillator with a position-dependent effective mass: exact solution, dynamical symmetry algebra and quasiprobability distribution functions

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Abstract We present some details of the new model of the quantum harmonic oscillator that is semiconfined due to a specific change of its mass by position. We succeeded in solving exactly the Schrödinger equation corresponding to this model. Moreover, we found its dynamical symmetry algebra and could compute analytically its Kirkwood, Husimi and Wigner joint quasiprobability functions.

1 Introduction

Many prominent physicists have expressed their opinions on the high complexity and mysteriousness of quantum mechanics. However, despite the challenges and many confusing solutions, quantum mechanics is known as an incredibly successful and accurate discipline for describing the behavior of physical systems at the submicron scales [1]. We are going to discuss in detail one of such confusing quantum mechanics solutions – it is the quantum harmonic oscillator that exhibits the semiconfinement effect.

The quantum harmonic oscillator in the framework of the non-relativistic approximation is one of the most attractive solutions of quantum mechanics. This problem being bounded at $\pm\infty$ configurational space, has elegant analytical solutions under the canonical commutation relations approach. Its wavefunctions of the stationary states being eigenfunctions of the correspond-

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ing Schrödinger equation are exactly expressed through the Hermite polynomials, whereas, the energy spectrum being fully discrete and it is eigenvalue of the same Schrödinger equation consists of the infinite number of equidistant energy levels with non-zero ground state. We decided slightly to modify this quantum mechanics problem and consider the quantum harmonic oscillator potential, which is bounded within the position values $-a \leq 0 < +\infty$, where we decided to call a taking finite positive values as a confinement parameter due to that the model now contains an additional infinitely high wall at the value of $x = -a$. We achieved such a semiconfinement effect in the non-relativistic quantum harmonic oscillator potential through a specific change of its mass by position. When we managed to solve the corresponding Schrödinger equation exactly, we observed very surprising and confusing analytical expressions – wavefunctions of the stationary states of such a semiconfined oscillator have been described through the generalized Laguerre polynomials, whereas, its energy spectrum completely overlaps with the equidistant energy spectrum of the so-called Hermite oscillator.

The paper is structured as follows – in the next section a brief description of the exactly solvable semiconfined oscillator model with the specific position-dependent mass is presented. Moreover, we extended this exact solution to the case, when the oscillator model suddenly becomes exposed to the action of an external homogeneous field. We also found that the dynamical symmetry algebra for this oscillator model is $\mathfrak{su}(1,1)$ Heisenberg-Lie algebra. Further, we constructed its phase space and could compute analytically its Kirkwood, Husimi, and Wigner functions of the joint quasiprobability distribution.

2 The semiconfined harmonic oscillator with a position-dependent effective mass – exact solution

Exact solution to the quantum harmonic oscillator in the framework of the non-relativistic approximation with the potential $V^{ho}(x) = \frac{m_0 \omega^2 x^2}{2}$ bounded at $x \to \pm\infty$ configurational space is well known within the canonical commutation relations [2]. The following second-order differential equation

$$\frac{\hbar^2}{2m_0} \frac{d^2\psi}{dx^2} + \left(E - \frac{m_0 \omega^2 x^2}{2} \right) \psi = 0 \tag{1}$$

being solved exactly leads to the discrete equidistant energy spectrum

$$E \equiv E_n^{ho} = \hbar\omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots \tag{2}$$

and the wavefunctions of the stationary states

$$\psi \equiv \psi_n^{ho}(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\lambda_0^2}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\lambda_0^2 x^2} H_n(\lambda_0 x), \quad \lambda_0 = \sqrt{\frac{m_0\omega}{\hbar}}, \tag{3}$$

analytically expressed through the $H_n(z)$ Hermite polynomials.

Even, an external homogeneous field $V^{ext}(x) = gx$ applied to this non-relativistic quantum system does not violate the property of its exact solubility. Then, the following extended version of eq.(1)

$$\frac{\hbar^2}{2m_0}\frac{d^2\psi}{dx^2} + \left(E - \frac{m_0\omega^2 x^2}{2} - gx\right)\psi = 0 \tag{4}$$

still being exactly solved leads to the discrete energy spectrum

$$E \equiv E_n^g = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{g^2}{2m_0\omega^2}, \quad n = 0, 1, 2, \dots \tag{5}$$

and the wavefunctions of the stationary states

$$\psi \equiv \psi_n^g(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\lambda_0^2}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\lambda_0^2\left(x + \frac{g}{m_0\omega^2}\right)^2} H_n\left(\lambda_0\left(x + \frac{g}{m_0\omega^2}\right)\right). \tag{6}$$

Now, one can apply a semiconfined effect to the non-relativistic quantum harmonic oscillator in the canonical approach assuming that its wavefunctions of the stationary states will tend to zero for position values $-\infty < x \leq -a$ and when $x \to +\infty$, where a is a positive constant ($a > 0$). It is shown that such a behavior of the quantum system can be achieved by replacing its constant mass with the mass varying with position:

$$m_0 \to M(x) = \begin{cases} \dfrac{am_0}{a+x}, & \text{for } -a < x < +\infty \\ +\infty, & \text{for } x \leq -a \end{cases}. \tag{7}$$

One needs to preserve the Hermiticity property of the Hamiltonian in the case of the introduction of the position-dependent mass $M(x)$. The simplest case of such a preservation is a replacement of the free Hamiltonian with the BenDaniel-Duke kinetic energy operator [3]:

$$-\frac{\hbar^2}{2m_0}\frac{d^2}{dx^2} \to -\frac{\hbar^2}{2}\frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx}. \tag{8}$$

Substitutions (7) and (8) generalize the Schrödinger equation (1) as follows:

$$\frac{\hbar^2}{2m_0}\left[(a+x)\frac{d^2}{dx^2} + \frac{d}{dx}\right]\psi + a\left(E - \frac{a}{a+x}\frac{m_0\omega_0^2 x^2}{2}\right)\psi = 0. \tag{9}$$

It is shown that the equation is still exactly solvable and the wavefunctions of the stationary states being its eigenfunctions are expressed by the generalized Laguerre polynomials as follows [4]:

$$\psi \equiv \psi_n^{scho}(x) = C_n \cdot \left(1+\frac{x}{a}\right)^{\lambda_0^2 a^2} e^{-\lambda_0^2 a(x+a)} L_n^{\left(2\lambda_0^2 a^2\right)}\left(2\lambda_0^2 a\,(x+a)\right), \quad (10)$$

where C_n is the following orthonormalization parameter:

$$C_n = (-1)^n \left(2\lambda_0^2 a^2\right)^{\lambda_0^2 a^2+\frac{1}{2}} \sqrt{\frac{n!}{a\Gamma\left(n+2\lambda_0^2 a^2+1\right)}}. \quad (11)$$

Surprising result here is the behavior of the energy spectrum of the semiconfined harmonic osicllator model. One observes that its energy spectrum completely overlaps with the energy spectrum (2), i.e.

$$E \equiv E_n^{scho} = E_n^{ho} = \hbar\omega\left(n+\frac{1}{2}\right), \quad n = 0,1,2,\ldots. \quad (12)$$

The semiconfined harmonic oscillator described by the Schrödinger equation (9) preserves its exact solvability even in the case of the suddenly applied external homogeneous field $V^{ext}(x) = gx$. The corresponding Schrödinger equation slightly generalizes eq.(9) as follows:

$$\frac{\hbar^2}{2m_0}\left[(a+x)\frac{d^2}{dx^2}+\frac{d}{dx}\right]\psi + a\left(E-\frac{a}{a+x}\frac{m_0\omega_0^2 x^2}{2}-gx\right)\psi = 0. \quad (13)$$

Its exact solution leads to the discrete energy spectrum [5]

$$E \equiv E_n^{scg} = \hbar\omega\sqrt{1+\frac{2g}{m_0\omega^2 a}}\left(n+\frac{1}{2}+\lambda_0^2 a^2\right) - m_0\omega^2 a^2 - ag, \quad n = 0,1,2,\ldots \quad (14)$$

and the wavefunctions of the stationary states

$$\begin{aligned}\psi \equiv \psi_n^{scg}(x) = {} & C_n^g \left(1+\tfrac{x}{a}\right)^{\lambda_0^2 a^2} e^{-\lambda_0^2 a\sqrt{1+\frac{2g}{m_0\omega^2 a}}(x+a)} \\ & \times L_n^{\left(2\lambda_0^2 a^2\right)}\left(2\lambda_0^2 a\sqrt{1+\tfrac{2g}{m_0\omega^2 a}}\,(x+a)\right).\end{aligned} \quad (15)$$

Here, the normalization constant is determined as follows:

$$C_n^g = (-1)^n \left(2\lambda_0^2 a^2\sqrt{1+\frac{2g}{m_0\omega^2 a}}\right)^{\lambda_0^2 a^2+\frac{1}{2}} \sqrt{\frac{n!}{a\Gamma\left(n+2\lambda_0^2 a^2+1\right)}}. \quad (16)$$

Both wavefunctions of the stationary states of the semiconfined harmonic oscillator model (10)&(15) easily recover their non-relativistic analogues (3)&(6) under the following known limit relation between the generalized Laguerre and Hermite polynomials [6]:

$$\lim_{\alpha\to\infty}\left(\frac{2}{\alpha}\right)^{\frac{1}{2}n} L_n^{(\alpha)}\left((2\alpha)^{\frac{1}{2}}x+\alpha\right)=\frac{(-1)^n}{n!}H_n(x). \tag{17}$$

Direct application of limit relation (17) requires first the following substitutions $x\to\lambda_0 x$ and $\alpha\to 2\lambda_0^2 a^2$. Then, $\alpha\to\infty$ under these substitutions is the same as $a\to\infty$. Thus, one obtains that

$$\lim_{a\to\infty} L_n^{\left(2\lambda_0^2 a^2\right)}\left(2\lambda_0^2 a\,(x+a)\right)=\frac{(-1)^n}{n!}H_n(\lambda_0 x).$$

One needs to note here that the ratio $\frac{x}{a}$ exists in the definition of the position-dependent mass (7). In fact, this ratio plays a vital role only in the recovery of the homogeneous nature m_0 of the position-dependent mass at $x=0$ equilibrium point and under the limit $a\to\infty$. This limit implies that $\frac{x}{a}\to 0$ also holds. It easily recovers the discussed above homogeneous mass nature. However, this ratio does not affect anyway to the general behavior of the harmonic oscillator potential.

3 The dynamical symmetry algebra of the semiconfined oscillator model

It is well known that the non-relativistic harmonic oscillator described by the Schrödinger equation (1) possesses the following closed Heisenberg-Weyl algebra as its dynamical symmetry algebra:

$$\left[\hat{H}^{ho},\hat{a}^{\pm}\right]=\pm\hbar\omega\hat{a}^{\pm},\qquad\left[\hat{a}^{-},\hat{a}^{+}\right]=1. \tag{18}$$

Here,

$$\hat{H}^{ho}=\hbar\omega\left(\hat{a}^{+}\hat{a}^{-}+\frac{1}{2}\right),\qquad\hat{a}^{\pm}=\sqrt{\frac{\hbar}{2m_0\omega}}\left(\frac{m_0\omega}{\hbar}x\mp\frac{d}{dx}\right). \tag{19}$$

Then, one can show that by analogy with the dynamical symmetry algebra of the non-relativistic harmonic oscillator, it is possible to introduce the three operators $\hat{K}_{\pm}$ and $\hat{K}_0$, which possess the following closed $\mathfrak{su}(1,1)$ Lie algebra:

$$\left[\hat{K}_0,\hat{K}_{\pm}\right]=\pm\hat{K}_{\pm},\qquad\left[\hat{K}_{-},\hat{K}_{+}\right]=2\hat{K}_0. \tag{20}$$

Here $\hat{K}_{\pm}$ operators are defined as $\hat{K}_{\pm} = \hat{K}_1 \pm i\hat{K}_2$. Correspondingly, the following definitions are valid for operators $\hat{K}_0$, $\hat{K}_1$ and $\hat{K}_2$:

$$\hat{K}_0 = \frac{1}{\hbar\omega}\left(m_0\omega^2a^2 + \hat{H}^{scho}\right), \quad \hat{K}_1 = \frac{1}{\hbar\omega}\left(m_0\omega^2a^2 - \hat{H}^{scho}\right), \quad \hat{K}_2 = i\left(\frac{1}{2} + a\frac{m_0}{M(x)}\frac{d}{dx}\right),$$

and Hamiltonian $\hat{H}^{scho} = -\frac{\hbar^2}{2}\frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + \frac{M(x)\omega^2x^2}{2}$.

Under the condition $a \to \infty$ the following correct limits hold:

$$\lim_{a\to\infty}\sqrt{\frac{\hbar}{2m_0\omega}}\frac{\hat{K}_{\pm}}{a} = \hat{a}^{\pm}, \lim_{a\to\infty}\left(\hat{K}_0 - m_0\omega^2a^2\right) = \hat{H}^{ho}.$$

4 The joint quasidistribution functions of the semiconfined oscillator model

We could compute analytically the Kirkwood, Husimi, and Wigner quasidistribution functions of the semiconfined harmonic oscillator model with the position-dependent mass. First of all, one needs to highlight that the Kirkwood function of the quantum system is defined as follows [7]:

$$K_n(x,p) = \frac{1}{2\pi\hbar}\psi_n(x)\,\psi_n^*(p)\,e^{-i\frac{px}{\hbar}}. \tag{21}$$

Then the exact expression of the wavefunction in the p-representation can be computed as a Fourier transform of the wavefunction (10). It has the following analytical expression in terms of the Jacobi polynomials:

$$\psi_n^{scho}(p) = C_n^p\left(\frac{m_0\omega a - ip}{p^2 + m_0^2\omega^2a^2}\right)^{\lambda_0^2a^2+1}e^{i\frac{ap}{\hbar}}$$
$$\times P_n^{\left(2\lambda_0^2a^2, -\lambda_0^2a^2-n\right)}\left(1 - 4m_0\omega a\frac{m_0\omega a - ip}{p^2 + m_0^2\omega^2a^2}\right).$$

Here,

$$C_n^p = (-1)^n\sqrt{\frac{m_0\hbar\omega a}{\pi}}\left(4m_0^2\omega^2a^3\right)^{\lambda_0^2a^2}\Gamma\left(\lambda_0^2a^2+1\right)\sqrt{\frac{n!}{\Gamma\left(n+\lambda_0^2a^2+1\right)}}.$$

Hence, the Kirkwood quasidistribution function can be easily deduced from eq.(21) as follows:

$$K_n(x,p) = \frac{C_nC_n^p}{2\pi\hbar}\left(1 + \frac{x}{a}\right)^{\lambda_0^2a^2}\left(\frac{m_0\omega a + ip}{p^2 + m_0^2\omega^2a^2}\right)^{\lambda_0^2a^2+1}e^{-\frac{(m_0\omega a + ip)(x+a)}{\hbar}}$$

$$\times L_n^{\left(2\lambda_0^2 a^2\right)}\left(\lambda_0^2 a\left(x+a\right)\right) P_n^{\left(2\lambda_0^2 a^2,-\lambda_0^2 a^2-n\right)}\left(1-4m_0\omega a\frac{m_0\omega a+ip}{p^2+m_0^2\omega^2 a^2}\right).$$

The Husimi function of the semiconfined harmonic oscillator model with the position-dependent mass is defined as follows [8]:

$$H_n\left(p,x\right)=\frac{1}{\left(2\pi\right)^{\frac{3}{2}}\hbar\Delta_x}\left|\int\psi_n\left(x'\right)e^{-i\frac{px'}{\hbar}-\frac{\left(x-x'\right)^2}{4\Delta_x^2}}dx'\right|^2. \tag{22}$$

We succeeded with the computation of the following exact expression of the Husimi function of the semiconfined quantum harmonic oscillator in terms of the parabolic cylinder functions (the value of the parameter Δ_x^2 is taken equal to $1/2\lambda_0^2$) [9]:

$$H_n\left(p,x\right)=\tfrac{1}{\pi\hbar}e^{-\frac{1}{\hbar\omega}\left(\frac{p^2}{2m_0}+\frac{m_0\omega^2}{2}\left(x^2+4ax+2a^2\right)\right)}\left(\lambda_0 a\right)^{2\lambda_0^2 a^2+1} \tag{23}$$
$$\times\tfrac{\Gamma\left(\lambda_0^2a^2+1\right)}{\Gamma\left(\lambda_0^2a^2+\frac{1}{2}\right)}\tfrac{\left(2\lambda_0^2a^2+1\right)_n}{n!}\sum_{k,s=0}^{n}\tfrac{\left(-n\right)_k\left(-n\right)_s\left(\lambda_0^2a^2+1\right)_k\left(\lambda_0^2a^2+1\right)_s}{\left(2\lambda_0^2a^2+1\right)_k\left(2\lambda_0^2a^2+1\right)_s}\tfrac{\left(2\lambda_0 a\right)^{k+s}}{k!s!}$$
$$\times D_{-\left(\lambda_0^2a^2+k+1\right)}\left(-\lambda_0\left(x-i\tfrac{p}{m_0\omega}\right)\right)D_{-\left(\lambda_0^2a^2+s+1\right)}\left(-\lambda_0\left(x+i\tfrac{p}{m_0\omega}\right)\right).$$

We also succeeded with the computation of the following exact expression of the Wigner function of the semiconfined quantum harmonic oscillator, which is in general defined as follows [10]:

$$W_n\left(p,x\right)=\frac{1}{2\pi\hbar}\int\psi_n^*\left(x-\frac{1}{2}x'\right)\psi_n\left(x+\frac{1}{2}x'\right)e^{-i\frac{px'}{\hbar}}dx'. \tag{24}$$

We applied a method, which removed a divergence of the integrand in the definition of the Wigner function and this allowed us to compute an analytical expression of the Wigner function within the integral border $-2\left(x+a\right)\leq x'\leq 2\left(x+a\right)$. It is analytically expressed as a product of the Bessel function of the first kind and the generalized Laguerre polynomials in the following manner [11]:

$$W_n\left(p,x\right)=2\hbar^{\lambda_0^2a^2-\frac{1}{2}}\frac{\left(\lambda_0^2a\right)^{2\lambda_0^2a^2+1}}{\Gamma\left(\lambda_0^2a^2+\frac{1}{2}\right)}e^{-2\lambda_0^2a\left(x+a\right)}\left(\tfrac{x+a}{p}\right)^{\lambda_0^2a^2+\frac{1}{2}}$$
$$\times\sum_{k=0}^{n}\frac{\left(2\lambda_0^2a\right)^{2k}\left(\lambda_0^2a^2+1\right)_k\left(\hbar\frac{x+a}{p}\right)^k}{\left(2\lambda_0^2a^2+1\right)_k k!}J_{\lambda_0^2a^2+k+\frac{1}{2}}\left(2\tfrac{p}{\hbar}\left(x+a\right)\right)L_{n-k}^{\left(2\lambda_0^2a^2+2k\right)}\left(4\lambda_0^2a\left(x+a\right)\right).$$

All these three quasiprobability distribution functions of the semiconfined oscillator model easily recover analytical expressions of their non-relativistic analogues under the limit $a\to+\infty$.

5 Conclusions

We presented details of the new model of the quantum harmonic oscillator. This model is semiconfined due to a specific change of its mass by position. We succeeded in solving exactly the Schrödinger equation corresponding to this model in the framework of the non-relativistic canonical quantum mechanics. Surprisingly, its energy spectrum completely overlaps with the energy spectrum of the known so-called Hermite oscillator, but the wavefunctions of the stationary states are expressed through the generalized Laguerre polynomials. We also extended this exact solution to the case, when the oscillator model suddenly becomes exposed to the action of the external homogeneous field. Moreover, we found that the dynamical symmetry algebra for this semiconfined harmonic oscillator model is $\mathfrak{su}(1,1)$ Heisenberg-Lie algebra. Further, we constructed the phase space of this oscillator model and could compute analytically its Husimi function in terms of the parabolic cylinder function, Kirkwood function in terms of the product of Laguerre and Jacobi polynomials, and Wigner function in terms of the product of the Bessel function of the first kind and the generalized Laguerre polynomials. Despite of existence of the confinement effect, its energy spectrum and wavefunction are still simple from the mathematics viewpoint, therefore, making it attractive for further studying of a lot of other useful properties of this model.

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