Bulg. J. Phys. **40** (2013) 115–120

The $\mathfrak{u}(2)_\alpha$ and $\mathfrak{su}(2)_\alpha$ Hahn Harmonic Oscillators*

E.I. Jafarov[1], **N.I. Stoilova**[2], **J. Van der Jeugt**[3]

[1]Institute of Physics, Azerbaijan National Academy of Sciences, Javid av. 33, AZ-1143 Baku, Azerbaijan

[2]Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria

[3]Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281-S9, B-9000 Gent, Belgium

Abstract. New models for the finite one-dimensional harmonic oscillator are proposed based upon the algebras $\mathfrak{u}(2)_\alpha$ and $\mathfrak{su}(2)_\alpha$. These algebras are deformations of the Lie algebras $\mathfrak{u}(2)$ and $\mathfrak{su}(2)$ extended by a parity operator, with deformation parameter α. Classes of irreducible unitary representations of $\mathfrak{u}(2)_\alpha$ and $\mathfrak{su}(2)_\alpha$ are constructed. It turns out that in these models the spectrum of the position operator can be computed explicitly, and that the corresponding (discrete) wavefunctions can be determined in terms of Hahn polynomials.

PACS codes: 03.67.Hk, 02.30.Gp

1 Introduction

Discrete and finite quantum systems have attracted much attention in recent years [1]. Such systems have been useful in areas such as quantum computing and quantum optics [2, 3]. However, in a finite-dimensional Hilbert space, the canonical commutation relations no longer hold. Several finite oscillator models have been proposed on different defining relations. Our interest comes from models related to a Lie algebra or its deformation.

For a one-dimensional finite oscillator, the position operator $\hat{q}$, its corresponding momentum operator $\hat{p}$ and the Hamiltonian $\hat{H}$, which is the generator of time evolution, should satisfy the compatibility of Hamilton's equations with the Heisenberg equations:

$$[\hat{H},\hat{q}] = -i\hat{p}, \qquad [\hat{H},\hat{p}] = i\hat{q}, \tag{1}$$

in units with mass and frequency both equal to 1, and $\hbar = 1$. Furthermore, one requires [4]:

*Talk at the Second Bulgarian National Congress in Physics, Sofia, September 2013.

- all operators $\hat{q}$, $\hat{p}$, $\hat{H}$ belong to some (Lie) algebra (or superalgebra) $\mathcal{A}$;
- the spectrum of $\hat{H}$ in (unitary) representations of $\mathcal{A}$ is equidistant.

The case with $\mathcal{A} = \mathfrak{su}(2)$ has been investigated in [4–6]. In the common $\mathfrak{su}(2)$ representations, labelled by an integer or half-integer j, the Hamiltonian is given by $\hat{H} = J_0 + j + \frac{1}{2}$, where $J_0 = J_z$ is the diagonal $\mathfrak{su}(2)$ operator and its spectrum is $n + \frac{1}{2}$ ($n = 0, 1, \ldots, 2j$). $\hat{q} = \frac{1}{2}(J_+ + J_-) = J_x$ and $\hat{p} = \frac{i}{2}(J_+ - J_-) = -J_y$, satisfying the relations (1), have a finite spectrum given by $\{-j, -j+1, \ldots, +j\}$ [4]. The discrete position wavefunctions are given by Krawtchouk functions and their shape is reminiscent of those of the canonical oscillator [4]. Under the limit $j \to \infty$ they coincide with the canonical wavefunctions in terms of Hermite polynomials [4, 7].

In the present paper we construct two new models for the finite one-dimensional harmonic oscillator based on the algebras $\mathfrak{u}(2)_\alpha$ and $\mathfrak{su}(2)_\alpha$, one-parameter deformations of the Lie algebras $\mathfrak{u}(2)$ and $\mathfrak{su}(2)$ (see [8, 9] for more details). In section 2 the deformed algebra $\mathfrak{u}(2)_\alpha$, its representations and the corresponding model are constructed. The spectrum of the position operator, and its eigenvectors are determined. The structure of these eigenvectors is studied, yielding position wavefunctions. Similarly, in section 3 we consider the model corresponding to $\mathfrak{su}(2)_\alpha$.

2 The $\mathfrak{u}(2)_\alpha$ model

Definition 1 *Let α be a parameter. The algebra $\mathfrak{u}(2)_\alpha$ is a unital algebra with basis elements J_0, J_+, J_-, C and P subject to the following relations:*

- *C commutes with all basis elements,*
- *P is a parity operator satisfying $P^2 = 1$ and*

$$[P, J_0] = PJ_0 - J_0P = 0, \qquad \{P, J_\pm\} = PJ_\pm + J_\pm P = 0. \tag{2}$$

- *The $\mathfrak{su}(2)$ relations are deformed as follows:*

$$[J_0, J_\pm] = \pm J_\pm, \tag{3}$$
$$[J_+, J_-] = 2J_0 - (2\alpha+1)^2 P - (2\alpha+1)CP. \tag{4}$$

Proposition 2 *Let j be a half-integer (i.e. $2j$ is odd), and consider the space V_j with basis vectors $|j,-j\rangle$, $|j,-j+1\rangle$, …, $|j,j\rangle$. Assume that $\alpha > -1$. Then* *the following action turns V_j into an irreducible representation space of $\mathfrak{u}(2)_\alpha$.*

$$C|j,m\rangle = (2j+1)\,|j,m\rangle, \tag{5}$$

$$P|j,m\rangle = (-1)^{j+m}\,|j,m\rangle, \tag{6}$$

$$J_0|j,m\rangle = m\,|j,m\rangle, \tag{7}$$

$$J_+|j,m\rangle = \begin{cases} \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle, & j+m\text{-odd;} \\ \sqrt{(j-m+2\alpha+1)(j+m+2\alpha+2)}\,|j,m+1\rangle, & j+m\text{-even.} \end{cases} \tag{8}$$

$$J_-|j,m\rangle = \begin{cases} \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle, & j+m\text{-even;} \\ \sqrt{(j+m+2\alpha+1)(j-m+2\alpha+2)}\,|j,m-1\rangle, & j+m\text{-odd.} \end{cases} \tag{9}$$

Straightforward calculations show that all defining relations from Definition 1 are valid when acting on an arbitrary vector $|j,m\rangle$. Irreducibility follows from the fact that $(J_+)^k|j,-j\rangle$ is nonzero and proportional to $|j,-j+k\rangle$ for $k = 1,2,\ldots,2j$, and similarly $(J_-)^k|j,j\rangle$ is nonzero.

If

$$\hat{q} = \frac{1}{2}(J_+ + J_-), \qquad \hat{p} = \frac{i}{2}(J_+ - J_-), \qquad \hat{H} = J_0 + \frac{1}{2}C, \tag{10}$$

it is easy to verify that (1) is satisfied and the spectrum of $\hat{H}$ is indeed linear and given by

$$n + \frac{1}{2} \qquad (n = 0,1,\ldots,2j). \tag{11}$$

From the actions (8)-(9), one finds that the operator $2\hat{q}$ takes the matrix form

$$2\hat{q} = \begin{pmatrix} 0 & M_0 & 0 & \cdots & 0 \\ M_0 & 0 & M_1 & \cdots & 0 \\ 0 & M_1 & 0 & \ddots & \\ \vdots & \vdots & \ddots & \ddots & M_{2j-1} \\ 0 & 0 & & M_{2j-1} & 0 \end{pmatrix}, \tag{12}$$

with

$$M_k = \begin{cases} \sqrt{(k+1)(2j-k)}, & \text{if } k \text{ is odd;} \\ \sqrt{(k+2\alpha+2)(2j-k+2\alpha+1)}, & \text{if } k \text{ is even.} \end{cases} \tag{13}$$

For this matrix, the eigenvalues are known explicitly [10, 11].

Proposition 3 *The $2j+1$ eigenvalues q of the position operator $\hat{q}$ in the representation V_j are given by*

$$-\alpha - j - \frac{1}{2}, -\alpha - j + \frac{1}{2}, \ldots, -\alpha - 1; \alpha+1, \alpha+2, \ldots, \alpha + j + \frac{1}{2}. \tag{14}$$

So in the $\mathfrak{u}(2)_\alpha$ oscillator model, there is a shift from the origin by $\alpha + \frac{1}{2}$.

The next result follows now from [11, Proposition 2]:

Proposition 4 *The orthonormal eigenvector of the position operator $\hat{q}$ in V_j for the eigenvalue q_k, denoted by $|j, q_k)$, is given by*

$$|j, q_k) = \sum_{m=-j}^{j} U_{j+m,j+k} |j, m\rangle. \tag{15}$$

Herein, $U = (U_{rs})_{0 \le r,s \le 2j}$ is a $(2j+1) \times (2j+1)$ matrix with elements

$$U_{2r,j-s-\frac{1}{2}} = U_{2r,j+s+\frac{1}{2}} = \frac{(-1)^r}{\sqrt{2}} \tilde{Q}_s(r; \alpha, \alpha+1, j-\frac{1}{2}), \tag{16}$$

$$U_{2r+1,j-s-\frac{1}{2}} = -U_{2r+1,j+s+\frac{1}{2}} = -\frac{(-1)^r}{\sqrt{2}} \tilde{Q}_s(r; \alpha+1, \alpha, j-\frac{1}{2}), \tag{17}$$

where $r, s \in \{0, 1, \ldots, j-\frac{1}{2}\}$. The functions $\tilde{Q}$ are normalized Hahn polynomials [12].

Corollary 5 *The $\mathfrak{u}(2)_\alpha$ oscillator wavefunctions are given by:*

$$\Phi_{2n}^{(\alpha)}(q_k) = \frac{(-1)^n}{\sqrt{2}} \tilde{Q}_{k-\frac{1}{2}}(n; \alpha, \alpha+1, j-\frac{1}{2}), \qquad k = \frac{1}{2}, \frac{3}{2}, \ldots, j;$$

$$\Phi_{2n+1}^{(\alpha)}(q_k) = \frac{(-1)^n}{\sqrt{2}} \tilde{Q}_{k-\frac{1}{2}}(n; \alpha+1, \alpha, j-\frac{1}{2}), \qquad k = \frac{1}{2}, \frac{3}{2}, \ldots, j.$$

3 The $\mathfrak{su}(2)_\alpha$ model

Definition 6 *Let α be a parameter. The algebra $\mathfrak{su}(2)_\alpha$ is a unital algebra with basis elements J_0, J_+, J_- and P subject to the following relations:*

- *P is a parity operator satisfying $P^2 = 1$ and*

$$[P, J_0] = PJ_0 - J_0P = 0, \qquad \{P, J_\pm\} = PJ_\pm + J_\pm P = 0. \tag{18}$$

- *The $\mathfrak{su}(2)$ relations are deformed as follows:*

$$[J_0, J_\pm] = \pm J_\pm, \tag{19}$$

$$[J_+, J_-] = 2J_0 + 2(2\alpha+1)J_0P. \tag{20}$$

Proposition 7 *Let j be an integer (i.e. $2j$ is even), and consider the space W_j with basis vectors $|j,-j\rangle$, $|j,-j+1\rangle, \ldots, |j,j\rangle$. Assume that $\alpha > -1$. Then the following action turns W_j into an irreducible representation space of $\mathfrak{su}(2)_\alpha$.*

$$P|j,m\rangle = (-1)^{j+m}\,|j,m\rangle, \tag{21}$$

$$J_0|j,m\rangle = m\,|j,m\rangle, \tag{22}$$

$$J_+|j,m\rangle = \begin{cases} \sqrt{(j-m)(j+m+2\alpha+2)}\,|j,m+1\rangle, & \textit{if } j+m \textit{ is even;} \\ \sqrt{(j-m+2\alpha+1)(j+m+1)}\,|j,m+1\rangle, & \textit{if } j+m \textit{ is odd,} \end{cases} \tag{23}$$

$$J_-|j,m\rangle = \begin{cases} \sqrt{(j+m+2\alpha+1)(j-m+1)}\,|j,m-1\rangle, & \textit{if } j+m \textit{ is odd;} \\ \sqrt{(j+m)(j-m+2\alpha+2)}\,|j,m-1\rangle, & \textit{if } j+m \textit{ is even.} \end{cases} \tag{24}$$

Quite similar as in the $\mathfrak{su}(2)_\alpha$ deformed case, the position, momentum and Hamiltonian operators defined by:

$$\hat{q} = \frac{1}{2}(J_+ + J_-), \qquad \hat{p} = \frac{i}{2}(J_+ - J_-), \qquad \hat{H} = J_0 + j + \frac{1}{2}, \tag{25}$$

satisfy (1). The spectrum of the position operator $\hat{q}$ is not equidistant, it is given by

$$-\sqrt{j(2\alpha+j+1)}, -\sqrt{(j-1)(2\alpha+j)}, \ldots, -\sqrt{2\alpha+2}; 0; \sqrt{2\alpha+2}, \ldots, \sqrt{j(2\alpha+j+1)}. \tag{26}$$

and the position wavefunctions by

$$\Phi_{2n}^{(\alpha)}(q_k) = \frac{(-1)^n}{\sqrt{2}}\tilde{Q}_k(n;\alpha,\alpha,j), \qquad n = 0,1,\ldots,j, \qquad k = 1,\ldots,j, \tag{27}$$

$$\Phi_{2n+1}^{(\alpha)}(q_k) = \frac{(-1)^n}{\sqrt{2}}\tilde{Q}_k(n;\alpha+1,\alpha+1,j-1),\; n = 0,1,\ldots,j,\; k = 1,\ldots,j. \tag{28}$$

These discrete wavefunctions, given in terms of Hahn polynomials, have nice properties, and they tend to the parabose wavefunctions when j is large. The analysis shows that for $\alpha \to -\frac{1}{2}$ they tend to the Krawtchouk wavefunctions of the $\mathfrak{su}(2)$ model; and for $\alpha \to -\frac{1}{2}$ and $j \to \infty$ they tend to the canonical oscillator wavefunctions in terms of Hermite polynomials.

References

[1] A. Vourdas (2004) *Rep. Progr. Phys.* **67** 267-320.
[2] S.L. Braunstein, V. Buzek and M. Hillery (2001) *Phys. Rev. A* **63** 052313.

[3] A. Miranowicz, W. Leonski and N. Imoto (2001) in *Modern Nonlinear Optics*, ed. M.W. Evans, *Adv. Chem. Phys*. **119** 155-193.

[4] N.M. Atakishiyev, G.S. Pososyan, L.E. Vicent and K.B. Wolf (2001) *J. Phys. A* **34** 9381-9398.

[5] N.M. Atakishiyev, G.S. Pososyan, L.E. Vicent and K.B. Wolf (2001) *J. Phys. A* **34** 9399-9415.

[6] N.M. Atakishiyev, G.S. Pososyan and K.B. Wolf (2005) *Phys. Part. Nuclei* **36** 247-265.

[7] N.M. Atakishiyev, G.S. Pososyan and K.B. Wolf (2003) *Int. J. Mod. Phys. A* **18** 317-327.

[8] E.I. Jafarov, N.I. Stoilova and J. Van der Jeugt (2011) *J. Phys. A* **44** 265203.

[9] E.I. Jafarov, N.I. Stoilova and J. Van der Jeugt (2011) *J. Phys. A* **44** 355205.

[10] T. Shi, Y. Li, Z. Song and C.P. Sun (2005) *Phys. Rev. A* **71** 032309.

[11] N.I. Stoilova and J. Van der Jeugt (2011) *SIGMA* **7** 033.

[12] R. Koekoek, P.A. Lesky and R.F. Swarttouw (2010) *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer-Verlag, Berlin.