A class of representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded special linear Lie superalgebra $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ and quantum statistics

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Abstract

The description of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded special linear Lie superalgebra $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ is carried out via generators $a_1^\pm, \ldots, a^\pm_{m_1+m_2+n_1+n_2}$ that satisfy triple relations and are called creation and annihilation operators. With respect to these generators, a class of Fock type representations of $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ is constructed. The properties of the underlying statistics are discussed and its Pauli principle is formulated.

Representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded special linear Lie superalgebra
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1 Introduction

A result from quantum field theory is that particles with half-integer spins are fermions, satisfying the Fermi-Dirac statistics, and particles with integer spins are bosons, satisfying the Bose-Einstein statistics. However, beyond Fermi-Dirac and Bose-Einstein statistics, various kinds of generalized quantum statistics have been introduced, investigated and discussed [1–9]. One of the first such generalizations are the so called parafermion and paraboson statistics [2]. The algebraic structure behind a system of $2m$-parafermion operators is the orthogonal Lie algebra $\mathfrak{so}(2m+1)$ [10], and behind a system of $2n$-parabosons is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [11]. For a mixed system of $2m$-parafermions and $2n$-parabosons there are two types of mutual nontrivial relations (from a physical point of view) [12] with algebraic structures the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ [13] and the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(1, 2m|2n, 0) \equiv \mathfrak{pso}(2m+1|2n)$ [14, 15]. All these algebras are of type B Lie algebras, Lie superalgebras or $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras. Furthermore, generalized statistics have been associated with all classical Lie algebras and basic classical Lie superalgebras from the infinite series A, B, C, and D and are refered to as A, B, C and D-(super)statistics [16–19]. Therefore it is natural to consider their $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded counterparts.

In recent years there has been an increased interests in colour algebras, especially in those with $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading and their applications [20–34]. In the following section we will remind the concept of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras, and define the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded special linear Lie superalgebra

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$\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$. In particular we shall give a set of $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ generators satisfying triple relations. Section 3 is devoted to a class of $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ irreducible representations, the so called Fock representations and in Section 4 we discuss the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded A statistics.

2 $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded special linear Lie superalgebra $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$

First it will be useful to recall the definition of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras [35–37].

As a linear space such algebras $\mathfrak{g}$ are direct sums of four subspaces:

$$\mathfrak{g} = \bigoplus_{\boldsymbol{a}} \mathfrak{g}_{\boldsymbol{a}} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(1,1)} \tag{2.1}$$

with $\boldsymbol{a} = (a_1, a_2)$ an element of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Writting $x_{\boldsymbol{a}}, y_{\boldsymbol{a}}, \ldots$, means that these elements belong to $\mathfrak{g}_{\boldsymbol{a}}$ and $\boldsymbol{a}$ is said to be the degree, $\deg x_{\boldsymbol{a}}$, of $x_{\boldsymbol{a}}$. Such elements $x_{\boldsymbol{a}}$ are called homogeneous elements. The bracket $[\![\cdot,\cdot]\!]$ on a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{g}$ satisfies the grading, symmetry and Jacobi identities:

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] \in \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}}, \tag{2.2}$$

$$[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!] = -(-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}[\![y_{\boldsymbol{b}}, x_{\boldsymbol{a}}]\!], \tag{2.3}$$

$$[\![x_{\boldsymbol{a}}, [\![y_{\boldsymbol{b}}, z_{\boldsymbol{c}}]\!]]\!] = [\![[\![x_{\boldsymbol{a}}, y_{\boldsymbol{b}}]\!], z_{\boldsymbol{c}}]\!] + (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}[\![y_{\boldsymbol{b}}, [\![x_{\boldsymbol{a}}, z_{\boldsymbol{c}}]\!]]\!], \tag{2.4}$$

where

$$\boldsymbol{a}+\boldsymbol{b} = (a_1+b_1, a_2+b_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2. \tag{2.5}$$

From (2.2) and (2.5) it follows that $\mathfrak{g}_{(0,0)}$ is a Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{g}_{(1,0)}, \mathfrak{g}_{(0,1)}, \mathfrak{g}_{(1,1)}$ are $\mathfrak{g}_{(0,0)}$-modules. In addition $\mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)}$ is also a Lie subalgebra of $\mathfrak{g}$, and the subspace $\mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}$ is a $\mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)}$-module. Furthermore, $\{\mathfrak{g}_{(1,1)}, \mathfrak{g}_{(0,1)}\} \subset \mathfrak{g}_{(1,0)}$ and $\{\mathfrak{g}_{(1,1)}, \mathfrak{g}_{(1,0)}\} \subset \mathfrak{g}_{(0,1)}$.

Let M be an arbitrary $(m_1+m_2+n_1+n_2+1 \times m_1+m_2+n_1+n_2+1)$-matrix of the following block form:

$$M = \begin{array}{cc} \begin{array}{cccc} m_1+1 & m_2 & n_1 & n_2 \end{array} & \\ \left(\begin{array}{cccc} a_{(0,0)} & a_{(1,1)} & a_{(1,0)} & a_{(0,1)} \\ b_{(1,1)} & b_{(0,0)} & b_{(0,1)} & b_{(1,0)} \\ c_{(1,0)} & c_{(0,1)} & c_{(0,0)} & c_{(1,1)} \\ d_{(0,1)} & d_{(1,0)} & d_{(1,1)} & d_{(0,0)} \end{array} \right) & \begin{array}{c} m_1+1 \\ m_2 \\ n_1 \\ n_2 \end{array} \end{array} \tag{2.6}$$

The indices of the matrix blocks show the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading, and the size of the blocks is indicated in the lines above and to the right of the matrix. Matrix M can be written as a sum of four matrices:

$$M = M_{(0,0)} + M_{(1,1)} + M_{(1,0)} + M_{(0,1)}, \tag{2.7}$$

with $M_{(a,b)}$ the corresponding block matrices $m_{(a,b)}$ and all other zeros. Defining the bracket $[\![\cdot,\cdot]\!]$ on the space of these matrices by:

$$[\![M_{(a_1,a_2)}, \tilde{M}_{(b_1,b_2)}]\!] = M_{(a_1,a_2)}\tilde{M}_{(b_1,b_2)} - (-1)^{a_1b_1+a_2b_2}\tilde{M}_{(b_1,b_2)}M_{(a_1,a_2)} \tag{2.8}$$

for the homogeneous elements $M_{(a_1,a_2)}$ and $\tilde{M}_{(b_1,b_2)}$ and extending it by linearity one obtains the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded general linear Lie superalgebra $\mathfrak{gl}(m_1+1, m_2|n_1, n_2)$.

It is straightforward to check that for the graded supertrace $\mathrm{Str}(A) = \mathrm{tr}(a_{(0,0)}) + \mathrm{tr}(b_{(0,0)}) - \mathrm{tr}(c_{(0,0)}) - \mathrm{tr}(d_{(0,0)})$ in terms of the ordinary trace tr, $\mathrm{Str}[\![A, B]\!] = 0$. Therefore, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded special linear Lie superalgebra $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ is defined as the subalgebra of $\mathfrak{gl}(m_1+1, m_2|n_1, n_2)$ with graded supertrace equal to 0.

Let

$$d_i = \begin{cases} (0,0); & i = 0, \dots, m_1 \\ (1,1); & i = m_1+1, \dots, m_1+m_2 = m \\ (1,0); & i = m_1+m_2+1, \dots, m_1+m_2+n_1 = m+n_1 \\ (0,1); & i = m_1+m_2+n_1+1, \dots, m_1+m_2+n_1+n_2 = m+n, \end{cases} \tag{2.9}$$

and let $e_{ij},\ i,j = 0,1,\dots,m_1+m_2+n_1+n_2 = m+n$ (where $m_1+m_2 = m, n_1+n_2 = n$) be the $(m+n+1 \times m+n+1)$ matrix (2.6) with 1 in the entry of row i, column j and 0 elsewhere. These matrices are homogeneous and the grading $\deg(e_{ij})$ is as follows:

$$\deg(e_{ij}) = d_i + d_j.$$

The bracket for such matrices is given by:

$$[\![e_{ij}, e_{kl}]\!] = \delta_{jk} e_{il} - (-1)^{(d_i+d_j)\cdot(d_k+d_l)} \delta_{il} e_{kj}.$$

The algebra $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ can be considered as the linear envelope of $e_{ij},\ i \neq j = 0,1,\dots,m+n,\ e_{00} - (-1)^{d_i\cdot d_i} e_{ii},\ i = 1,\dots,m+n$. A set of generators of $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ is given by:

$$a_i^+ = e_{i0},\ \ a_i^- = e_{0i},\ i = 1,\dots,m+n,\ \ (\deg(a_i^\pm) = d_i) \tag{2.10}$$

since

$$[\![a_i^+, a_j^-]\!] = e_{ij},\ i \neq j = 1,\dots,m+n;\ [\![a_k^-, a_k^+]\!] = e_{00} - (-1)^{d_k\cdot d_k} e_{kk},\ k = 1,\dots,m+n. \tag{2.11}$$

Denote these generators by

$$a_i^\pm \equiv b_i^\pm \in \mathfrak{g}_{(0,0)},\ \ i = 1,\dots,m_1, \tag{2.12}$$
$$a_i^\pm \equiv \tilde{b}_{i-m_1}^\pm \in \mathfrak{g}_{(1,1)},\ \ i = m_1+1,\dots,m, \tag{2.13}$$
$$a_i^\pm \equiv f_{i-m}^\pm \in \mathfrak{g}_{(1,0)},\ \ i = m+1,\dots,m+n_1, \tag{2.14}$$
$$a_i^\pm \equiv \tilde{f}_{i-m-n_1}^\pm \in \mathfrak{g}_{(0,1)},\ \ i = m+n_1+1,\dots,m+n. \tag{2.15}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded special linear Lie superalgebra $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$ can be defined in terms of the generators $a_i^\pm,\ i = 1,\dots,m+n$ and the following relations:

$$\begin{aligned} &[\![a_i^\xi, a_j^\xi]\!] = 0,\ \ \xi = \pm,\ i,j = 1,\dots,m+n, \\ &[\![[\![a_i^+, a_j^-]\!], a_k^+]\!] = \delta_{jk} a_i^+ + (-1)^{d_i\cdot d_i} \delta_{ij} a_k^+, \\ &[\![[\![a_i^+, a_j^-]\!], a_k^-]\!] = -(-1)^{(d_i+d_j)\cdot d_k} \delta_{ik} a_j^- - (-1)^{d_i\cdot d_i} \delta_{ij} a_k^-,\ \ i,j,k = 1,\dots,m+n. \end{aligned} \tag{2.16}$$

Definition 1. *We will call the generators $a_i^\pm,\ i = 1,\dots,m+n$, creation and annihilation operators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$.*

3 Representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$

We now proceed to construct a class of representations, Fock type representations, of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$. The irreducible Fock representations are labelled by one non-negative integer $p = 1, 2, \dots$, called an order of the statistics. To construct them assume that the corresponding representation space W_p contains (up to a multiple) a cyclic vector $|0\rangle$, such that

$$\begin{aligned} &a_i^-|0\rangle = 0, \quad i = 1, 2, \ldots, n+m, \\ &[\![a_i^-, a_j^+]\!]|0\rangle = \delta_{ij} p|0\rangle, \quad p \in \mathbb{N}, \quad i, j = 1, 2, \ldots, n+m. \end{aligned} \tag{3.1}$$

The Fock spaces are finite-dimensional irreducible $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$-modules. The vectors

$$(b_1^+)^{r_1} \ldots (b_{m_1}^+)^{r_{m_1}} (\tilde{b}_1^+)^{l_1} \ldots (\tilde{b}_{m_2}^+)^{l_{m_2}} (f_1^+)^{\theta_1} \ldots (f_{n_1}^+)^{\theta_{n_1}} (\tilde{f}_1^+)^{\lambda_1} \ldots (\tilde{f}_{n_2}^+)^{\lambda_{n_2}} |0\rangle \tag{3.2}$$

subject to the following restriction

$$r_i, l_i \in \mathbb{Z}_+, \qquad \theta_i, \lambda_i \in \{0, 1\}, \quad R = \sum_{i=1}^{m_1} r_i + \sum_{i=1}^{m_2} l_i + \sum_{i=1}^{n_1} \theta_i + \sum_{i=1}^{n_2} \lambda_i \leq p \tag{3.3}$$

constitute a basis in W_p. The linear space of all vectors (3.2) for any $r_i, l_i \in \mathbb{Z}_+$, $\theta_i, \lambda_i \in \{0, 1\}$, i.e. without the restriction (3.3), is an infinite-dimensional $\mathfrak{sl}(m_1+1, m_2|n_1, n_2)$-module $\bar{W}_p$. It is however not irreducible and $\bar{W}_p$ contains an infinite-dimensional invariant subspace W_p^{inv}, linear envelope of all the vectors (3.2) with $R = \sum_{i=1}^{m_1} r_i + \sum_{i=1}^{m_2} l_i + \sum_{i=1}^{n_1} \theta_i + \sum_{i=1}^{n_2} \lambda_i > p$. Then W_p is a factor module of $\bar{W}_p$ with respect to W_p^{inv}.

Define a Hermitian form $\langle\, ,\, \rangle$ on W_p with the standard Fock space technique. So, postulate that

$$\langle 0|0\rangle = 1, \tag{3.4}$$

$$\langle a_i^\pm v|w\rangle = \langle v|a_i^\mp w\rangle, \qquad \forall v, w \in W_p. \tag{3.5}$$

With respect to this form, the different vectors in (3.2) are orthogonal, and the following vectors form an orthonormal basis of W_p :

$$\begin{aligned} &|p; r_1, \ldots, r_{m_1}, l_1, \ldots, l_{m_2}, \theta_1, \ldots, \theta_{n_1}, \lambda_1, \ldots, \lambda_{n_2}) = \sqrt{\frac{(p-R)!}{p! r_1! \ldots \lambda_{n_2}!}} \\ &\times (b_1^+)^{r_1} \ldots (b_{m_1}^+)^{r_{m_1}} (\tilde{b}_1^+)^{l_1} \ldots (\tilde{b}_{m_2}^+)^{l_{m_2}} (f_1^+)^{\theta_1} \ldots (f_{n_1}^+)^{\theta_{n_1}} (\tilde{f}_1^+)^{\lambda_1} \ldots (\tilde{f}_{n_2}^+)^{\lambda_{n_2}} |0\rangle, \end{aligned} \tag{3.6}$$

with

$$r_i, l_i \in \mathbb{Z}_+, \qquad \theta_i, \lambda_i \in \{0, 1\}, \quad R = \sum_{i=1}^{m_1} r_i + \sum_{i=1}^{m_2} l_i + \sum_{i=1}^{n_1} \theta_i + \sum_{i=1}^{n_2} \lambda_i \leq p. \tag{3.7}$$

Proposition 2. *The transformation of the orthonormal basis of W_p under the action of the creation and annihilation operators $a_i^\pm$ reads:*

$$b_i^+|p; \ldots, r_{i-1}, r_i, r_{i+1}, \ldots) = \sqrt{(r_i+1)(p-R)}|p; \ldots, r_{i-1}, r_i+1, r_{i+1}, \ldots), \tag{3.8}$$

$$\tilde{b}_i^+|p; \ldots, l_{i-1}, l_i, l_{i+1}, \ldots) = \sqrt{(l_i+1)(p-R)}|p; \ldots, l_{i-1}, l_i+1, l_{i+1}, \ldots), \tag{3.9}$$

$$f_i^+|p; \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots) = (1-\theta_i)(-1)^{l_1+\ldots+l_{m_2}}(-1)^{\theta_1+\ldots+\theta_{i-1}}\sqrt{p-R}|p; \ldots, \theta_{i-1}, \theta_i+1, \theta_{i+1}, \ldots), \tag{3.10}$$

$$\tilde{f}_i^+|p; \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots) = (1-\lambda_i)(-1)^{l_1+\ldots+l_{m_2}}(-1)^{\lambda_1+\ldots+\lambda_{i-1}}\sqrt{p-R}|p; \ldots, \theta_{i-1}, \theta_i+1, \theta_{i+1}, \ldots), \tag{3.11}$$

$$b_i^-|p;\ldots,r_{i-1},r_i,r_{i+1},\ldots) = \sqrt{r_i(p-R+1)}|p;\ldots,r_{i-1},r_i-1,r_{i+1},\ldots), \tag{3.12}$$

$$\tilde{b}_i^-|p;\ldots,l_{i-1},l_i,l_{i+1},\ldots) = \sqrt{l_i(p-R+1)}|p;\ldots,l_{i-1},l_i-1,l_{i+1},\ldots), \tag{3.13}$$

$$f_i^-|p;\ldots,\theta_{i-1},\theta_i,\theta_{i+1},\ldots) = \theta_i(-1)^{l_1+\ldots+l_{m_2}}(-1)^{\theta_1+\ldots+\theta_{i-1}}\sqrt{p-R+1}|p;\ldots,\theta_{i-1},\theta_i-1,\theta_{i+1},\ldots), \tag{3.14}$$

$$\tilde{f}_i^-|p;\ldots,\lambda_{i-1},\lambda_i,\lambda_{i+1},\ldots) = \lambda_i(-1)^{l_1+\ldots+l_{m_2}}(-1)^{\lambda_1+\ldots+\lambda_{i-1}}\sqrt{p-R+1}|p;\ldots,\theta_{i-1},\theta_i+1,\theta_{i+1},\ldots). \tag{3.15}$$

The above transformations are obtained by applying the defining relations (2.16) of $\mathfrak{sl}(m_1+1,m_2|n_1,n_2)$. However, in order to prove that these explicit actions (3.8)-(3.15) give a representation of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded special linear Lie superalgebra $\mathfrak{sl}(m_1+1,m_2|n_1,n_2)$ it is sufficient to show that (3.8)-(3.15) satisfy the defining relations of the algebra which is a long but straightforward matter. The irreducibility then follows from the fact that for any nonzero vector $x\in W_p$ there exists a polynomial $\mathfrak{P}$ of $\mathfrak{sl}(m_1+1,m_2|n_1,n_2)$ generators such that $\mathfrak{P}x = W_p$.

4 $\mathbb{Z}_2\times\mathbb{Z}_2$-graded A statistics

The operators $b_i^\pm \equiv a_i^\pm \in \mathfrak{g}_{(0,0)}$, $i=1,\ldots,m_1$ and $\tilde{b}_{i-m_1}^\pm \equiv a_i^\pm \in \mathfrak{g}_{(1,1)}$, $i=m_1+1,\ldots,m$ generate the Lie algebra $\mathfrak{sl}(m+1)$ and satisfy the relations:

$$\begin{aligned}
&[a_i^\xi, a_j^\xi] = 0, \;\; \xi=\pm,\; i,j=1,\ldots,m,\\
&[\,[a_i^+,a_j^-],a_k^+] = \delta_{jk}a_i^+ + \delta_{ij}a_k^+,\\
&[\,[a_i^+,a_j^-],a_k^-] = -\delta_{ik}a_j^- - \delta_{ij}a_k^-.
\end{aligned} \tag{4.1}$$

Relations (4.1) are the defining triple relations of A-statistics [38].

There are other two sets of operators $f_{i-m}^\pm \equiv a_i^\pm \in \mathfrak{g}_{(1,0)}$ for $i=m+1,\ldots,m+n_1$ and $\tilde{f}_{i-m-n_1}^\pm \equiv a_i^\pm \in \mathfrak{g}_{(0,1)}$ for $i=m+n_1+1,\ldots,m+n$ satisfying the common defining triple relations of A-superstatistics [39]:

$$\begin{aligned}
&\{a_i^+,a_j^+\} = \{a_i^-,a_j^-\} = 0,\\
&[\{a_i^+,a_j^-\},a_k^+] = \delta_{jk}a_i^+ - \delta_{ij}a_k^+,\\
&[\{a_i^+,a_j^-\},a_k^-] = -\delta_{ik}a_j^- + \delta_{ij}a_k^-.
\end{aligned} \tag{4.2}$$

Here, either $i,j,k=m+1,\ldots,m+n_1$ or else $i,j,k=m+n_1+1,\ldots,m+n$.

It is easy to write down the mixed relations between the three families of operators $a_i^\pm$, $i=1,\ldots,m+n$ (the operators of A-statistics and the two sets of operators of A-superstatistics), in terms of (anti)commutators using (2.16) and the $\mathbb{Z}_2\times\mathbb{Z}_2$-grading. In particular the mixed relations between the two families of A-superstatistics operators are as follows:

$$\begin{aligned}
&[a_i^+,a_j^+] = [a_i^-,a_j^-] = 0,\\
&\{[a_i^+,a_j^-],a_k^+\} = \delta_{jk}a_i^+,\\
&\{[a_i^+,a_j^-],a_k^-\} = \delta_{ik}a_j^-,
\end{aligned} \tag{4.3}$$

with in (4.3), either $i=m+1,\ldots,m+n_1$, $j=m+n_1+1,\ldots,m+n$, $k=m+1,\ldots,m+n$, or else $i=m+n_1+1,\ldots,m+n$, $j=m+1\ldots,m+n_1$, $k=m+1,\ldots,m+n$;

and

$$\begin{aligned}
&\{a_i^+, a_j^+\} = \{a_i^-, a_j^-\} = 0,\\
&[\{a_i^+, a_j^-\}, a_k^+] = -\delta_{ij} a_k^+,\\
&[\{a_i^+, a_j^-\}, a_k^-] = +\delta_{ij} a_k^-.
\end{aligned} \tag{4.4}$$

with in (4.4), either $i, j = m+1, \ldots, m+n_1,\ k = m+n_1+1, \ldots, m+n$, or else $i, j = m+n_1+1, \ldots, m+n,\ k = m+1, \ldots, m+n_1$.

The operators a_i^+ (resp. a_i^-) can be interpreted as operators in a state space W_p, the Fock space, creating "a particle" (resp. annihilating "a particle"), with energy ε_i . Let for simplicity $m = n$ and consider a Hamiltonian

$$H = \sum_{i=1}^{m} \varepsilon_i([a_i^+, a_i^-] + \{a_i^+, a_i^-\}). \tag{4.5}$$

Then

$$[H, a_i^{\pm}] = \pm\varepsilon_i a_i^{\pm}, \quad [H, a_{i+m}^{\pm}] = \pm\varepsilon_i a_{i+m}^{\pm}. \tag{4.6}$$

This result together with (3.8)-(3.15) allows one to interpret $r_i,\ i = 1, \ldots, m_1,\ l_i,\ i = 1, \ldots, m_2,\ \theta_i,\ i = 1, \ldots, n_1$ and $\lambda_i,\ i = 1, \ldots, n_2$ as the number of particles on the corresponding orbital. The operator a_i^+ increases this number by one, it "creates" a particle in the one-particle state (orbital). Similarly, the operator a_i^- diminishes this number by one, it "kills" a particle on the corresponding orbital. However on every orbital of the last n there cannot be more than one particle, whereas such restriction does not hold for the first m orbitals. Therefore we have generalized fermions and bosons in this case. However there is an extra property. Since $\sum_{i=1}^{m_1} r_i + \sum_{i=1}^{m_2} l_i + \sum_{i=1}^{n_1} \theta_i + \sum_{i=1}^{n_2} \lambda_i \leq p$ no more than p particles can be accomodated in the system. This is the *Pauli principle* for this statistics.

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