On characters and superdimensions of some infinite-dimensional irreducible representations of $\mathfrak{osp}(m|n)$

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Chiral spinors and self dual tensors of the Lie superalgebra $\mathfrak{osp}(m|n)$ are infinite dimensional representations belonging to the class of representations with Dynkin labels $[0,\ldots,0,p]$. We show that the superdimension of $[0,\ldots,0,p]$ coincides with the dimension of a $\mathfrak{so}(m-n)$ representation. When the superdimension is finite, these representations could play a role in supergravity models. Our technique is based on expansions of characters in terms of supersymmetric Schur functions. In the process of studying these representations, we obtain new character expansions.

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I. INTRODUCTION

Models of supergravity theory [1, 2] are often implicitly or explicitly based upon tensor representations of the orthosymplectic Lie superalgebra $\mathfrak{osp}(m|n)$ [3, 4]. Chiral spinors and self dual tensors of $\mathfrak{osp}(m|n)$ play an important role in such models. These tensors are, however, infinite-dimensional. Nonetheless, the so-called superdimension of these tensors corresponds to the dimension of a finite-dimensional tensor of $\mathfrak{so}(m-n)$ [5] (to be interpreted appropriately when $m-n$ is negative [6]), thus paving the way for new covariant quantization schemes.

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In [5] we initiated the study of this correspondence between certain infinite-dimensional representations of $\mathfrak{osp}(m|n)$ and finite-dimensional representations of $\mathfrak{so}(m - n)$. Let us be more precise. In terms of (the distinguished) Dynkin diagrams of $\mathfrak{osp}(m|n)$, the spinor representation has Dynkin labels $[0, 0, \ldots, 0, 1]$ and the self dual tensor $[0, 0, \ldots, 0, 2]$. In [5], we treated the irreducible representations (irreps) with Dynkin labels $[0, 0, \ldots, 0, p]$, where $p$ is a positive integer (a convention followed throughout this paper).

In the present paper, we shall first review some of the results of [5], and for this we need to recall some definitions and notations. For all these developments, characters of a class of representations of $\mathfrak{osp}(m|n)$ play a prominent role. Since the Lie superalgebras $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ both contain the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ as a subalgebra, it is convenient to express the characters of the infinite-dimensional $\mathfrak{osp}$-irreps as an infinite sum of $\mathfrak{gl}(m|n)$ characters (given by supersymmetric Schur functions). In [5] this was done for the irreps $[0, 0, \ldots, 0, p]$ of $\mathfrak{osp}(2m+1|2n)$ (leading to a new character formula for the case of $\mathfrak{osp}(2m|2n)$). In the current paper, we can extend this and obtain new character formulas for irreps of type $[0, \ldots, 0, r, p - r]$ for $\mathfrak{osp}(2m|2n)$.

II. DEFINITIONS AND NOTATIONS

A. Partitions and (super)symmetric functions

We need some basic notions on partitions and symmetric functions, see [7] as a standard reference. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of weight $|\lambda|$ and length $\ell(\lambda) \leq n$ is a sequence of non-negative integers satisfying the condition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, such that their sum is $|\lambda|$, and $\lambda_i > 0$ if and only if $i \leq \ell(\lambda)$. To each such partition there corresponds a Young diagram $F^\lambda$ consisting of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left-adjusted rows of lengths $\lambda_i$ for $i = 1, 2, \ldots, \ell(\lambda)$. For example, the Young diagram of $\lambda = (5, 4, 4, 2)$ is given by

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The conjugate partition $\lambda'$ corresponds to the Young diagram of $\lambda$ reflected about the main diagonal. For the above example, $\lambda' = (4, 4, 3, 3, 1)$.

If $\lambda, \mu$ are partitions, one writes $\lambda \supset \mu$ if the diagram of $\lambda$ contains that of $\mu$. The
difference $\lambda - \mu$ is called a skew diagram [7]. For example, if $\mu = (4, 4, 3)$, then the boxes of the skew diagram $\lambda - \mu$ are crossed in the following picture:

A skew diagram is a horizontal strip if it has at most one box in each column. The number of boxes of the horizontal strip is its length. The above example is a horizontal strip of length 4.

Partitions are used to label symmetric functions. The Schur functions [7] or $S$-functions

$$s_\lambda(x) = \det(x_i^\lambda + n-j)_{1 \leq i, j \leq n}/\det(x_i^{n-j})_{1 \leq i, j \leq n}.$$

In terms of two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, one can define the ring $\Lambda_{m,n}$ of supersymmetric polynomials with integer coefficients [8]. This ring consists of all double symmetric polynomials in $x$ and $y$ (elements of $\Lambda_m \otimes \Lambda_n$) that satisfy the so-called cancellation property (i.e. when the substitution $x_1 = t$, $y_1 = -t$ is made in an element $p$ of $\Lambda_m \otimes \Lambda_n$, the resulting polynomial is independent of $t$). For a partition $\lambda$, one can define supersymmetric Schur polynomials $s_\lambda(x|y)$ belonging to $\Lambda_{m,n}$ [8, 9]. These polynomials $s_\lambda(x|y)$ are zero when $\lambda_{m+1} > n$. Denote by $\mathcal{H}_{m,n}$ the set of all partitions with $\lambda_{m+1} \leq n$, i.e. the partitions (with their Young diagram) inside the $(m, n)$-hook. The set of $s_\lambda(x|y)$ with $\lambda \in \mathcal{H}_{m,n}$ forms a $\mathbb{Z}$-basis of $\Lambda_{m,n}$.

### B. Dimension, superdimension, $t$-dimension

A finite-dimensional irreducible representation of the Lie algebra $\mathfrak{gl}(n)$ is characterized by a partition $\lambda$ with $\ell(\lambda) \leq n$. In terms of the standard basis $\epsilon_1, \ldots, \epsilon_n$ of the weight space of $\mathfrak{gl}(n)$, the highest weight of this representation is $\sum_{i=1}^n \lambda_i \epsilon_i$, and the representation space will be denoted by $V^\lambda_{\mathfrak{gl}(n)}$. Weyl’s character formula for such representations yields

$$\text{char} V^\lambda_{\mathfrak{gl}(n)} = s_\lambda(x),$$

where $x_i = e^{\epsilon_i}$.

Just as the functions $s_\lambda(x)$ are characters of irreducible representations (or simple modules) of the Lie algebra $\mathfrak{gl}(n)$, the supersymmetric Schur functions are characters of a class of
simple modules of the Lie superalgebra $\mathfrak{gl}(m|n)$, namely of the covariant representations [8].

For a partition $\lambda \in \mathcal{H}_{m,n}$, the corresponding covariant representation will be denoted by $V^\lambda_{\mathfrak{gl}(m|n)}$. In terms of the standard basis $\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n$ of the weight space of $\mathfrak{gl}(m|n)$, the highest weight of this representation is $\sum_{i=1}^m \lambda_i \epsilon_i + \sum_{j=1}^n \max(\lambda'_j - m, 0) \delta_j$. The main result of [8] is

$$\text{char } V^\lambda_{\mathfrak{gl}(m|n)} = s_\lambda(x|y),$$

(1)

where $x_i = e^{\epsilon_i}$ and $y_j = e^{\delta_j}$.

Any Lie superalgebra $\mathfrak{g}$ is $\mathbb{Z}_2$-graded: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. A Lie superalgebra module or representation $V$ is also $\mathbb{Z}_2$-graded: $V = V_0 \oplus V_1$. In our convention, the highest weight vector $v$ of $V$ will always be an even vector ($v \in V_0$). When $V$ is finite-dimensional, one can speak of the dimension and superdimension of $V$:

$$\dim V = \dim V_0 + \dim V_1, \quad \text{sdim } V = \dim V_0 - \dim V_1.$$ 

Superdimension formulas for covariant representations of $\mathfrak{gl}(m|n)$ are known [9]. The result depends on whether $m$ is greater than, equal to, or less than $n$. It can be summarized as follows:

$$\text{sdim } V^\lambda_{\mathfrak{gl}(n+k|n)} = \dim V^\lambda_{\mathfrak{gl}(k)}, \quad \text{sdim } V^\lambda_{\mathfrak{gl}(m|n+k)} = (-1)^{|\lambda|} \dim V^\lambda_{\mathfrak{gl}(k)}. \quad (2)$$

In particular, when $m = n$, sdim $V^\lambda_{\mathfrak{gl}(n|n)} = 0$ unless $\lambda$ is the zero partition (0) (then $V^{(0)}_{\mathfrak{gl}(n|n)}$ is the trivial module with sdim $V^{(0)}_{\mathfrak{gl}(n|n)} = 1$). Note that (2) implies: when $\ell(\lambda) > k$ then sdim $V^\lambda_{\mathfrak{gl}(n+k|n)} = 0$; when $\lambda_1 > k$ then sdim $V^\lambda_{\mathfrak{gl}(m|n+k)} = 0$.

Finally, let us introduce the notion of $t$-dimension of a Lie (super)algebra highest weight representation $V$. This is nothing else but a specialization of the character of $V$, just like the $q$-dimension [10, Chapter 10]. Recall that the $q$-dimension of $V$, with highest weight $\Lambda$, is the specialization $F_1(e^{-\Lambda} \text{char } V)$, where $F_1$ is determined by

$$F_1(e^{-\alpha_i}) = q,$$

and the $\alpha_i$'s are the simple roots of the Lie (super)algebra. So this corresponds to a gradation with respect to the simple roots.

The $t$-dimension is again a specialization $F(e^{-\Lambda} \text{char } V)$ of the character, but now $F$ is determined in a different way. For a Lie algebra, of which the simple roots are commonly expressed in terms of the standard basis $\epsilon_1, \ldots, \epsilon_n$, one puts $F(e^{-\epsilon_i}) = t$. For a Lie superalgebra, of which the simple roots are commonly expressed in terms of the standard
basis \( \epsilon_1, \ldots, \epsilon_n, \delta_1, \ldots, \delta_n \), one puts \( F(e^{-\epsilon_i}) = t \) and \( F(e^{-\delta_i}) = t \) for the \( t \)-dimension, and \( F(e^{-\epsilon_i}) = t \) and \( F(e^{-\delta_i}) = -t \) for the \( t \)-superdimension.

Let us clarify the meaning by means of an example. Consider the orthogonal Lie algebra \( \mathfrak{so}(2n+1) \), with simple roots \( \epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n \), and the representation \( V \) with Dynkin labels \([0, \ldots, 0, p]\), for which the highest weight is \( (\frac{p}{2}, \ldots, \frac{p}{2}) \) in the \( \epsilon \)-basis. For this representation, the character reads [11, 12]

\[
\text{char}[0, \ldots, 0, p]_{\mathfrak{so}(2n+1)} = (x_1 \cdots x_n)^{-p/2} \sum_{\lambda_1 \leq p, \ell(\lambda) \leq n} s_{\lambda}(x). \tag{3}
\]

So the sum is over all partitions \( \lambda \) such that the Young diagram of \( \lambda \) fits inside the \( n \times p \) rectangle, of width \( p \) and height \( n \). Specializing this character according to \( F \), one finds:

\[
\dim_t[0, \ldots, 0, p]_{\mathfrak{so}(2n+1)} = \sum_{\lambda_1 \leq p, \ell(\lambda) \leq n} \dim V^\lambda_{\mathfrak{gl}(n)} t^{[\lambda]} \tag{4}
\]

When the character is expressed in terms of Schur functions, as in (3), it yields in fact the branching of the representation according to \( \mathfrak{so}(2n+1) \rightarrow \mathfrak{gl}(n) \). When the character is specialized as in (4), it is a polynomial in \( t \) (or, in case of an infinite-dimensional representation, a formal power series in \( t \)) such that the coefficient of \( t^k \) counts the dimension “at level \( k \)” according to the \( \mathbb{Z} \)-gradation induced by the \( \mathfrak{gl}(n) \) subalgebra of \( \mathfrak{so}(2n+1) \).

### C. \( t \)-dimension for \( \mathfrak{osp}(1|2n) \)

In this subsection we shall consider the \( t \)-dimension for a class of representations of \( \mathfrak{g} = \mathfrak{osp}(1|2n) \). Let us first fix some notation [13–15]. In the common basis \( \delta_j \) for the weight space of \( \mathfrak{osp}(1|2n) \), the odd roots are given by \( \pm \delta_j \) \((j = 1, \ldots, n)\), and the even roots are \( \delta_i - \delta_j \) \((i \neq j)\) and \( \pm (\delta_i + \delta_j) \). The simple roots are

\[
\delta_1 - \delta_2, \ \delta_2 - \delta_3, \ldots, \delta_{n-1} - \delta_n, \ \delta_n. \tag{5}
\]

The character specialization of the previous subsection corresponds to the following \( \mathbb{Z} \)-gradation of \( \mathfrak{g} = \mathfrak{osp}(1|2n) \): \( \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \mathfrak{g}_{+2} \), where each \( \mathfrak{g}_j \) is spanned by the root vectors corresponding to the following roots:

\[
\begin{align*}
\mathfrak{g}_{-2} & \quad \mathfrak{g}_{-1} & \quad \mathfrak{g}_0 & \quad \mathfrak{g}_{+1} & \quad \mathfrak{g}_{+2} \\
-\delta_i - \delta_j & \quad -\delta_i & \quad \delta_i - \delta_j & \quad \delta_i & \quad \delta_i + \delta_j
\end{align*}
\]
Note that $\mathfrak{g}_0 = \mathfrak{gl}(n)$.

We will consider infinite-dimensional highest weight representations $V$ of $\mathfrak{g}$, such that the action of $\mathfrak{g}_0 = \mathfrak{gl}(n)$ on the highest weight vector $v$ of $V$ corresponds to a finite-dimensional $\mathfrak{g}_0$ module $V_0$. Then the $\mathbb{Z}$-gradation of $\mathfrak{g}$ induces a $\mathbb{Z}$-gradation of $V$:

$$V = V_0 \oplus V_{-1} \oplus V_{-2} \oplus \cdots$$

in terms of finite-dimensional $\mathfrak{g}_0$ modules, and the $t$-(super)dimension gives

$$\dim_t(V) = \sum_{i=0}^{\infty} \dim V_{-i} t^i, \quad \text{sdim}_t(V) = \sum_{i=0}^{\infty} \dim V_{-i} (-t)^i = \dim_{-t}(V).$$  \hfill (6)

For reasons that will become clear, we will consider the irreducible highest weight representation with highest weight given by $(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2})$ in the $\delta$-basis. For this representation, the Dynkin labels are $[0, 0, \ldots, 0, -p]$. The structure and character of this representation have been determined in [16]. Using the notation $x_i = e^{-\delta_i}$, one has:

$$\text{char}[0, 0, \ldots, 0, -p]_{osp(1|2n)} = (x_1 \cdots x_n)^{p/2} \sum_{\lambda, \ell(\lambda) \leq p} s_{\lambda}(x).$$  \hfill (7)

This is an infinite sum over all partitions of length at most $p$. Since $s_{\lambda}(x) = 0$ if $\ell(\lambda) > n$, the sum is actually over all partitions satisfying $\ell(\lambda) \leq \min(n, p)$. Thus:

$$\dim_t[0, 0, \ldots, 0, -p]_{osp(1|2n)} = \sum_{\lambda, \ell(\lambda) \leq \min(n, p)} \dim V_{\mathfrak{gl}(n)}^{\lambda t^{\ell(\lambda)}}.$$

This infinite sum can be rewritten in an alternative form, see [5].

III. SUPERDIMENSIONS FOR $osp(2m+1|2n)$

Consider the Lie superalgebra $B(m, n) = osp(2m+1|2n)$, with the distinguished set of simple roots in the $\epsilon$-$\delta$-basis [13, 15]

$$\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m.$$  \hfill (9)

Also in this case there exists a useful $\mathbb{Z}$-gradation of $\mathfrak{g} = osp(2m+1|2n)$: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \mathfrak{g}_{+2}$, where each $\mathfrak{g}_j$ is spanned by the root vectors corresponding to the following roots:

$$\mathfrak{g}_{-2} \quad \mathfrak{g}_{-1} \quad \mathfrak{g}_0 \quad \mathfrak{g}_{+1} \quad \mathfrak{g}_{+2}$$

$$-\delta_i - \delta_j \quad -\delta_i \quad \delta_i - \delta_j \quad \delta_i \quad \delta_i + \delta_j$$

$$-\epsilon_i - \epsilon_j \quad (i \neq j) \quad -\epsilon_i \quad \epsilon_i - \epsilon_j \quad \epsilon_i \quad \epsilon_i + \epsilon_j \quad (i \neq j)$$

$$-\epsilon_i - \delta_j \quad \pm(\epsilon_i - \delta_j) \quad \epsilon_i.$$
So \( g_0 = \mathfrak{gl}(m|n) \), and this gradation corresponds to the \( t \)-dimension introduced earlier.

Let us consider the irreducible highest weight representation with highest weight given by \( (\frac{p}{2}, \ldots, \frac{p}{2}; -\frac{p}{2}, \ldots, -\frac{p}{2}) \) in the \( \epsilon - \delta \)-basis. This representation has Dynkin labels \([0, 0, \ldots, 0, p]\). Using \( x_i = e^{-\epsilon_i}, y_i = e^{-\delta_i} \), the following character formula holds [5, 17]:

\[
\text{char}\{0, \ldots, 0, p\}_{\mathfrak{osp}(2m+1|2n)} = (y_1 \cdots y_n/x_1 \cdots x_m)^{p/2} \sum_{\lambda, \lambda_1 \leq p} s_\lambda(x|y).
\]  

(10)

So here the sum is over all partitions \( \lambda \) inside the \((m, n)\)-hook (otherwise \( s_\lambda(x|y) \) is zero anyway) with \( \lambda_1 \leq p \), or equivalently \( \ell(\lambda') \leq p \).

In order to determine \( \text{sdim}\{0, \ldots, 0, p\}_{\mathfrak{osp}(2m+1|2n)} \), one should (apart from the factor in front of the above sum) specify \( x_i = t \) and \( y_j = -t \) in the above character, and so one finds

\[
\text{sdim}\{0, \ldots, 0, p\}_{\mathfrak{osp}(2m+1|2n)} = \sum_{\lambda, \lambda_1 \leq p} s_\lambda(t, \ldots, t) - t, \ldots, -t \\
= \sum_{\lambda, \lambda_1 \leq p} s_\lambda(1, \ldots, 1) - 1, \ldots, -1) t^{\lambda} \\
= \sum_{\lambda, \lambda_1 \leq p} \text{sdim} V^\lambda_{\mathfrak{gl}(m|n)} t^{\lambda}.
\]  

(11)

Using the properties of \( \mathfrak{gl}(m|n) \) superdimensions, this leads to the following three cases.

**Case 1:** \( m = n, \mathfrak{osp}(2n+1|2n) \). All superdimensions of covariant representations of \( \mathfrak{gl}(n|n) \) are zero, except when \( \lambda = (0) \). Hence:

\[
\text{sdim}\{0, \ldots, 0, p\}_{\mathfrak{osp}(2n+1|2n)} = 1.
\]  

(12)

**Case 2:** \( m = n + k, \mathfrak{osp}(2n + 2k + 1|2n) \). Now it follows directly from (11) and (2) that

\[
\text{sdim}\{0, \ldots, 0, p\}_{\mathfrak{osp}(2n+1|2n)} = \sum_{\lambda, \lambda_1 \leq p} \dim V^\lambda_{\mathfrak{gl}(k)} t^{\lambda} = \sum_{\lambda, \lambda_1 \leq p, \ell(\lambda) \leq k} \dim V^\lambda_{\mathfrak{gl}(k)} t^{\lambda}.
\]  

(13)

This coincides with expression (4). Hence we can write

\[
\text{sdim}\{0, 0, \ldots, 0, p\}_{\mathfrak{osp}(2n+2k+1|2n)} = \text{dim}\{0, \ldots, 0, p\}_{\mathfrak{so}(2k+1)}.
\]  

(14)

**Case 3:** \( n = m + k, \mathfrak{osp}(2m + 1|2m + 2k) \). One finds:

\[
\text{sdim}\{0, \ldots, 0, p\}_{\mathfrak{osp}(2m+1|2n)} = \sum_{\lambda, \lambda_1 \leq p, \lambda_1 \leq k} (-1)^{\lambda} \dim V^\lambda_{\mathfrak{gl}(k)} t^{\lambda} = \sum_{\mu, \ell(\mu) \leq \min(p,k)} \dim V^\mu_{\mathfrak{gl}(k)} (-t)^{[\mu]}.
\]  

(15)
The right hand side is the same expression as (8), so
\[ \text{sdim}_t[0,0,\ldots,0,p]_{\text{osp}(2m+1|2m+2k)} = \dim_{-t}[0,\ldots,0,-p]_{\text{osp}(1|2k)}. \] (16)

So in all three cases, the superdimension for \( \text{osp}(2m+1|2n) \) simplifies and reduces to a dimension of \( \text{so}(2m+1-2n) \) or \( \text{osp}(1|2n-2m) \).

**IV. SUPERDIMENSIONS FOR \( \text{osp}(2m|2n) \) AND NEW CHARACTERS**

For \( D(m,n) = \text{osp}(2m|2n) \), the distinguished set of simple roots in the \( \epsilon-\delta \)-basis is
\[ \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-2} - \epsilon_{m-1}, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \epsilon_1. \] (17)

It will be helpful to see \( D(m,n) \) as a subalgebra of \( B(m,n) \). In fact, using the \( \mathbb{Z} \)-gradation \( g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_{+1} \oplus g_{+2} \) of \( g = \text{osp}(2m+1|2n) \) introduced in the previous section, it is easy to see that \( \text{osp}(2m|2n) = g_{-2} \oplus g_0 \oplus g_{+2} \), with root structure as in Section III.

For the irreducible highest weight representation of \( \text{osp}(2m|2n) \) with highest weight given by \( (\frac{p}{2}, \ldots, \frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2}) \), with Dynkin labels are \( [0,0,\ldots,0,p] \), the character was determined in [5]:
\[ \text{char}[0,\ldots,0,p]_{\text{osp}(2m|2n)} = (y_1 \cdots y_n/x_1 \cdots x_m)^{p/2} \sum_{\lambda \in \mathcal{B}, \lambda \leq p} s_{\lambda}(x|y). \] (18)

Herein, \( \mathcal{B} \) denotes the set of partitions for which each part appears twice (including the zero partition). Thus, one finds
\[ \text{sdim}_t[0,\ldots,0,p]_{\text{osp}(2m|2n)} = \sum_{\lambda \in \mathcal{B}, \lambda \leq p} \text{sdim} V^\lambda_{\text{gl}(m|n)} t^{||\lambda||}. \] (19)

This expression allows once again to deduce superdimension formulas in three cases: \( m = n \), \( m > n \) and \( m < n \), see [5]. Let us give here the formula for \( m > n \), i.e. \( m = n + k \), or \( \text{osp}(2n + 2k|2n) \). From (19) one has:
\[ \text{sdim}_t[0,\ldots,0,p]_{\text{osp}(2m|2n)} = \sum_{\lambda \in \mathcal{B}, \lambda \leq p} \dim V^\lambda_{\text{gl}(k)} t^{||\lambda||} = \sum_{\lambda \in \mathcal{B}, \lambda \leq p, \ell(\lambda) \leq k} \dim V^\lambda_{\text{gl}(k)} t^{||\lambda||}. \] (20)

And thus, using known characters of \( \text{so}(2k) \) [5]:
\[ \text{sdim}_t[0,0,\ldots,0,p]_{\text{osp}(2n+2k|2n)} = \begin{cases} \dim_t[0,\ldots,0,0,p]_{\text{so}(2k)} & \text{for } k \text{ even}, \\ \dim_t[0,\ldots,0,p,0]_{\text{so}(2k)} & \text{for } k \text{ odd}. \end{cases} \] (21)
Here, the convention for the order of the simple roots of $\mathfrak{so}(2k)$ is $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{k-1} - \epsilon_k, \epsilon_{k-1} + \epsilon_k$.

At this point, we can make some interesting observations and additions to the results obtained in [5]. For this, let us first consider the representations appearing here for $\mathfrak{so}(2k+1)$ and $\mathfrak{so}(2k)$. In (3) we obtained

$$\text{char}[0, \ldots, 0, p]_{\mathfrak{so}(2k+1)} = (x_1 \cdots x_k)^{-p/2} \sum_{\lambda_1 \leq p, \ell(\lambda) \leq k} s_\lambda(x).$$

(22)

Essentially, this is the branching $\mathfrak{so}(2k+1) \supset \mathfrak{gl}(k)$. But for this inclusion, there is an intermediate subalgebra: $\mathfrak{so}(2k+1) \supset \mathfrak{so}(2k) \supset \mathfrak{gl}(k)$. From Weyl's character formula, it is easy to deduce the branching of the above $\mathfrak{so}(2k+1)$ representation with respect to $\mathfrak{so}(2k)$:

$$\text{char}[0, \ldots, 0, p]_{\mathfrak{so}(2k+1)} = \sum_{r=0}^{p} \text{char}[0, \ldots, r, p-r]_{\mathfrak{so}(2k)}.$$  

(23)

The $\mathfrak{so}(2k)$ representations that appeared earlier, with expressions in terms of Schur functions, were $[0, \ldots, 0, p]$ and $[0, \ldots, 0, p, 0]$. So the question is now: how to write the character of the other $\mathfrak{so}(2k)$ representations $[0, \ldots, r, p-r]$ as a sum of Schur functions? Or in other words, what is the branching $\mathfrak{so}(2k) \supset \mathfrak{gl}(k)$ for these representations? The answer is:

**Theorem.** For $k$ even, one has

$$\text{char}[0, \ldots, 0, r, p-r]_{\mathfrak{so}(2k)} = (x_1 \cdots x_k)^{-p/2} \sum_{\lambda_1 \leq p, \ell(\lambda) \leq k; \lambda \in B_r} s_\lambda(x).$$

(24)

Herein, $B_r$ stands for the set of partitions of $B$ to which a horizontal strip of length $r$ is attached. (Recall that $B$ is the set of partitions for which each part appears twice.) The first condition ($\lambda_1 \leq p, \ell(\lambda) \leq k$) means that (the Young diagram of) $\lambda$ fits inside the $k \times p$ rectangle. Similarly, for $k$ odd:

$$\text{char}[0, \ldots, 0, r, p-r]_{\mathfrak{so}(2k)} = (x_1 \cdots x_k)^{-p/2} \sum_{\lambda_1 \leq p, \ell(\lambda) \leq k; \lambda \in B_{p-r}} s_\lambda(x).$$

(25)

We have not found the above result in the literature. The actual proof is rather technical. It can be obtained using the branching rules for $\mathfrak{so}(2k) \supset \mathfrak{gl}(k)$ described in [18]. Note that, in accordance with (23), the union of all partitions of $B_r$ in the $k \times p$ rectangle, for $r = 0, 1, \ldots, p$, is equal to the set of all partitions in the rectangle.

But now we can extend the analogy that we observed between representations $[0, \ldots, 0, p]$ of $\mathfrak{osp}(2m|2n)$ and those of $\mathfrak{so}(2k)$ for $m = n + k$. This leads to the following
Conjecture. For $|m - n|$ even, one has

$$\text{char}[0, \ldots, 0, r, p - r]_{\text{osp}(2m|2n)} = (y_1 \cdots y_n/x_1 \cdots x_m)^{p/2} \sum_{\lambda_1 \leq p, \lambda \in B_r} s_{\lambda}(x/y).$$ (26)

So in this case we have an expansion as an infinite sum of supersymmetric Schur functions, labeled by partitions $\lambda$ inside the $(m, n)$-hook, of width at most $p$, and belonging to $B_r$.

For $|m - n|$ odd, the result is similar, with $B_r$ replaced by $B_{p-r}$.

To conclude the paper, we have analyzed characters and superdimensions for representations of the form $[0, \ldots, 0, p]$ for $\text{osp}(2m + 1|2n)$, and of the form $[0, \ldots, 0, r, p - r]$ for $\text{osp}(2m|2n)$. It should be noted that characters for more general $\text{osp}(m|n)$ tensors have been studied in [19]. However, the formulas in [19] lead to alternating series of $S$-functions, which are not as easy to handle as the characters obtained here.

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