# Algebraic generalization of quantum statistics

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**Abstract.**  Generalized quantum statistics such as para-Bose and para-Fermi statistics are related to the basic classical Lie superalgebras $B(0|n)$ and $B_n$. We give a quite general definition of "a generalized quantum statistics associated to a Lie superalgebra $G$". This definition is closely related to a certain $\mathbb{Z}$-grading of $G$. The generalized quantum statistics is determined by a set of root vectors (the creation and annihilation operators of the statistics) and the set of algebraic relations for these operators. Then we give a complete classification of all generalized quantum statistics associated to the Lie superalgebras $A_n$, $B_n$, $C_n$, $D_n$, $G_2$, $F_4$, $E_6$, $E_7$, $E_8$, $A(m|n)$, $B(m|n)$, $C(n)$, $D(m|n)$, $G(3)$, $F(4)$ and $D(2,1;\alpha)$.

## 1. Introduction

Green [1] extended the ordinary Bose and Fermi statistics to parastatistics. Both for para-Bose and para-Fermi operators, the bilinear commutators or anticommutators for bosons and fermions are replaced by trilinear relations. These trilinear relations are closely related to a Lie algebra or Lie superalgebra. For example, a set of $2n$ para-Fermi operators $f_i^\xi$ ($\xi = \pm$, $i = 1, \ldots, n$)

$$[[f_j^\xi, f_k^\eta], f_l^\epsilon] = \frac{1}{2}(\epsilon - \eta)^2 \delta_{kl} f_j^\xi - \frac{1}{2}(\epsilon - \xi)^2 \delta_{jl} f_k^\eta, \ \xi, \eta, \epsilon = \pm \text{ or } \pm 1; \ j, k, l = 1, \ldots, n \ (1)$$

generates the Lie algebra $B_n = so(2n+1)$ [2], whereas a set of $2n$ para-Bose operators $b_i^\xi$ ($\xi = \pm$, $i = 1, \ldots, n$)

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi, \ \xi, \eta, \epsilon = \pm \text{ or } \pm 1; \ j, k, l = 1, \ldots, n \qquad (2)$$

considered as odd elements, generates the orthosymplectic Lie superalgebra $B(0|n) = osp(1|2n)$ [3]. Therefore parastatistics can be associated with representations of the Lie superalgebras of class $B$ [4]. Then it is natural to expect that new types of generalized quantum statistics (GQS) can be associated with other Lie superalgebras. Note that, for conveniece, we often consider Lie algebras as a subset of Lie superalgebras (namely as Lie superalgebras with even elements only).

Examples of alternative types of GQS have been considered by Palev [5]-[13]. Inspired by the examples of parastatistics and Palev's examples we give a classification of all GQS associated

with the Lie superalgebras $A_n$, $B_n$, $C_n$, $D_n$, $G_2$, $F_4$, $E_6$, $E_7$, $E_8$, $A(m|n)$, $B(m|n)$, $C(n)$, $D(m|n)$, $G(3)$, $F(4)$ and $D(2,1;\alpha)$. More details of the classification presented here can be found in the series of papers [14]-[16].

GQS associated to a Lie superalgebra $G$ is related to a $\mathbb{Z}$-grading of $G$ of the type $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$. Such a $\mathbb{Z}$-grading is said to be of length 5 if $G_{\pm 2} \neq 0$; if $G_{\pm 2} = 0$, but $G_{\pm 1} \neq 0$, then the $\mathbb{Z}$-grading is of length 3. These $\mathbb{Z}$-gradings imply that one is dealing with Lie (super)triple systems (in the case of Lie algebras, see [17] and in the case of Lie superalgebras, see [18]). Such systems have also been referred to as "(super)ternary algebras" in [19]. Another mathematical structure, $F$-Lie algebra, also called ternary algebra for $F = 3$ was introduced in [20]. However the systems considered here are in the sense of [19]. In the last paper, examples and explicit constructions of (super)ternary algebras are given for many simple Lie algebras and Lie superalgebras. In section VII of [19] one notes: "It would be interesting to embark on a complete classification of ternary algebras and superternary algebras and provide a list of all possible constructions of a given Lie (super)algebra from (super)ternary algebras." The results of the present paper provide such a complete classification.

In the next section we give a definition of generalized quantum statistics associated with a Lie superalgebra $G$ and the corresponding creation and annihilation operators. This notion is closely related to gradings of $G$, and to regular subalgebras of $G$. Following the definition, we describe the classification method. In the process of investigations Dynkin diagram techniques [21, 22] play a crucial role. In the remaining sections of the paper, the classification results are summarised.

## 2. Definition and classification method

Let $G$ be a Lie superalgebra [4], with bracket $[\![x, y]\!]$, where (in $U(G)$)

$$[\![x, y]\!] = xy - (-1)^{\deg(x)\deg(y)} yx,$$

if $x$ and $y$ are homogeneous (for a Lie algebra, all elements are even and have degree 0; for a Lie superalgebra, homogeneous elements are even or odd, having degree 0 or 1).

**Definition 1** *Let $G$ be a Lie superalgebra, with antilinear anti-involutive mapping $\omega$. A set of $2N$ root vectors $x_i^{\pm}$ $(i = 1, \ldots, N)$ is called a set of creation and annihilation operators (CAOs) for $G$ if:*

- $\omega(x_i^{\pm}) = x_i^{\mp}$,
- $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$ *is a $\mathbb{Z}$-grading of $G$, with $G_{\pm 1} = span\{x_i^{\pm}, \; i = 1 \ldots, N\}$ and $G_{j+k} = [\![G_j, G_k]\!]$.*

*The algebraic relations $\mathcal{R}$ satisfied by the operators $x_i^{\pm}$ are the relations of a generalized quantum statistics (GQS) associated with $G$.*

A consequence of this definition is that $G$ is generated by $G_{-1}$ and $G_{+1}$, i.e. by the set of CAOs, and since $G_{j+k} = [\![G_j, G_k]\!]$, it follows that

$$G = \text{span}\{x_i^{\xi}, \; [\![x_i^{\xi}, x_j^{\eta}]\!]; \quad i, j = 1, \ldots, N, \; \xi, \eta = \pm\}. \tag{3}$$

The second condition in Definition 1 implies that $G_0$ itself is a subalgebra of $G$ spanned by root vectors of $G$. So $G_0$ is a regular subalgebra containing the Cartan subalgebra $H$ of $G$. By the adjoint action, the remaining $G_i$'s are $G_0$-modules. This implies the following technique in order to obtain a complete classification of all GQS associated with $G$:

(i) Determine all regular subalgebras $G_0$ of $G$. If not yet contained in $G_0$, replace $G_0$ by $G_0 + H$.

(ii) For each regular subalgebra $G_0$, determine the decomposition of $G$ into simple $G_0$-modules $g_k$ $(k = 1, 2, \ldots)$.

(iii) Investigate whether there exists a $\mathbb{Z}$-grading of $G$ of the form

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}, \tag{4}$$

where each $G_i$ is either directly a module $g_k$ or else a sum of such modules $g_1 \oplus g_2 \oplus \cdots$, such that $\omega(G_{+i}) = G_{-i}$.

To find regular subalgebras one can use the method of deleting nodes from (extended) Dynkin diagrams [21, 22]. The second stage is straightforward by means of representation theoretical techniques. The third stage requires most of the work: one must try out all possible combinations of the $G_0$-modules $g_k$, and see whether it is possible to obtain a grading of the type (4). In this process, if one of the simple $G_0$-modules $g_k$ is such that $\omega(g_k) = g_k$, then it follows that this module should be part of $G_0$. In other words, such a case reduces essentially to another case with a larger regular subalgebra.

In the following sections we shall give a summary of the classification process for the simple Lie algebras and basic classical Lie superalgebras.

## 3. Simple Lie algebras

*3.1. The Lie algebra $A_n$*

Let $G$ be the special linear Lie algebra $sl(n+1) \equiv A_n$, consisting of traceless $(n+1) \times (n+1)$ matrices. The Cartan subalgebra $H$ of $G$ is the subspace of diagonal matrices. The root vectors of $G$ are the elements $e_{jk}$ $(j \neq k = 1, \ldots, n+1)$, where $e_{jk}$ is a matrix with zeros everywhere except a 1 on the intersection of row $j$ and column $k$. The corresponding root is $\epsilon_j - \epsilon_k$, in the usual basis. The anti-involution is such that $\omega(e_{jk}) = e_{kj}$. The simple roots, the Dynkin diagram and the extended Dynkin diagram of $A_n$ are given in Table 1.

In order to find regular subalgebras of $G = A_n$, one should delete nodes from the Dynkin diagram of $G$ or from its extended Dynkin diagram.

**Step 1.** Delete node $i$ from the Dynkin diagram. The corresponding diagram is the Dynkin diagram of $sl(i) \oplus sl(n-i+1)$, so $G_0 = H + sl(i) \oplus sl(n-i+1)$. In this case, there are only two $G_0$ modules and we can put

$$G_{-1} = \text{span}\{e_{kl}; \; k = 1, \ldots, i, \; l = i+1, \ldots, n+1\}, \quad G_{+1} = \omega(G_{-1}). \tag{5}$$

Therefore $sl(n+1)$ has the following grading:

$$sl(n+1) = G_{-1} \oplus G_0 \oplus G_{+1}, \tag{6}$$

the number of creation and annihilation operators is $N = i(n-i+1)$. Note that the cases $i$ and $n+1-i$ are isomorphic.

The most interesting cases are those with $i = 1$ and $i = 2$, for which we shall explicitly give the relations $\mathcal{R}$ between the CAOs.

For $i = 1$, $N = n$, the rank of $A_n$. Putting

$$a_j^- = e_{1,j+1}, \qquad a_j^+ = e_{j+1,1}, \; j = 1, \ldots, n, \tag{7}$$

the corresponding relations $\mathcal{R}$ read $(j, k, l = 1, \ldots, n)$:

$$\begin{aligned}
&[a_j^+, a_k^+] = [a_j^-, a_k^-] = 0, \\
&[[a_j^+, a_k^-], a_l^+] = \delta_{jk} a_l^+ + \delta_{kl} a_j^+, \\
&[[a_j^+, a_k^-], a_l^-] = -\delta_{jk} a_l^- - \delta_{jl} a_k^-.
\end{aligned} \tag{8}$$

**Table 1**. Simple Lie algebras, their (extended) Dynkin diagrams with a labelling of the nodes and the corresponding simple roots.

| LA | Dynkin diagram | extended Dynkin diagram |
|---|---|---|
| $A_n$ | $\underset{1}{\overset{\epsilon_1-\epsilon_2}{\circ}}-\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\cdots-\underset{n-1}{\overset{\epsilon_{n-1}-\epsilon_n}{\circ}}-\underset{n}{\overset{\epsilon_n-\epsilon_{n+1}}{\circ}}$ | top node $\underset{0}{\overset{\epsilon_{n+1}-\epsilon_1}{\circ}}$ connected to ends of $\underset{1}{\overset{\epsilon_1-\epsilon_2}{\circ}}-\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\cdots-\underset{n-1}{\overset{\epsilon_{n-1}-\epsilon_n}{\circ}}-\underset{n}{\overset{\epsilon_n-\epsilon_{n+1}}{\circ}}$ |
| $B_n$ | $\underset{1}{\overset{\epsilon_1-\epsilon_2}{\circ}}-\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\cdots-\underset{n-1}{\overset{\epsilon_{n-1}-\epsilon_n}{\circ}}\Rightarrow\underset{n}{\overset{\epsilon_n}{\circ}}$ | $\overset{\epsilon_1-\epsilon_2}{\underset{1}{\circ}}$ and $\overset{-\epsilon_1-\epsilon_2}{\underset{0}{\circ}}$ joined to $\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\cdots-\underset{n-1}{\overset{\epsilon_{n-1}-\epsilon_n}{\circ}}\Rightarrow\underset{n}{\overset{\epsilon_n}{\circ}}$ |
| $C_n$ | $\underset{1}{\overset{\epsilon_1-\epsilon_2}{\circ}}-\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\cdots-\underset{n-1}{\overset{\epsilon_{n-1}-\epsilon_n}{\circ}}\Leftarrow\underset{n}{\overset{\epsilon_n}{\circ}}$ | $\underset{0}{\overset{-2\epsilon_1}{\circ}}\Rightarrow\underset{1}{\overset{\epsilon_1-\epsilon_2}{\circ}}-\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\cdots-\underset{n-1}{\overset{\epsilon_{n-1}-\epsilon_n}{\circ}}\Leftarrow\underset{n}{\overset{\epsilon_n}{\circ}}$ |
| $D_n$ | $\underset{1}{\overset{\epsilon_1-\epsilon_2}{\circ}}-\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\cdots-\underset{n-2}{\overset{\epsilon_{n-2}-\epsilon_{n-1}}{\circ}}$ forking to $\overset{\epsilon_{n-1}-\epsilon_n}{\underset{n-1}{\circ}}$ and $\underset{n}{\overset{\epsilon_{n-1}+\epsilon_n}{\circ}}$ | $\overset{\epsilon_1-\epsilon_2}{\underset{1}{\circ}}$, $\overset{-\epsilon_1-\epsilon_2}{\underset{0}{\circ}}$ joined to $\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\cdots-\underset{n-2}{\overset{\epsilon_{n-2}-\epsilon_{n-1}}{\circ}}$ forking to $\overset{\epsilon_{n-1}-\epsilon_n}{\underset{n-1}{\circ}}$ and $\underset{n}{\overset{\epsilon_{n-1}+\epsilon_n}{\circ}}$ |
| $G_2$ | $\underset{1}{\overset{\epsilon_2+\epsilon_3-2\epsilon_1}{\circ}}\Rrightarrow\underset{2}{\overset{\epsilon_1-\epsilon_2}{\circ}}$ | $\underset{0}{\overset{\epsilon_1+\epsilon_2-2\epsilon_3}{\circ}}-\underset{1}{\overset{\epsilon_2+\epsilon_3-2\epsilon_1}{\circ}}\Rrightarrow\underset{2}{\overset{\epsilon_1-\epsilon_2}{\circ}}$ |
| $F_4$ | $\underset{1}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\underset{2}{\overset{\epsilon_3-\epsilon_4}{\circ}}\Rightarrow\underset{3}{\overset{\epsilon_4}{\circ}}-\underset{4}{\overset{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)}{\circ}}$ | $\underset{0}{\overset{-\epsilon_1-\epsilon_2}{\circ}}-\underset{1}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\underset{2}{\overset{\epsilon_3-\epsilon_4}{\circ}}\Rightarrow\underset{3}{\overset{\epsilon_4}{\circ}}-\underset{4}{\overset{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)}{\circ}}$ |
| $E_6$ | $\underset{1}{\overset{\epsilon_3-\epsilon_4}{\circ}}-\underset{2}{\overset{\epsilon_4-\epsilon_5}{\circ}}-\underset{3}{\overset{\epsilon_5-\epsilon_6}{\circ}}-\underset{4}{\overset{\epsilon_6-\epsilon_7}{\circ}}-\underset{5}{\overset{\epsilon_7-\epsilon_8}{\circ}}$ with node 6 $\overset{6}{\underset{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8)}{\circ}}$ below node 3 | $\underset{1}{\overset{\epsilon_3-\epsilon_4}{\circ}}-\underset{2}{\overset{\epsilon_4-\epsilon_5}{\circ}}-\underset{3}{\overset{\epsilon_5-\epsilon_6}{\circ}}-\underset{4}{\overset{\epsilon_6-\epsilon_7}{\circ}}-\underset{5}{\overset{\epsilon_7-\epsilon_8}{\circ}}$ with node 6 $\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8)$ then node 0 $-\epsilon_1+\epsilon_2$ below node 3 |
| $E_7$ | $\underset{1}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\underset{2}{\overset{\epsilon_3-\epsilon_4}{\circ}}-\underset{3}{\overset{\epsilon_4-\epsilon_5}{\circ}}-\underset{4}{\overset{\epsilon_5-\epsilon_6}{\circ}}-\underset{5}{\overset{\epsilon_6-\epsilon_7}{\circ}}-\underset{6}{\overset{\epsilon_7-\epsilon_8}{\circ}}$ with node 7 $\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8)$ below node 4 | $\underset{1}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\underset{2}{\overset{\epsilon_3-\epsilon_4}{\circ}}-\underset{3}{\overset{\epsilon_4-\epsilon_5}{\circ}}-\underset{4}{\overset{\epsilon_5-\epsilon_6}{\circ}}-\underset{5}{\overset{\epsilon_6-\epsilon_7}{\circ}}-\underset{6}{\overset{\epsilon_7-\epsilon_8}{\circ}}-\underset{0}{\overset{-\epsilon_1+\epsilon_8}{\circ}}$ with node 7 $\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8)$ below node 4 |
| $E_8$ | $\underset{1}{\overset{-\epsilon_1-\epsilon_2}{\circ}}-\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\underset{3}{\overset{\epsilon_3-\epsilon_4}{\circ}}-\underset{4}{\overset{\epsilon_4-\epsilon_5}{\circ}}-\underset{5}{\overset{\epsilon_5-\epsilon_6}{\circ}}-\underset{6}{\overset{\epsilon_6-\epsilon_7}{\circ}}-\underset{7}{\overset{\epsilon_7-\epsilon_8}{\circ}}$ with node 8 $\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8)$ below node 5 | $\underset{0}{\overset{\frac{1}{2}\sum_{i=1}^{8}\epsilon_i}{\circ}}-\underset{1}{\overset{-\epsilon_1-\epsilon_2}{\circ}}-\underset{2}{\overset{\epsilon_2-\epsilon_3}{\circ}}-\underset{3}{\overset{\epsilon_3-\epsilon_4}{\circ}}-\underset{4}{\overset{\epsilon_4-\epsilon_5}{\circ}}-\underset{5}{\overset{\epsilon_5-\epsilon_6}{\circ}}-\underset{6}{\overset{\epsilon_6-\epsilon_7}{\circ}}-\underset{7}{\overset{\epsilon_7-\epsilon_8}{\circ}}$ with node 8 $\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8)$ below node 5 |

These are the relations of the so called $A$-statistics [5]-[6], [9]-[11].

For $i = 2$, $N = 2(n-1)$, let

$$a^-_{-j} = e_{1,j+2}, \quad a^-_{+j} = e_{2,j+2}, \qquad j = 1, \dots, n-1,$$
$$a^+_{-j} = e_{j+2,1}, \quad a^+_{+j} = e_{j+2,2}, \qquad j = 1, \dots, n-1. \tag{9}$$

Now the corresponding relations are ($\xi, \eta, \epsilon = \pm$; $j, k, l = 1, \dots, n-1$):

$$[a^+_{\xi j}, a^+_{\eta k}] = [a^-_{\xi j}, a^-_{\eta k}] = 0,$$
$$[a^+_{\xi j}, a^-_{-\xi k}] = 0, \qquad j \neq k,$$
$$[a^+_{-j}, a^-_{-k}] = [a^+_{+j}, a^-_{+k}], \qquad j \neq k,$$
$$[a^+_{+j}, a^-_{-j}] = [a^+_{+k}, a^-_{-k}], \tag{10}$$
$$[a^+_{-j}, a^-_{+j}] = [a^+_{-k}, a^-_{+k}],$$
$$[[a^+_{\xi j}, a^-_{\eta k}], a^+_{\epsilon l}] = \delta_{\eta\epsilon}\delta_{jk}a^+_{\xi l} + \delta_{\xi\eta}\delta_{kl}a^+_{\epsilon j},$$
$$[[a^+_{\xi j}, a^-_{\eta k}], a^-_{\epsilon l}] = -\delta_{\xi\epsilon}\delta_{jk}a^-_{\eta l} - \delta_{\xi\eta}\delta_{jl}a^-_{\epsilon k}.$$

These relations are already more complicated than (8). But they are still defining relations for the Lie algebra $A_n$.

**Step 2.** Delete node $i$ and $j$ from the Dynkin diagram. By the symmetry of the Dynkin diagram, it is sufficient to consider $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $i < j < n + 1 - i$. We have $G_0 = H + sl(i) \oplus sl(j - i) \oplus sl(n + 1 - j)$. In this case, there are six simple $G_0$-modules. All the possible combinations of these modules give rise to gradings of the form

$$sl(n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}.$$

There are essentially three different ways in which these $G_0$-modules can be combined. To characterize these three cases, it is sufficient to give only $G_{-1}$:

$$G_{-1} = \text{span}\{e_{kl}, e_{lp}; \; k = 1, \dots, i, \; l = i+1, \dots, j, \; p = j+1, \dots, n+1\}, \tag{11}$$
$$\text{with } N = (j-i)(n+1-j+i);$$
$$G_{-1} = \text{span}\{e_{kl}, e_{pk}; \; k = 1, \dots, i, \; l = i+1, \dots, j, \; p = j+1, \dots, n+1\}, \tag{12}$$
$$\text{with } N = i(n+1-i);$$
$$G_{-1} = \text{span}\{e_{kl}, e_{lp}; \; k = 1, \dots, i, \; p = i+1, \dots, j, \; l = j+1, \dots, n+1\}, \tag{13}$$
$$\text{with } N = j(n+1-j).$$

It turns out that the sets of CAOs corresponding to (12) and (13) are isomorphic to (11), so it is sufficient to consider only (11). Each case of (11) with $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $i < j < n + 1 - i$ gives rise to a distinct GQS. For reasons explained earlier, we shall give the corresponding set of relations explicitly only for small $N$. In this case, it is interesting to give $\mathcal{R}$ for $j - i = 1$, because then the number of creation or annihilation operators is $N = n$. One can label the CAOs as follows:

$$a^-_k = e_{k,i+1}, \quad a^+_k = e_{i+1,k}, \qquad k = 1, \dots, i;$$
$$a^-_k = e_{i+1,k+1}, \quad a^+_k = e_{k+1,i+1}, \qquad k = i+1, \dots, n. \tag{14}$$

Using

$$\langle k \rangle = \begin{cases} 0 & \text{if } k = 1, \dots, i \\ 1 & \text{if } k = i+1, \dots, n \end{cases} \tag{15}$$

the quadratic and triple relations read:

$$[a_k^+, a_l^+] = [a_k^-, a_l^-] = 0, \qquad k, l = 1, \ldots, i \text{ or } k, l = i+1, \ldots, n,$$
$$[a_k^-, a_l^+] = [a_k^+, a_l^-] = 0, \qquad k = 1, \ldots, i, \ l = i+1, \ldots, n, \tag{16}$$
$$[[a_k^+, a_l^-], a_m^+] = (-1)^{\langle l \rangle + \langle m \rangle} \delta_{kl} a_m^+ + (-1)^{\langle l \rangle + \langle m \rangle} \delta_{lm} a_k^+, \ k, l = 1, \ldots, i \text{ or } k, l = i+1, \ldots, n,$$
$$[[a_k^+, a_l^-], a_m^-] = -(-1)^{\langle l \rangle + \langle m \rangle} \delta_{kl} a_m^- - (-1)^{\langle l \rangle + \langle m \rangle} \delta_{km} a_l^-, \ k, l = 1, \ldots, i \text{ or } k, l = i+1, \ldots, n,$$
$$[[a_k^\xi, a_l^\xi], a_m^{-\xi}] = -\delta_{km} a_l^\xi + \delta_{lm} a_k^\xi, \ k = 1, \ldots, i, \ l = i+1, \ldots, n,$$
$$[[a_k^\xi, a_l^\xi], a_m^\xi] = 0, \qquad (\xi = \pm; \ k, l, m = 1, \ldots, n).$$

The existence of the set of CAOs (14) is pointed out in [5] as a possible example. The relations (16) with $n = 2m$ and $i = m$ are the commutation relations of the so called causal A-statistics investigated in [8].

**Step 3.** If we delete 3 or more nodes from the Dynkin diagram, the resulting $\mathbb{Z}$-gradings of $sl(n+1)$ are no longer of the form $sl(n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}$, but there would be non-zero $G_i$ with $|i| > 2$, so these cases are not relevant for our classification.

**Step 4.** Next, we move on to the extended Dynkin diagram of $G$. If we delete node $i$ from the extended Dynkin diagram, then remaining diagram is again of type $A_n$, so $G_0 = G$, and there are no CAOs.

**Step 5.** If we delete node $i$ and $j$ from the extended Dynkin diagram $(0 \leq i < j \leq n+1)$, then $sl(n+1) = G_{-1} \oplus G_0 \oplus G_{+1}$ with $G_0 = H + sl(j-i) \oplus sl(n-j+i+1)$, and

$$G_{-1} = \text{span}\{e_{kl}; \ k = i+1 \ldots, j, \ l \neq i+1, \ldots, j\}.$$

The number of annihilation operators is $N = (j-i)(n+1-j+i)$. It is not difficult to see that all these cases are isomorphic to those of Step 1. This can also be deduced from the symmetry of the Dynkin diagram.

**Step 6.** If we delete nodes $i$, $j$ and $k$ from the extended Dynkin diagram $(i < j < k)$, then the corresponding $\mathbb{Z}$-gradings are of the form

$$sl(n+1) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_{+1} \oplus G_{+2}.$$

All the corresponding CAOs, however, are isomorphic to those of Step 2 (which can again be seen from the remaining Dynkin diagram).

**Step 7.** If we delete 4 or more nodes from the extended Dynkin diagram, the corresponding $\mathbb{Z}$-grading of $sl(n+1)$ has no longer the required properties (i.e. there are non-zero subspaces $G_i$ with $|i| > 2$).

*3.2. The Lie algebras $B_n$, $C_n$ and $D_n$*
$G = so(2n+1) \equiv B_n$ is the subalgebra of $sl(2n+1)$ consisting of matrices of the form:

$$\begin{pmatrix} a & b & c \\ d & -a^t & e \\ -e^t & -c^t & 0 \end{pmatrix}, \tag{17}$$

where $a$ is any $(n \times n)$-matrix, $b$ and $d$ are antisymmetric $(n \times n)$-matrices, and $c$ and $e$ are $(n \times 1)$-matrices. The Cartan subalgebra $H$ of $G$ is again the subspace of diagonal matrices.

The root vectors and corresponding roots of $G$ are given by:

$$
\begin{aligned}
e_{jk} - e_{k+n,j+n} &\leftrightarrow \epsilon_j - \epsilon_k, & j \neq k = 1, \ldots, n, \\
e_{j,k+n} - e_{k,j+n} &\leftrightarrow \epsilon_j + \epsilon_k, & j < k = 1, \ldots, n, \\
e_{j+n,k} - e_{k+n,j} &\leftrightarrow -\epsilon_j - \epsilon_k, & j < k = 1, \ldots, n, \\
e_{j,2n+1} - e_{2n+1,j+n} &\leftrightarrow \epsilon_j, & j = 1, \ldots, n, \\
e_{n+j,2n+1} - e_{2n+1,j} &\leftrightarrow -\epsilon_j, & j = 1, \ldots, n.
\end{aligned}
$$

$G = sp(2n) \equiv C_n$ is the subalgebra of $sl(2n)$ consisting of matrices of the form:

$$
\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \tag{18}
$$

where $a$ is any $(n{\times}n)$-matrix, and $b$ and $c$ are symmetric $(n{\times}n)$-matrices. The Cartan subalgebra $H$ consist of the diagonal matrices, and the root vectors and corresponding roots of $G$ are:

$$
\begin{aligned}
e_{jk} - e_{k+n,j+n} &\leftrightarrow \epsilon_j - \epsilon_k, & j \neq k = 1, \ldots, n, \\
e_{j,k+n} + e_{k,j+n} &\leftrightarrow \epsilon_j + \epsilon_k, & j \leq k = 1, \ldots, n, \\
e_{j+n,k} + e_{k+n,j} &\leftrightarrow -\epsilon_j - \epsilon_k, & j \leq k = 1, \ldots, n.
\end{aligned}
$$

$G = so(2n) \equiv D_n$ is the subalgebra of $sl(2n)$ consisting of matrices of the form:

$$
\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \tag{19}
$$

where $a$ is any $(n \times n)$-matrix, and $b$ and $c$ are antisymmetric $(n \times n)$-matrices. The Cartan subalgebra $H$ consist of the diagonal matrices, and the root vectors and corresponding roots of $G$ are:

$$
\begin{aligned}
e_{jk} - e_{k+n,j+n} &\leftrightarrow \epsilon_j - \epsilon_k, & j \neq k = 1, \ldots, n, \\
e_{j,k+n} - e_{k,j+n} &\leftrightarrow \epsilon_j + \epsilon_k, & j < k = 1, \ldots, n, \\
e_{j+n,k} - e_{k+n,j} &\leftrightarrow -\epsilon_j - \epsilon_k, & j < k = 1, \ldots, n.
\end{aligned}
$$

The simple roots, Dynkin diagrams and extended Dynkin diagrams of $B_n$, $C_n$ and $D_n$ are given in Table 1. The anti-involution is such that $\omega(e_{jk}) = e_{kj}$. Just as for $A_n$, we have performed the process of deleting nodes from the Dynkin diagrams and from the extended Dynkin diagrams of $B_n$, $C_n$ and $D_n$. The results are summarized in Table 2.

*3.3. The exceptional Lie algebras*

The Lie algebra $G_2$ of rank 2 has dimension 14. In terms of the orthonormal vectors $\epsilon_1, \epsilon_2, \epsilon_3$ such that $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$, the root system is given by

$$
\Delta = \{\epsilon_i - \epsilon_j, \ \epsilon_i + \epsilon_j - 2\epsilon_k \ (1 \leq i \neq j \neq k \leq 3)\}. \tag{20}
$$

The simple root system is

$$
\Pi = \{\alpha_1 = \epsilon_2 + \epsilon_3 - 2\epsilon_1, \ \alpha_2 = \epsilon_1 - \epsilon_2\} \tag{21}
$$

and the corresponding Dynkin diagram and extended Dynkin diagram are given in Table 1. The process described in the previous section, deleting nodes from the (extended) Dynkin diagram, leads to the results in Table 2.

**Table 2**. Summary of the classification for simple Lie algebras: all non-isomorphic GQS are given. For each GQS, we list: the subalgebra $G_0$ (each $G_0$ contains the complete Cartan subalgebra $H$, so we only list the remaining part of $G_0 = H + \cdots$); the length $l$ of the $\mathbb{Z}$-grading (3 or 5), $\dim G_1 = \dim G_{-1}$, which is also the number $N$ of creation and annihilation operators.

| Lie algebra | $G_0 = H+$ | $\ell$ | $\dim G_1 = N$ |
|---|---|---|---|
| $A_n$ | $sl(i) \oplus sl(n-i+1), i \leq \lfloor \frac{n+1}{2} \rfloor$ | 3 | $i(n+1-i)$ |
| | $sl(i) \oplus sl(j-i) \oplus sl(n+1-j), (i \leq \lfloor \frac{n}{2} \rfloor, i < j < n+1-i)$ | 5 | $(j-i)(n+1-j+i)$ |
| $B_n$ | $so(2n-1)$ | 3 | $2n-1$ |
| | $sl(i) \oplus so(2(n-i)+1), (2 \leq i \leq n)$ | 5 | $2i(n-i)+i$ |
| $C_n$ | $sl(i) \oplus sp(2(n-i)), (1 \leq i \leq n-1)$ | 3 | $2i(n-i)$ |
| | $sl(n)$ | 3 | $\frac{n(n+1)}{2}$ |
| $D_n$ | $so(2n-2)$ | 3 | $2(n-1)$ |
| | $sl(i) \oplus so(2(n-i)), (2 \leq i \leq n-3)$ | 5 | $2i(n-i)$ |
| | $sl(n-2) \oplus sl(2) \oplus sl(2)$ | 5 | $4(n-2)$ |
| | $sl(n-1)$ | 5 | $2(n-1)$ and $\frac{n(n-1)}{2}$ |
| | $sl(n)$ | 3 | $\frac{n(n-1)}{2}$ |
| $G_2$ | $sl(2)$ | 5 | 4 |
| $F_4$ | $sp(6)$ | 5 | 14 |
| | $so(7)$ | 5 | 8 |
| $E_6$ | $so(10)$ | 3 | 16 |
| | $sl(2) \oplus sl(5)$ | 5 | 20 |
| | $sl(6)$ | 5 | 20 |
| | $so(8)$ | 5 | 16 |
| $E_7$ | $E_6$ | 3 | 27 |
| | $sl(2) \oplus so(10)$ | 5 | 32 |
| | $so(12)$ | 5 | 32 |
| | $sl(7)$ | 5 | 35 |
| $E_8$ | $E_7$ | 5 | 56 |
| | $so(14)$ | 5 | 64 |

Let us next consider the Lie algebra $F_4$, of rank 4 and dimension 52. In terms of the orthonormal vectors $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ the root system is given by

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j \, (1 \leq i \neq j \leq 4); \, \pm\epsilon_j \, (1 \leq j \leq 4); \, \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}. \tag{22}$$

The simple root system is

$$\Pi = \{\alpha_1 = \epsilon_2 - \epsilon_3, \, \alpha_2 = \epsilon_3 - \epsilon_4, \, \alpha_3 = \epsilon_4, \, \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\} \tag{23}$$

and the corresponding Dynkin diagram and the extended Dynkin diagram are given in Table 1. We have considered now the various ways of deleting nodes from these diagrams, and investigate whether they give rise to $\mathbb{Z}$-gradings of the type (4) giving the result in Table 2.

For the remaining exceptional Lie algebras $E_6$, $E_7$ and $E_8$, our labeling of simple roots is again the usual one. But our choice of (simple) roots in terms of vectors $\epsilon_i$ is slightly different. For $E_8$, our choice is essentially the same as in [23, Table 1], except that we work with an independent basis $\epsilon_i$ in $R^8$ (and not a redundant basis in $R^9$). The roots for $E_8$ are the same as in [24] (but in this last textbook the choice of simple roots is different). For $E_7$ and $E_6$ it is

convenient for us to take the same root space as for $E_8$. The simple roots of $E_7$ are then those of $E_8$ with the first one deleted, and the simple roots of $E_6$ are then those of $E_7$ with the first one deleted.

The Lie algebra $E_6$ of rank 6 has dimension 78. We will use the following root system of $E_6$. Consider the 8-dimensional real vector space $R^8$ with orthonormal basis vectors $\epsilon_i$ $(i = 1, \ldots, 8)$. The roots of $E_6$ are elements of the 6-dimensional subspace $V$ of $R^8$ consisting of those elements $\sum_{i=1}^{8} c_i \epsilon_i$ with $c_1 + c_2 = 0$ and $\sum_{i=3}^{8} c_i = 0$. A set of simple roots of $E_6$ is then given by the elements

$$\alpha_i = \epsilon_{i+2} - \epsilon_{i+3} \ (i = 1, \ldots, 5), \quad \alpha_6 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8). \tag{24}$$

All 72 nonzero roots are given by

$$\pm(\epsilon_i - \epsilon_j), \quad (1 \leq i \neq j \leq 2 \text{ or } 3 \leq i \neq j \leq 8)$$
$$\frac{1}{2}(\sum_{i=1}^{8}(-1)^{a_i}\epsilon_i), \quad (a_i \in \{0,1\}; \ \sum_{i=1}^{2} a_i = 1, \ \sum_{i=3}^{8} a_i = 3). \tag{25}$$

The corresponding Dynkin diagram and the extended Dynkin diagram are given in Table 1.

The roots of $E_7$ are elements of the 7-dimensional subspace $V'$ consisting of elements $\sum_{i=1}^{8} c_i \epsilon_i$ with $\sum_{i=1}^{8} c_i = 0$. A set of simple roots of $E_7$ consists of the six simple roots $\alpha_i$ $(i = 1, \ldots, 6)$ of $E_6$ plus the extra root

$$\epsilon_2 - \epsilon_3. \tag{26}$$

The corresponding Dynkin diagram and the extended Dynkin diagram are given in Table 1.

By construction, the $E_6$ subsystem of $E_7$ is evident. The nonzero roots of $E_7$ consist of

$$\pm(\epsilon_i - \epsilon_j), \quad (1 \leq i < j \leq 8);$$
$$\frac{1}{2}(\sum_{i=1}^{8}(-1)^{a_i}\epsilon_i), \quad (a_i \in \{0,1\}; \ \sum_{i=1}^{8} a_i = 4). \tag{27}$$

Note that the 72 nonzero roots of $E_6$ are indeed part of the 126 nonzero roots (27).

In terms of the orthonormal vectors $\epsilon_i$, $i = 1, \ldots, 8$ the root system of $E_8$ is given by

$$\pm\epsilon_i \pm \epsilon_j, \quad (1 \leq i < j \leq 8);$$
$$\frac{1}{2}\sum_{i=1}^{8} \xi_i \epsilon_i, \quad \xi_i = \pm 1 \text{ and the number of } \xi_i = +1 \text{ is even.} \tag{28}$$

A set of simple roots of $E_8$ consists of the seven simple roots $\alpha_i$ $(i = 1, \ldots, 7)$ of $E_7$ plus the extra root

$$-\epsilon_1 - \epsilon_2. \tag{29}$$

The corresponding Dynkin diagram and the extended Dynkin diagram are given in Table 1, whereas summary of the GQSs associated with the Lie algebras $E_6$, $E_7$ and $E_8$ are presented in Table 2.
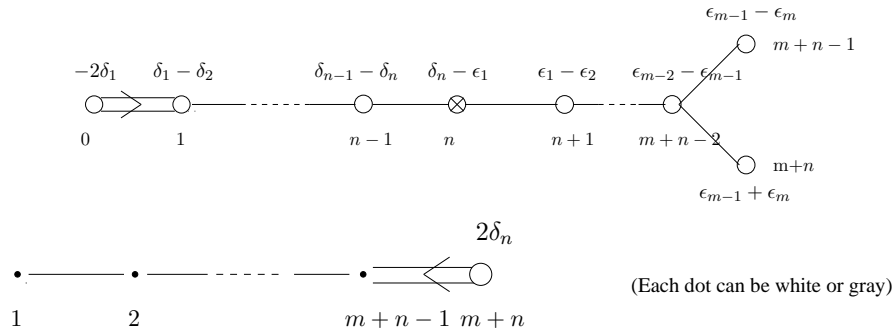
**4. The basic classical Lie superalgebras**

Let us note first that for the basic classical Lie superalgebras, the description by means of a Dynkin diagram is not unique: besides the so-called distinguished Dynkin diagram, other non-equivalent Dynkin diagrams exist [4], [25]. This feature makes it harder to obtain a
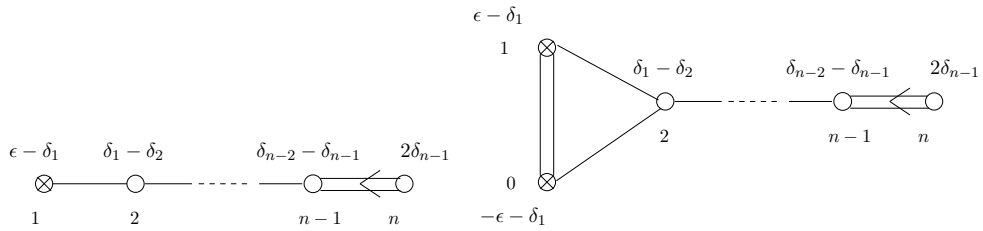
complete classification of all generalized quantum systems since one should repeat the process of investigation for all non distinguished Dynkin diagrams and their extensions. In Table 3 we give all (extended) Dynkin diagrams of basic classical Lie superalgebras, relevant for the classification process. All GQS associated with the basic classical Lie superalgebras are listed in Table 4.

**Table 3**. Basic classical Lie superalgebras, the (extended) Dynkin diagrams, relevant for the classification process, with a labeling of the nodes and the corresponding simple roots.
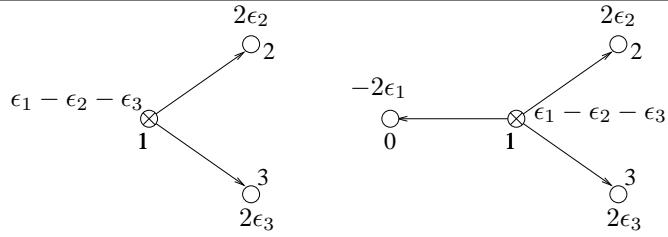
| LSA | Dynkin diagram and extended Dynkin diagram |
|---|---|

Top diagram:

$$-2\delta_1 \quad \delta_1-\delta_2 \qquad \delta_{n-1}-\delta_n \quad \delta_n-\epsilon_1 \quad \epsilon_1-\epsilon_2 \quad \epsilon_{m-2}-\epsilon_{m-1} \qquad \epsilon_{m-1}-\epsilon_m$$

$$m+n-1$$
$$0 \qquad 1 \qquad\qquad n-1 \qquad n \qquad n+1 \qquad m+n-2$$
$$\epsilon_{m-1}+\epsilon_m \qquad \text{m}+n$$

$$2\delta_n$$

(Each dot can be white or gray)

$$1 \qquad 2 \qquad\qquad m+n-1 \quad m+n$$

---

$C(n)$

$$\epsilon-\delta_1 \quad \delta_1-\delta_2 \qquad \delta_{n-2}-\delta_{n-1} \quad 2\delta_{n-1}$$
$$1 \qquad 2 \qquad\qquad n-1 \qquad n$$

$$\epsilon-\delta_1$$
$$1$$
$$\delta_1-\delta_2 \qquad \delta_{n-2}-\delta_{n-1} \quad 2\delta_{n-1}$$
$$2 \qquad\qquad n-1 \qquad n$$
$$0$$
$$-\epsilon-\delta_1$$

---

$D(2,1;\alpha)$

$$2\epsilon_2 \qquad\qquad 2\epsilon_2$$
$$2 \qquad\qquad 2$$
$$\epsilon_1-\epsilon_2-\epsilon_3 \qquad\qquad -2\epsilon_1 \qquad \epsilon_1-\epsilon_2-\epsilon_3$$
$$1 \qquad\qquad 0 \qquad 1$$
$$3 \qquad\qquad 3$$
$$2\epsilon_3 \qquad\qquad 2\epsilon_3$$

---

$G(3)$

$$\delta+\epsilon_1 \qquad \epsilon_2 \qquad -\epsilon_2+\epsilon_3 \qquad -2\delta \quad \delta+\epsilon_1 \qquad \epsilon_2 \qquad -\epsilon_2+\epsilon_3$$
$$1 \qquad 2 \qquad 3 \qquad 0 \qquad 1 \qquad 2 \qquad 3$$

$$-\delta-\epsilon_1 \quad \delta-\epsilon_3 \quad -\epsilon_2+\epsilon_3$$
$$1 \qquad 2 \qquad 3$$
$$\epsilon_1-\delta$$
$$0$$

$$-\delta-\epsilon_1 \quad \delta-\epsilon_3 \quad -\epsilon_2+\epsilon_3$$
$$1 \qquad 2 \qquad 3$$

$$-\epsilon_2+\epsilon_3 \quad -\delta+\epsilon_2 \quad \delta \qquad \epsilon_1-\epsilon_3 \quad -\epsilon_2+\epsilon_3 \quad -\delta+\epsilon_2 \quad \delta$$
$$1 \qquad 2 \qquad 3 \qquad 0 \qquad 1 \qquad 2 \qquad 3$$

---

$F(4)$

$$\tfrac{1}{2}(-\epsilon_1-\epsilon_2-\epsilon_3+\delta) \quad \epsilon_3 \quad \epsilon_2-\epsilon_3 \quad \epsilon_1-\epsilon_2$$
$$1 \qquad 2 \qquad 3 \qquad 4$$

$$\delta \quad \tfrac{1}{2}(-\sum_{i=1}^3 \epsilon_i+\delta) \quad \epsilon_3 \quad \epsilon_2-\epsilon_3 \quad \epsilon_1-\epsilon_2$$
$$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4$$

$$-\tfrac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3-\delta)$$
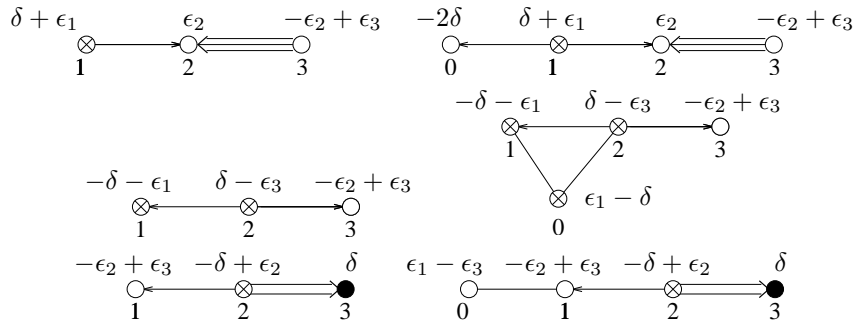$$\tfrac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3-\delta) \qquad \epsilon_2-\epsilon_3 \quad \epsilon_1-\epsilon_2$$
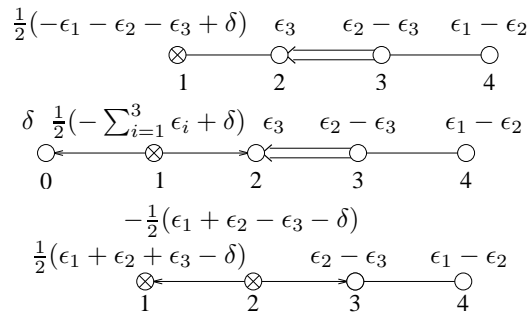$$1 \qquad 2 \qquad 3 \qquad 4$$

**Table 4**. Summary of the classification for basic classical Lie superalgebras: all non-isomorphic GQS are given. For each GQS, we list: the subalgebra $G_0$ (each $G_0$ contains the complete Cartan subalgebra $H$, so we only list the remaining part of $G_0 = H + \cdots$); the length $\ell$ of the $\mathbb{Z}$-grading; $\dim G_{-1} = \dim G_{+1}$, which is also the number $N$ of creation or annihilation operators

| LSA | $G_0 = H + \cdots$ | $\ell$ | $\dim G_1 = N$ |
|---|---|---|---|
| $A(m|n)$ | $sl(k|l) \oplus sl(p|q)$ | 3 | $(k+l)(p+q)$ |
| | $(k+p=m+1,\ l+q=n+1,$ | | |
| | $k+l \neq 0,\ p+q \neq 0)$ | | |
| | $sl(k|l) \oplus sl(p|q) \oplus sl(r|s)$ | 5 | $(k+l)(p+q+r+s)$ |
| | $(k+p+r=m+1,$ | 5 | $(p+q)(k+l+r+s)$ |
| | $l+q+s=n+1,$ | 5 | $(r+s)(k+l+p+q)$ |
| | $k+l \neq 0,\ p+q \neq 0,\ r+s \neq 0)$ | | |
| $B(m|n)$ | $sl(k|l) \oplus B(m-k|n-l)$ | 5 | $(k+l)(2m-2k+2n-2l+1)$ , |
| | $(k=0,\ldots,m;\ l=0,\ldots,n;$ | | |
| | $(k,l) \notin \{(0,0),(1,0)\})$ | | |
| | $B(m-1|n) \qquad [(k,l)=(1,0)]$ | 3 | $2m+2n-1$ |
| $B(0|n)$ | $sl(i) \oplus B(0|n-i)$ | 5 | $i(2n-2i+1)$ |
| | $(i=1,\ldots,n)$ | | |
| $D(m|n)$ | $sl(k|l) \oplus D(m-k|n-l)$ | 5 | $2(k+l)(m+n-k-l)$ , |
| | $(k=0,1,\ldots,m;$ | | |
| | $l=0,1,\ldots,n;$ | | |
| | $(k,l) \notin \{(0,0),(1,0),(m-1,n),(m,n)\})$ | | |
| | $D(m-1|n) \qquad [(k,l)=(1,0)]$ | 3 | $2(m+n-1)$ |
| | $sl(m|n) \qquad [(k,l)=[m,n]]$ | 3 | $(m+n)(m+n+1)/2 - m$ |
| | $sl(m-1|n) \qquad [(k,l)=(m-1,n)]$ | 5 | $(m+n)(m+n+1)/2 - m$ |
| | $sl(m-1|n) \qquad [(k,l)=(m-1,n)]$ | 5 | $2(m+n-1)$ |
| $C(n)$ | $sl(k|l) \oplus D(1-k|n-1-l)$ | 5 | $2(k+l)(n-k-l)$ |
| | $(k=0,1;\ l=1,\ldots,n-2)$ | | |
| | $C_{n-1} \qquad [(k,l)=(1,0)]$ | 3 | $2(n-1)$ |
| | $sl(1|n-1) \qquad [(k,l)=(1,n-1)]$ | 3 | $n(n+1)/2 - 1$ |
| | $sl(n-1) \qquad [(k,l)=(0,n-1)]$ | 5 | $n(n+1)/2 - 1$ |
| | $sl(n-1) \qquad [(k,l)=(0,n-1)]$ | 5 | $2(n-1)$ |
| $D(2,1;\alpha)$ | $sl(2) \oplus sl(2)$ | 5 | 4 |
| | $sl(1|2)$ | 3 | 4 |
| | $sl(1|1)$ | 5 | 4 |
| $G(3)$ | $G_2$ | 5 | 7 |
| | $sl(1|2)$ | 5 | 7 |
| | $sl(3|1)$ | 3 | 8 |
| | $osp(3|2)$ | 5 | 8 |
| | $sl(3) \oplus osp(1|2)$ | 3 | 9 |
| $F(4)$ | $so(7)$ | 5 | 8 |
| | $sl(1|2) \oplus sl(2)$ | 5 | 10 |
| | $osp(2|4)$ | 3 | 10 |
| | $sl(2) \oplus so(5)$ | 5 | 8 |
| | $D(2,1;-1/3)$ | 5 | 10 |
| | $sl(3|1)$ | 5 | 8 |

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