Macroscopic properties of A-statistics and A-superstatistics

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Abstract. In the context of Lie (super)algebras of type A, generalized quantum statistics is considered. For $sl(n+1)$ this is A-statistics, and for $sl(1|n)$ this is A-superstatistics. The creation and annihilation operators, their triple relations, and their Fock spaces are given. We consider a 'free' Hamiltonian and some macroscopic properties are studied : the grand partition function, average number of particles, etc. Symmetric functions play a role in expressions for these thermodynamical quantities.

Many years after Green [1] introduced para-Bose and para-Fermi statistics, as generalizations of Bose-Einstein and Fermi-Dirac statistics, the algebraic structure of these statistics became clear. It was shown [2, 3] that n pairs of parafermions generate the Lie algebra $so(2n+1) = B_n$, and that n pairs of parabosons generate [4] the Lie superalgebra $osp(1|2n) = B(0, n)$. So para-Fermi and para-Bose statistics fall into the class of B-statistics. In this paper, we shall consider the generalized statistics related to the Lie algebra $A_n = sl(n + 1)$ (A-statistics) and to the Lie superalgebra $A(0, n) = sl(1|n)$ (A-superstatistics) and some of their thermodynamical quantities. We first summarize the properties of A-statistics, given in [5], in order to compare them with those of A-superstatistics (the details of which will be published elsewhere).

The CAO's (creation and annihilation operators) of A-statistics are defined [6] as Jacobson generators $a_1^\pm, a_2^\pm, \ldots, a_n^\pm$ of $sl(n + 1)$ satisfying

$$\begin{aligned}
&[[a_i^+, a_j^-], a_k^+] = \delta_{kj} a_i^+ + \delta_{ij} a_k^+,\\
&[[a_i^+, a_j^-], a_k^-] = -\delta_{ki} a_j^- - \delta_{ij} a_k^-,\\
&[a_i^+, a_j^+] = [a_i^-, a_j^-] = 0.
\end{aligned} \tag{1}$$

The Lie algebra generated by the $2n$ elements $a_i^\pm$ ($i = 1, \ldots, n$) subject to the relations (1) is $sl(n + 1)$ [6]. In terms of the usual Weyl generators e_{ij} ($i, j = 0, 1, \ldots, n$) of $sl(n + 1)$, one can take $a_i^+ = e_{i0}$ and $a_i^- = e_{0i}$. In [5], the Fock spaces $W(p, n)$ of A-statistics are constructed. These are labelled by $p \in \mathbb{N}$, and can be defined by a vacuum vector $|0\rangle$ and the relations

$$a_k^-|0\rangle = 0, \qquad a_i^- a_j^+|0\rangle = \delta_{ij} p|0\rangle. \tag{2}$$

$W(p, n)$ is a finite-dimensional irreducible $sl(n + 1)$-module with a basis consisting of all vectors $(a_1^+)^{l_1}(a_2^+)^{l_2} \ldots (a_n^+)^{l_n}|0\rangle$ subject to the restriction

$$|l| = l_1 + l_2 + \cdots + l_n \le p. \tag{3}$$

The usual Hermitian form on $W(p, n)$ gives an inner product, and the vectors

$$|p; l_1, \ldots, l_n\rangle = \sqrt{\frac{(p - |l|)!}{p!}} \frac{(a_1^+)^{l_1} \cdots (a_n^+)^{l_n}}{\sqrt{l_1! l_2! \cdots l_n!}} |0\rangle, \tag{4}$$

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($|l| = l_1 + l_2 + \cdots + l_n \leq p$) constitute an orthonormal basis. The action of the CAO's of A_n on this basis is given by :

$$a_i^+|p;l_1,\ldots,l_n\rangle = \sqrt{(l_i+1)(p-|l|)}\,|p;l_i+1\rangle, \tag{5}$$

$$a_i^-|p;l_1,\ldots,l_n\rangle = \sqrt{l_i(p-|l|+1)}\,|p;l_i-1\rangle, \tag{6}$$

where in the rhs we have written only the label that changes. The Fock space $W(p,n)$ is also an irreducible $gl(n+1)$ module, and denoting the Weyl generators e_{ii} by N_i, one has

$$N_i|p;l_1,\ldots,l_n\rangle = l_i|p;l_1,\ldots,l_n\rangle, \quad i=1,\ldots,n. \tag{7}$$

The Hamiltonian for which macroscopic properties will be studied is the "free" Hamiltonian $H = \sum_{i=1}^n \epsilon_i N_i$. This Hamiltonian satisfies $[H, a_i^\pm] = \pm\epsilon_i a_i^\pm$, so a_i^+ (a_i^-) can be interpreted as an operator creating (annihilating) a particle (quasiparticle, excitation) on orbital i with energy ϵ_i. Since

$$H|p;l_1,l_2,\ldots,l_n\rangle = (\epsilon_1 l_1 + \epsilon_2 l_2 + \cdots + \epsilon_n l_n)|p;l_1,l_2,\ldots,l_n\rangle, \tag{8}$$

the state $|p;l_1,\ldots,l_n\rangle$ can be interpreted as a state with l_1 particles on the first orbital, l_2 particles on the second, etc. In our picture, H has the form of a free Hamiltonian, and the interaction is inherently introduced via the statistical properties of A-statistics. These statistical properties follow from the structure of the Fock spaces; in particular, (3) leads to the Pauli principle of A-statistics : if the system is in the representation $W(p,n)$, it cannot accommodate more than p particles.

In order to study macroscopic properties of A-statistics, denote by τ the (fundamental) temperature of the system and let μ_i be the chemical potential for particles on orbital i. The probability $\mathcal{P}(p,n;r)$ for the system to be in a (quantum) state $r = (l_1,\ldots,l_n)$ (with $N_r = l_1 + \cdots + l_n$ and energy $E_r = l_1\epsilon_1 + \cdots + l_n\epsilon_n$) is given by :

$$\mathcal{P}(p,n;r) = \exp\Big(\sum_{i=1}^n (\frac{\mu_i-\epsilon_i}{\tau}) l_i\Big)/Z(p,n), \tag{9}$$

where $Z(p,n)$ is the grand partition function (GPF) :

$$Z(p,n) = \sum_{0\leq l_1+\cdots+l_n\leq p} \Big(\exp(\frac{\mu_1-\epsilon_1}{\tau})\Big)^{l_1} \cdots \Big(\exp(\frac{\mu_n-\epsilon_n}{\tau})\Big)^{l_n}. \tag{10}$$

Using the notation $x_i = \exp(\frac{\mu_i-\epsilon_i}{\tau})$, $i=1,\ldots,n$, one can show that this is equal to

$$\sum_{0\leq l_1+\cdots+l_n\leq p} x_1^{l_1}x_2^{l_2}\cdots x_n^{l_n} = \sum_{k=0}^p h_k(x_1,\ldots,x_n) = h_p(x_1,\ldots,x_n,1), \tag{11}$$

where h_k are the *complete symmetric functions* [7]. Once this has been realized, also various other thermodynamical quantities (distribution function, average energy, ...) can be expressed in terms of these symmetric functions [5]. In that same paper, some special cases are considered. One of these is the degenerate case, where all orbitals i have the same energy and chemical potential; then the various thermodynamical functions simplify and they can easily be compared with the case of Bose-Einstein statistics. Another special case is that of equidistant energy levels, also leading to interesting specializations of the general thermodynamical functions.

Let us now describe the situation of A-superstatistics. Here, the CAO's are defined as (odd) Jacobson generators $f_i^\pm$ ($i=1,\ldots,n$) of the Lie superalgebra $sl(1|n)$ satisfying

$$\begin{aligned}
&\{f_i^+, f_j^+\} = \{f_i^-, f_j^-\} = 0, \\
&[\{f_i^+, f_j^-\}, f_k^+] = \delta_{jk} f_i^+ - \delta_{ij} f_k^+, \\
&[\{f_i^+, f_j^-\}, f_k^-] = -\delta_{ik} f_j^- + \delta_{ij} f_k^-.
\end{aligned} \tag{12}$$

The Lie superalgebra generated by the $2n$ odd elements $f_i^{\pm}$ $(i = 1, \ldots, n)$ subject to the relations (12) is $A(0, n) = sl(1|n)$ [8]. In terms of the usual Weyl generators e_{ij} $(i, j = 0, 1, \ldots, n)$ of $sl(1|n)$, one can again take $a_i^+ = e_{i0}$ and $a_i^- = e_{0i}$. The Fock spaces $V(p, n)$ of A-superstatistics are also labelled by $p \in \mathbb{N}$, and defined by

$$f_i^-|0\rangle = 0, \qquad f_i^- f_j^+|0\rangle = p\,\delta_{ij}|0\rangle. \tag{13}$$

These Fock spaces are finite-dimensional unitary irreducible $sl(1|n)$-modules, with a basis consisting of vectors $(f_1^+)^{\theta_1}(f_2^+)^{\theta_2}\cdots(f_n^+)^{\theta_n}|0\rangle$, where

$$\theta_i \in \{0, 1\}, \qquad |\theta| \equiv \sum_{i=1}^{n} \theta_i \leq p. \tag{14}$$

For $p < n$, these are so-called atypical representations of $sl(1|n)$; for $p \geq n$, these representations are typical and in that case $\dim V(p, n) = 2^n$. The usual Hermitian form on $V(p, n)$ gives again an inner product, and the vectors

$$|p; \theta\rangle \equiv |p; \theta_1, \ldots, \theta_n\rangle = \sqrt{\frac{(p - |\theta|)!}{p!}}(f_1^+)^{\theta_1}(f_2^+)^{\theta_2}\cdots(f_n^+)^{\theta_n}|0\rangle, \tag{15}$$

subject to the condition (14), constitute an orthonormal basis of $V(p, n)$. Under the action of the CAO's, this basis transforms as follows :

$$\begin{aligned} f_i^-|p; \theta\rangle &= \theta_i(-1)^{\theta_1+\cdots+\theta_{i-1}}\sqrt{p - |\theta| + 1}\;|p; \theta_i - 1\rangle, \\ f_i^+|p; \theta\rangle &= (1 - \theta_i)(-1)^{\theta_1+\cdots+\theta_{i-1}}\sqrt{p - |\theta|}\;|p; \theta_i + 1\rangle. \end{aligned} \tag{16}$$

If we denote again the Weyl generator e_{ii} by N_i, and introduce the Hamiltonian $H = \sum_{i=1}^n \epsilon_i N_i$, then $[H, f_i^{\pm}] = \pm\epsilon_i f_i^{\pm}$, so f_i^+ (f_i^-) can be interpreted as an operator creating (annihilating) a particle on orbital i with energy ϵ_i. Here, the statistics is closer related to Fermi-Dirac statistics in the sense that on each orbital there can be at most one particle. On the other hand, A-superstatistics differs from Fermi-Dirac statistics in the sense that also the order of statistics p plays a role, and no more than p particles can be accommodated if the system is in the representation $V(p, n)$.

Let us now investigate some macroscopic properties of A-superstatistics. With a notation similar to that of A-statistics, the grand partition function becomes :

$$Z(p, n) = \sum_{\substack{0 \leq \theta_1 + \cdots + \theta_n \leq p \\ \theta_i \in \{0,1\}}} \Big(\exp(\frac{\mu_1 - \epsilon_1}{\tau})\Big)^{\theta_1} \cdots \Big(\exp(\frac{\mu_n - \epsilon_n}{\tau})\Big)^{\theta_n}. \tag{17}$$

Putting $x_i = \exp(\frac{\mu_i - \epsilon_i}{\tau})$ $(i = 1, \ldots, n)$, this becomes

$$Z(p, n) = \sum_{\substack{0 \leq \theta_1 + \cdots + \theta_n \leq p \\ \theta_i \in \{0,1\}}} x_1^{\theta_1} x_2^{\theta_2} \cdots x_n^{\theta_n} = \sum_{k=0}^{p} e_k(x_1, \ldots, x_n). \tag{18}$$

Herein, e_k are the *elementary symmetric functions* [7]. So where the complete symmetric functions play a role in A-statistics, the elementary symmetric functions take over this role in A-superstatistics. For the typical case with $p \geq n$, one finds $Z(p \geq n, n) = (1 + x_1)\cdots(1 + x_n)$.

Next, let us also construct the distribution function, i.e. the average number of particles of A-superstatistics of order p. From the definition, one has

$$\bar{N}(p, n) = \sum_{\substack{0 \leq \theta_1 + \cdots + \theta_n \leq p \\ \theta_i \in \{0,1\}}} |\theta| \frac{x_1^{\theta_1} x_2^{\theta_2} \cdots x_n^{\theta_n}}{Z(p, n)}. \tag{19}$$

Using the symmetric functions, this can be written as

$$\bar{N}(p,n) = p - \frac{\sum_{k=0}^{p-1}(p-k)e_k(x_1,\ldots,x_n)}{\sum_{k=0}^{p} e_k(x_1,\ldots,x_n)}, \qquad (20)$$

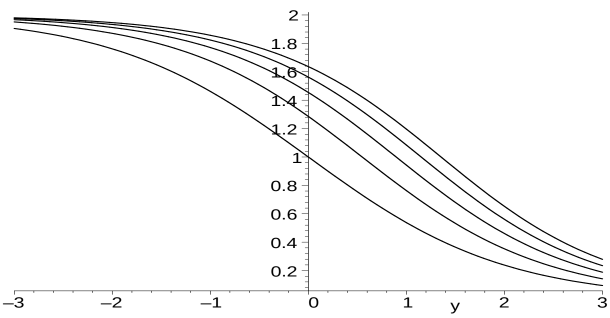
from which it is also clear that the average number of particles is less than p. In the typical case with $p \geq n$, one finds $\bar{N}(p \geq n, n) = \sum_{i=1}^{n} x_i/(1+x_i)$.

As a special example of A-superstatistics, let us consider the degenerate case where all orbitals have the same energy and chemical potential, so where all x_i are the same and denoted by x. Then the earlier computed functions simplify. For example, $Z(p,n) = \sum_{k=0}^{p}\binom{n}{k}x^k$. This reduces further to $(1+x)^n$ when $p \geq n$, i.e. it coincides in that case with the GPF for a Fermi system with n distinct orbitals having the same energy. Let us also consider the specialization of the distribution function :

$$\bar{N}(p,n) = \frac{\sum_{k=0}^{p} k\binom{n}{k}x^k}{\sum_{k=0}^{p}\binom{n}{k}x^k} = \frac{nx}{1+x} - n\binom{n-1}{p}\frac{x^{p+1}}{(1+x)Z(p,n)}. \qquad (21)$$

Clearly, for $p \geq n$, this becomes $\bar{N}(p \geq n, n) = nx/(1+x)$, coinciding with the distribution function of a Fermi system with n distinct orbitals having the same energy. For $p < n$, the last term in (21) is responsible for the deviation from such a Fermi system. To have an idea of such deviations, it is instructive to consider an example. Let us fix $p = 2$ (so at most two particles present), and consider the graph of $\bar{N}(p,n)$ for some n-values, say $n = 2,3,4,5,6$. This gives us the average number of particles of A-superstatistics of order $p = 2$ when n orbitals of equal energy (but distinct "internal labels") are available. These graphs are shown in Figure 1. As energy variable we have taken $y = (\epsilon - \mu)/\tau$, where $x = e^{-y}$. For $n = p = 2$, the graph coincides with the distribution function of a Fermi system with 2 distinct orbitals; for $n > p$, the function is similar but the half-filling is shifted to the right.

Figure 1. Graphs of $\bar{N}(p,n)$, for fixed $p = 2$, and $n = 2,3,4,5,6$. The graph of $\bar{N}(p,2)$ is the closest to the (horizontal) y-axis, then $\bar{N}(p,3)$, etc.



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