Symmetries of the S_3 Dirac-Dunkl operator

Hendrik De Bie, Roy Oste, and Joris Van der Jeugt

Abstract We work in three-dimensional Euclidean space on which the symmetric group S_3 acts in a natural way. Here, we consider the Dunkl operators, a generalization of partial derivatives in the form of differential-difference operators associated to a reflection group, S_3 in our case. In this setting, the main object of study is the Dunkl version of the Dirac operator. We determine the classes of symmetries of the Dirac-Dunkl operator and present the algebra they generate.

1 Introduction

We present a different view on a recently considered specific case of more general abstract results. In a first paper [3], we determined the symmetries, and the algebraic structure they generate, for a class of Laplace-like and Dirac-like operators in the framework of Wigner quantization. These operators include, in particular, the Dunkl version of Laplace and Dirac operators associated to arbitrary reflection group or root system. The term Dunkl refers to the Dunkl operators [5, 8], a generalization of partial derivatives in the form of differential-difference operators associated to a reflection group. In a second article [4], we moved from the abstract setting to a concrete example being the S_3 Dirac-Dunkl operator, which appears in the Dirac Hamiltonian for

Hendrik De Bie
Department of Mathematical Analysis, Faculty of Engineering and Architecture, Ghent University, Krijgslaan 281-S8, 9000 Gent, Belgium
e-mail: Hendrik.DeBie@UGent.be

Roy Oste, Joris Van der Jeugt
Department of Applied Mathematics, Computer Science and Statistics, Faculty of Sciences, Ghent University, Krijgslaan 281-S9, 9000 Gent, Belgium
e-mail: Roy.Oste@UGent.be; e-mail: Joris.VanderJeugt@UGent.be

the S_3 Dunkl Dirac equation. Here, S_3 is the symmetric group on three elements which acts in a natural way on three-dimensional Euclidean space by coordinate permutation, and occurs as the reflection group associated to the root system A_2. The structure of the symmetries and symmetry algebra in this case followed by substituting the corresponding Dunkl operators in the expressions obtained for the abstract Dirac-like operator, void of references to Dunkl operators or reflection groups. In the current paper, we will show that, though requiring more tedious computations, the algebraic relations for the symmetries of the S_3 Dirac-Dunkl operator can be obtained also in the concrete setting of Dunkl operators and reflection groups. The reason for this is not only to further clarify these results, but also to validate their correctness and moreover highlight the power and beauty of approaching and dealing with a problem in a more abstract general framework.

In fact, the calculations in the current paper served as an inspiration for, and predate the work on the generalized version. They were in turn inspired by results on Dirac-Dunkl operators for other classes of reflection groups and root systems. We mention in particular the $(\mathbb{Z}_2)^n$ and B_3 cases [1, 2, 7] where the symmetry algebra was shown to generalize the so-called Bannai-Ito algebra.

In the subsequent section, we go over the definition and notions required to introduce our main object of study, the Dirac-Dunkl operator related to S_3. In section 3, we present the symmetries of this Dirac-Dunkl operator with explicit expressions and also give the algebra generated by them.

2 The S_3 Dirac-Dunkl operator

We consider three-dimensional space $\mathbb{R}^3$ with coordinates x_1, x_2, x_3. The symmetric group S_3 is generated by the transpositions g_{12}, g_{23}, g_{31} which act on functions in a natural way, e.g. $g_{12} f(x_1, x_2, x_3) = f(x_2, x_1, x_3)$. Denoting $g_{123} = g_{12}g_{23} = g_{31}g_{12} = g_{23}g_{31}$ and $g_{321} = g_{23}g_{12} = g_{12}g_{31} = g_{31}g_{23}$, the six elements of S_3 are $\{1, g_{12}, g_{23}, g_{31}, g_{123}, g_{321}\}$.

For a parameter κ, usually assumed to be positive, the Dunkl operators [5, 8] associated to S_3 are given by

$$\mathcal{D}_1 = \partial_{x_1} + \kappa \left(\frac{1 - g_{12}}{x_1 - x_2} + \frac{1 - g_{13}}{x_1 - x_3} \right), \; \mathcal{D}_2 = \partial_{x_2} + \kappa \left(\frac{1 - g_{12}}{x_2 - x_1} + \frac{1 - g_{23}}{x_2 - x_3} \right),$$

$$\mathcal{D}_3 = \partial_{x_3} + \kappa \left(\frac{1 - g_{31}}{x_3 - x_1} + \frac{1 - g_{23}}{x_3 - x_2} \right).$$

The property that makes these generalizations of partial derivatives so special is that they commute with one another, $[\mathcal{D}_i, \mathcal{D}_j] = 0$ for $i, j \in \{1, 2, 3\}$. Furthermore, the action of S_3 on the Dunkl operators is simply given by

$$g_{12}\mathcal{D}_1 = \mathcal{D}_2 g_{12}, \quad g_{12}\mathcal{D}_2 = \mathcal{D}_1 g_{12}, \quad g_{12}\mathcal{D}_3 = \mathcal{D}_3 g_{12}, \tag{1}$$

and similarly for g_{23} and g_{31}. The commutation relations with the coordinate variables are readily shown to be

$$[\mathcal{D}_i, x_j] = \mathcal{D}_i x_j - x_j \mathcal{D}_i = \begin{cases} 1 + \kappa \sum\limits_{k \neq i} g_{ik} & i = j \\ -\kappa g_{ij} & i \neq j \end{cases} \tag{2}$$

for $i, j, k \in \{1, 2, 3\}$. Note that when $\kappa = 0$ these reduce to $[\mathcal{D}_i, x_j] = \delta_{ij}$, as the Dunkl operators then reduce to ordinary partial derivatives.

In this setting, the Laplace-Dunkl operator is given by

$$\Delta_\kappa = (\mathcal{D}_1)^2 + (\mathcal{D}_2)^2 + (\mathcal{D}_3)^2.$$

The Dirac-Dunkl operator $\underline{D}$ is defined as a square root of the Dunkl Laplacian as follows:

$$\underline{D} = e_1\mathcal{D}_1 + e_2\mathcal{D}_2 + e_3\mathcal{D}_3,$$

where e_1, e_2, e_3 generate the three-dimensional Euclidean Clifford algebra and hence satisfy $\{e_i, e_j\} = 2\delta_{ij}$ for $i, j \in \{1, 2, 3\}$. They can be realized by means of the well-known Pauli matrices: $e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Together with the vector variable $\underline{x} = e_1x_1 + e_2x_2 + e_3x_3$, the operator $\underline{D}$ generates a realization of the $\mathfrak{osp}(1|2)$ Lie superalgebra, governed by the relations

$$[\{\underline{D}, \underline{x}\}, \underline{D}] = -2\underline{D}, \qquad [\{\underline{D}, \underline{x}\}, \underline{x}] = 2\underline{x}.$$

Here, the operator $\{\underline{D}, \underline{x}\} = \underline{D}\,\underline{x} + \underline{x}\,\underline{D}$ can be written as $2(\mathbb{E} + 3\kappa)$ where $\mathbb{E} = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}$ is the Euler operator, which measures the degree of a homogeneous polynomial in x_1, x_2, x_3.

3 Symmetries of the Dirac-Dunkl operator

We now turn to the subject of symmetries of the operator $\underline{D}$. By the word symmetry, we have in mind an operator which "supercommutes" (commutes or anticommutes) with the operator in question.

A first symmetry of the Dirac-Dunkl operator is the so-called Scasimir operator of the $\mathfrak{osp}(1|2)$ realization above, which anticommutes with $\underline{D}$. It is given explicitly by

$$\frac{1}{2}\big([\underline{D}, \underline{x}] - 1\big) = 1 + \kappa(g_{12} + g_{23} + g_{31}) + e_1e_2L_{12} + e_2e_3L_{23} + e_3e_1L_{31}, \tag{3}$$

where the right-hand side is obtained through the commutation relations (2). The Scasimir operator is usually denoted by $\Gamma+1$, where the notation Γ refers to the "angular Dirac operator", appearing when $\underline{D}$ is written in spherical coordinates. Moreover, in the right-hand side of (3)

$$L_{12} = x_1\mathcal{D}_2 - x_2\mathcal{D}_1, \quad L_{23} = x_2\mathcal{D}_3 - x_3\mathcal{D}_2, \quad L_{31} = x_3\mathcal{D}_1 - x_1\mathcal{D}_3$$

are the Dunkl versions of the angular momentum operators, which commute with Δ_κ. They are the main object of study in a related paper on the Dunkl angular momentum algebra [6].

The square of the Scasimir operator yields the $\mathfrak{osp}(1|2)$ Casimir operator which commutes with $\underline{D}$:

$$(\Gamma+1)^2 = \Gamma + \kappa(g_{12}+g_{23}+g_{31}) + 3\kappa^2(1+g_{123}+g_{321}) - (L_{12}^2 + L_{23}^2 + L_{13}^2).$$

Another symmetry of $\underline{D}$ inherent to the Clifford algebra, is the pseudo-scalar $e_1e_2e_3$. Because of the anti-commutation relations of e_1, e_2, e_3, one immediately sees that $[\underline{D}, e_1e_2e_3] = 0$ and moreover $(e_1e_2e_3)^2 = -1$. In fact, in the realization by means of the Pauli matrices, $e_1e_2e_3$ is just i times the identity matrix.

While the Dunkl Laplacian is invariant under the action of S_3, following the interaction (1), the Dirac-Dunkl operator is not, as the group action leaves the Clifford elements e_1, e_2, e_3 unchanged. To get symmetries of $\underline{D}$, we extend the S_3 action to affect also Clifford elements. We do this by appending a group element of S_3 with an appropriate element in the Pin group of the Clifford algebra. In this way, we arrive at the symmetries

$$G_{12} = \frac{1}{\sqrt{2}} g_{12}(e_1 - e_2), \ G_{23} = \frac{1}{\sqrt{2}} g_{23}(e_2 - e_3), \ G_{31} = \frac{1}{\sqrt{2}} g_{31}(e_3 - e_1). \quad (4)$$

One readily verifies that they satisfy

$$G_{12}e_1 = -e_2G_{12}, \quad G_{12}e_2 = -e_1G_{12}, \quad G_{12}e_3 = -e_3G_{12}, \quad (G_{12})^2 = 1,$$

with analogous relations for G_{23} and G_{31}. Hence, they anti-commute with $\underline{D}$. We also have symmetries which commute with the S_3 Dirac-Dunkl operator, corresponding to the two even elements of S_3:

$$G_{123} = G_{12}G_{23} = \frac{1}{2} g_{123}(e_1e_2 + e_2e_3 + e_3e_1 - 1) = G_{23}G_{31} = G_{31}G_{12},$$

$$G_{321} = G_{23}G_{12} = \frac{1}{2} g_{321}(e_2e_1 + e_3e_2 + e_1e_3 - 1) = G_{31}G_{23} = G_{12}G_{31}.$$

Finally, we present the most interesting class of symmetries of $\underline{D}$. They extend the notion of Dunkl angular momentum operators as symmetries of the Dunkl Laplacian to Dunkl *total* angular momentum operators in the context of Dirac theory.

Theorem 1. *The operators*

$$
\begin{aligned}
O_{12} &= x_1\mathcal{D}_2 - x_2\mathcal{D}_1 + \frac{1}{2}e_1e_2 + \frac{\kappa}{\sqrt{2}}(G_{12}e_1 + G_{12}e_2 - G_{23}e_1 - G_{31}e_2),\\
O_{23} &= x_2\mathcal{D}_3 - x_3\mathcal{D}_2 + \frac{1}{2}e_2e_3 + \frac{\kappa}{\sqrt{2}}(G_{23}e_2 + G_{23}e_3 - G_{31}e_2 - G_{12}e_3),\\
O_{31} &= x_3\mathcal{D}_1 - x_1\mathcal{D}_3 + \frac{1}{2}e_3e_1 + \frac{\kappa}{\sqrt{2}}(G_{31}e_3 + G_{31}e_1 - G_{12}e_3 - G_{23}e_1)
\end{aligned} \tag{5}
$$

commute with $\underline{D}$ *and with* $\underline{x}$.

Proof. We show that $[O_{12}, \underline{D}] = 0$, the other results are completely analogous. Using the commutation relations (2), we have

$$
\begin{aligned}
[x_1, \underline{D}]\, \mathcal{D}_2 &= -(1 + \kappa g_{12} + \kappa g_{13})e_1\mathcal{D}_2 + \kappa g_{12}e_2\mathcal{D}_2 + \kappa g_{13}e_3\mathcal{D}_2\,,\\
-[x_2, \underline{D}]\, \mathcal{D}_1 &= -\kappa g_{12}e_1\mathcal{D}_1 + (1 + \kappa g_{21} + \kappa g_{23})e_2\mathcal{D}_1 - \kappa g_{23}e_3\mathcal{D}_1\,,\\
\frac{1}{2}[e_1e_2, \underline{D}] &= -e_2\mathcal{D}_1 + e_1\mathcal{D}_2\,.
\end{aligned}
$$

Finally, for $i \neq j$ and k elements of $\{1, 2, 3\}$ we have

$$
[G_{ij}e_k, \underline{D}] = G_{ij}\{e_k, \underline{D}\} = 2G_{ij}\mathcal{D}_k\,, \tag{6}
$$

where we used $\underline{D}G_{ij} = -G_{ij}\underline{D}$ and the anticommutation relations of the Clifford algebra. Using relation (6) to compute the final terms of $[O_{12}, \underline{D}]$ and plugging in the definition (4), all components of $[O_{12}, \underline{D}]$ cancel out. □

Theorem 2. *The algebra generated by the symmetries* $\Gamma + 1$, $e_1e_2e_3$, G_{12}, G_{23}, G_{31} *and* O_{12}, O_{23}, O_{31} *is governed by the following relations:*

- $\Gamma + 1$ *and* $e_1e_2e_3$ *commute with the other symmetries,*
- G_{12}, G_{23}, G_{31} *generate a copy of* S_3 *and act on the indices of* O_{12}, O_{23}, O_{31} *by an* S_3 *action with minus sign, i.e.*

  $$
  G_{12}O_{12} = -O_{12}G_{12}, \quad G_{12}O_{23} = -O_{31}G_{12}, \quad G_{12}O_{31} = -O_{23}G_{12},
  $$

  and analogous actions of G_{23} *and* G_{31},
- *the commutation relations*

  $$
  \begin{aligned}
  [O_{23}, O_{12}] &= O_{31} + \sqrt{2}\kappa(\Gamma + 1)e_1e_2e_3(G_{23} - G_{12}) + \frac{3}{2}\kappa^2(G_{123} - G_{321})\\
  [O_{31}, O_{23}] &= O_{12} + \sqrt{2}\kappa(\Gamma + 1)e_1e_2e_3(G_{31} - G_{23}) + \frac{3}{2}\kappa^2(G_{123} - G_{321})\\
  [O_{12}, O_{31}] &= O_{23} - \sqrt{2}\kappa(\Gamma + 1)e_1e_2e_3(G_{12} - G_{31}) + \frac{3}{2}\kappa^2(G_{123} - G_{321}).
  \end{aligned}
  $$

Proof. We work out $[O_{23}, O_{12}]$ — the other relations follow in a similar manner — by going over the commutators of the different components of the operators (5). By means of the commutation relations (2) we have

$$[L_{23}, L_{12}] = L_{31}(1 + \kappa g_{12} + \kappa g_{23}) - \kappa L_{12} g_{23} - \kappa L_{23} g_{12},$$

which can also be found in ref. [6]. Here, we recognize a first ingredient to make O_{31}, namely L_{31}. Recalling the expression (3) for $\Gamma + 1$, the terms accompanied with a factor κ will form part of the product $\sqrt{2}\kappa(\Gamma + 1)e_1e_2e_3(G_{23} - G_{12})$. Another part of this product is given by the commutators $[L_{ij}, G_{kl}e_m] = (L_{ij} - L_{g_{kl}(i)g_{kl}(j)})G_{kl}e_m$, where the indices in the second term are permuted by the S_3-action.

Next, we have $[L_{23}, e_1e_2] = 0 = [e_2e_3, L_{12}]$, while $[\frac{1}{2}e_2e_3, \frac{1}{2}e_1e_2] = \frac{1}{2}e_3e_1$. The final ingredients to make O_{31} follow from the terms in the commutators $[e_ie_j, G_{kl}e_m] = G_{kl}(e_{g_{kl}(i)}e_{g_{kl}(j)}e_m - e_me_ie_j)$ — with suitable values for i, j, k, l, m — where two of the three indices of the Clifford elements are equal and thus cancel out. The remaining terms of the latter commutators serve as parts of the product $\sqrt{2}\kappa(\Gamma + 1)e_1e_2e_3(G_{23} - G_{12})$, whose final ingredient follows from the commutators $[G_{ij}e_k, G_{lm}e_n] = G_{ij}G_{lm}e_{g_{lm}(k)}e_n - G_{lm}G_{ij}e_{g_{ij}(n)}e_k$ — again with appropriate values for i, j, k, l, m, n — when $g_{lm}(k) \neq n$ or $g_{ij}(n) \neq k$. This leaves eight terms such as $G_{23}e_2G_{12}e_1 = -G_{23}G_{12} = -G_{321}$ of which two cancel out and the remaining six terms form $\frac{3}{2}\kappa^2(G_{123} - G_{321})$, so we arrive at the desired result. □

For a further analysis of the symmetry algebra of Theorem 2, and its representations, the reader is referred to Ref. [4].

References

1. H. De Bie, V.X. Genest, L. Vinet, Commun. Math. Phys. **344** (2016) 447–464
2. H. De Bie, V.X. Genest, L. Vinet, Adv. Math. **303** (2016) 390–414
3. H. De Bie, R. Oste, J. Van der Jeugt, On the algebra of symmetries of Laplace and Dirac operators, arXiv:1701.05760
4. H. De Bie, R. Oste, J. Van der Jeugt, The total angular momentum algebra related to the S_3 Dunkl Dirac equation, arXiv:1705.08751
5. C.F. Dunkl, Trans. Amer. Math. Soc. **311** (1989) 167–183
6. M. Feigin, T. Hakobyan, J. High Energy Phys. **11** (2015) 107
7. V.X. Genest, L. Lapointe, L. Vinet, $\mathfrak{osp}(1,2)$ and generalized Bannai-Ito algebras, arXiv:1705.03761
8. M. Rösler, in *Orthogonal Polynomials and Special Functions*, Lecture Notes in Mathematics vol 1817 (Springer, Berlin, 2003), pp. 93–135