The N-particle Wigner Quantum Oscillator: non-commutative coordinates and physical properties

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Abstract. After introducing Wigner Quantum Systems, we give a short review of the one-dimensional Wigner Quantum Oscillator. Then we define the three-dimensional N-particle Wigner Quantum Oscillator, and its relation to the Lie superalgebra $\mathfrak{sl}(1|3N)$. In this framework (and first for $N = 1$), energy, coordinates, momentum and the angular momentum of the particles are investigated.

Wigner Quantum Systems (WQSs) are quantum systems in which the canonical commutation relations (CCRs) are replaced by a compatibility condition between the Heisenberg equations and Hamilton's equations. By dropping the CCRs, WQSs offer a natural framework for quantum mechanics with non-commutative coordinates. Here we consider in particular the three-dimensional N-particle Wigner Quantum Oscillator (WQO). As an introductory example, some properties of the one-dimensional WQO are reviewed, with an emphasis on the different particle probability distributions (as compared to the ordinary one-dimensional oscillator). For the three-dimensional N-particle WQO, a solution related to the Lie superalgebra $\mathfrak{sl}(1|3N)$ is considered. For $N = 1$ we investigate the operators corresponding to coordinates, momentum and angular momentum of the particle in a class of representation spaces. Remarkable properties include the discrete spectrum of coordinate operators and their non-commutativity (and the same for the momentum operators). This has some unconventional consequences for the particle localisation. We compare some of these properties with those of the canonical quantum oscillator (CQO). Finally, the case for general N is described.

Let $\hat{H}$ be the Hamiltonian of a system expressed in terms of the coordinates and momenta. This system is a WQS [1, 2] if all postulates of ordinary quantum mechanics hold, i.e.

P1 The state space W is a Hilbert space. To every physical observable O there corresponds a Hermitian (self-adjoint) operator $\hat{O}$ acting in W.

P2 The observable O can take on only those values which are eigenvalues of $\hat{O}$. The expectation value of the observable O in a state ψ is given by $\langle\hat{O}\rangle_\psi = (\psi,\hat{O}\psi)/(\psi,\psi)$.

But the postulate on canonical commutation relations (CCRs) is replaced by:

P3 In the Heisenberg picture, Hamilton's equations and the Heisenberg equations hold and are identical (as operator equations) in W.

Let us consider the harmonic oscillator Hamiltonian for a system of N particles in three dimensions,

$$\hat{H} = \sum_{\alpha=1}^{N} \Big(\frac{\hat{\mathbf{P}}_\alpha^2}{2m} + \frac{m\omega^2}{2}\hat{\mathbf{R}}_\alpha^2 \Big), \tag{1}$$

in terms of the $6N$ variables $\hat{R}_{\alpha k}$ and $\hat{P}_{\alpha k}$ ($k=1,2,3$, $\alpha = 1,2\ldots,N$) interpreted as the Cartesian coordinates and momenta of the N particles. By postulate **P3**, the 3D vector operators $\hat{\mathbf{R}}_1,\ldots,\hat{\mathbf{R}}_N$ and $\hat{\mathbf{P}}_1,\ldots,\hat{\mathbf{P}}_N$ must satisfy Hamilton's equations

$$\dot{\hat{\mathbf{P}}}_\alpha = -m\omega^2\hat{\mathbf{R}}_\alpha, \ \ \dot{\hat{\mathbf{R}}}_\alpha = \frac{1}{m}\hat{\mathbf{P}}_\alpha, \tag{2}$$

and the Heisenberg equations

$$\dot{\hat{\mathbf{P}}}_\alpha = \frac{i}{\hbar}[\hat{H},\hat{\mathbf{P}}_\alpha], \ \ \dot{\hat{\mathbf{R}}}_\alpha = \frac{i}{\hbar}[\hat{H},\hat{\mathbf{R}}_\alpha], \tag{3}$$

for $\alpha = 1,2,\ldots,N$. Moreover (2) and (3) should be identical as operator equations. This leads to the following compatibility conditions (CCs):

$$[\hat{H},\hat{\mathbf{P}}_\alpha] = i\hbar m\omega^2\hat{\mathbf{R}}_\alpha, \ \ [\hat{H},\hat{\mathbf{R}}_\alpha] = -\frac{i\hbar}{m}\hat{\mathbf{P}}_\alpha. \tag{4}$$

The task is now to find operator solutions of (4). This turns out to be a difficult problem, for which not all solutions are known, except for the case of one particle in one dimension. Let us therefore first consider the Hamiltonian $\hat{H}$ for the one-dimensional WQO,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{r}^2, \tag{5}$$

depending upon the 2 variables (operators) $\hat{r}$ and $\hat{p}$. The compatibility conditions (4) are

$$[\hat{H},\hat{p}] = i\hbar m\omega^2\hat{r}, \qquad [\hat{H},\hat{r}] = -\frac{i\hbar}{m}\hat{p}. \tag{6}$$

Wigner solved these equations [3], leading to the first example of a "Wigner quantum system". The solution can be obtained by introducing two new operators b^- and b^+:

$$b^\pm = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{r} \mp i\sqrt{\frac{1}{2m\omega\hbar}}\,\hat{p}. \tag{7}$$

Then it follows that $\hat{H} = \frac{\hbar\omega}{2}\{b^-,b^+\}$, and the compatibility conditions (6) read

$$[\{b^-,b^+\},b^\pm] = \pm 2b^\pm. \tag{8}$$

These relations correspond to the defining relations of the Lie superalgebra $\mathfrak{osp}(1|2)$. Furthermore, the self-adjointness of $\hat{H}$, $\hat{r}$ and $\hat{p}$ lead to the star conditions $\hat{H}^\dagger = \hat{H}$ and $(b^\pm)^\dagger = b^\mp$.

Thus one is led to the classification of star representations of the Lie superalgebra $\mathfrak{osp}(1|2)$. These representations are known: they are of type $\ell^2(\mathbb{Z}_+)$ and labelled by a positive real number a. The actions of the operators $b^\pm$ on the orthonormal basis vectors $|n\rangle$ ($n \geq 0$) are:

$$b^+|2n\rangle = \sqrt{2(n+a)}\,|2n+1\rangle, \qquad b^+|2n+1\rangle = \sqrt{2(n+1)}\,|2n+2\rangle, \tag{9}$$
$$b^-|2n\rangle = \sqrt{2n}\,|2n-1\rangle, \qquad b^-|2n+1\rangle = \sqrt{2(n+a)}\,|2n\rangle. \tag{10}$$

Only for $a = 1/2$ one finds back the canonical case. In general, one has

$$\hat{H}\,|n\rangle = (n+a)\hbar\omega\,|n\rangle. \tag{11}$$

So the energy levels are equidistant, and in units of $\hbar\omega$ they are given by $E_n = n+a$. From (7), one finds $[\hat{p},\hat{r}] = i\hbar(b^+b^- - b^-b^+)$. Then the action (9)-(10) yields

$$[\hat{p},\hat{r}]\,|2n\rangle = -2ai\hbar\,|2n\rangle, \qquad [\hat{p},\hat{r}]\,|2n+1\rangle = -2(1-a)i\hbar\,|2n+1\rangle. \tag{12}$$

So again it is easy to see that for $a = 1/2$ one gets the canonical commutation relation $[\hat{p},\hat{r}] = -i\hbar$. But this also shows that there are non-canonical solutions for this WQS.

In order to discuss position probabilities for this WQO, set $\hbar = m = \omega = 1$. Using $\hat{r} = (b^+ + b^-)/\sqrt{2}$, one finds now from (9) and (10)

$$\hat{r}\,|2n\rangle = \frac{1}{\sqrt{2}}(\sqrt{2n}\,|2n-1\rangle + \sqrt{2(n+a)}\,|2n+1\rangle),$$
$$\hat{r}\,|2n+1\rangle = \frac{1}{\sqrt{2}}(\sqrt{2(n+a)}\,|2n\rangle + \sqrt{2(n+1)}\,|2n+2\rangle). \tag{13}$$

This means that $\hat{r}$ is (or extends to) an unbounded Jacobi operator on $\ell^2(\mathbb{Z}_+)$. In fact, this Jacobi operator corresponds to the generalized Hermite polynomials $H_n^{(a-1/2)}(x)$, orthogonal over $\mathbb{R}$ with respect to the weight function $|x|^{2a-1}e^{-x^2}$ [4, p. 157] ($a = 1/2$ corresponds to the usual Hermite polynomials). So the spectrum of $\hat{r}$ is $\mathbb{R}$. In order to see this, consider the formal eigenvectors $v(x)$ of $\hat{r}$, for the eigenvalue x, and write them as $v(x) = \sum_{n=0}^\infty \alpha_n(x)\,|n\rangle$. The equation $\hat{r}v(x) = x\,v(x)$ leads, using (13), to

$$x\alpha_{2n}(x) = \sqrt{n}\,\alpha_{2n-1}(x) + \sqrt{n+a}\,\alpha_{2n+1}(x), \quad x\alpha_{2n+1}(x) = \sqrt{n+a}\,\alpha_{2n}(x) + \sqrt{n+1}\,\alpha_{2n+2}(x). \tag{14}$$

Comparing this with the recurrence relations of $H_n^{(a-1/2)}(x)$ yields

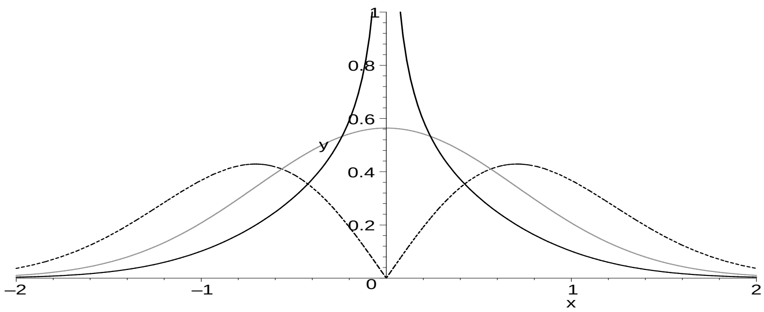
$$v(x) = \sum_{n=0}^\infty h_n^{(a-1/2)}(x)|x|^{a-1/2}\exp(-x^2/2)\,|n\rangle, \tag{15}$$

where $h_n^{(a-1/2)}(x)$ is the *normalized* generalized Hermite polynomial. By orthogonality it follows that (see also [5])

$$|n\rangle = \int_{-\infty}^{+\infty} h_n^{(a-1/2)}(x)|x|^{a-1/2}\exp(-x^2/2)v(x)dx. \tag{16}$$

This means that the probability distribution for the WQO being located at position x, when the system is in the stationary state $|n\rangle$, is given by $\left(h_n^{(a-1/2)}(x)\right)^2|x|^{2a-1}e^{-x^2}$. The position probability distributions for the WQO show deviations with respect to those of the CQO. For example, the position probabilities for the ground state ($n = 0$) are plotted in Figure 1.

Figure 1. Position probability distribution for the ground state $n = 0$ for $a = 1/4$ (full line), for $a = 1/2$ (the canonical case, grey line) and for $a = 1$ (dotted line).



Let us next consider the simple case of one particle in three dimensions. The general compatibility conditions (4), with $N = 1$, also admit solutions in the context of $\mathfrak{osp}(1|6)$ that can be seen as deviations of the canonical solutions. However, the state spaces of general $\mathfrak{osp}(1|6)$ solutions are not easy to describe. Another, and easier, class of solutions was first observed by Palev [1], and is related to the Lie superalgebra $\mathfrak{sl}(1|3)$. To describe these, consider new (unkown) operators as linear combination of the old ones ($k = 1,2,3$),

$$A_k^{\pm} = \sqrt{\frac{m\omega}{2\hbar}}\hat{R}_k \pm i\sqrt{\frac{1}{2m\omega\hbar}}\hat{P}_k, \tag{17}$$

where we have dropped the index α (since there is only one $\alpha = 1$). In terms of these operators, (1) becomes $\hat{H} = \frac{\omega\hbar}{2}\sum_{k=1}^{3}\{A_k^+, A_k^-\}$, and the compatibility conditions (4) read:

$$\sum_{j=1}^{3}[\{A_j^+, A_j^-\}, A_k^{\pm}] = \mp 2A_k^{\pm}. \tag{18}$$

The following yields an important family of solutions:

$$[\{A_i^+, A_j^-\}, A_k^+] = \delta_{jk}A_i^+ - \delta_{ij}A_k^+, \quad [\{A_i^+, A_j^-\}, A_k^-] = -\delta_{ik}A_j^- + \delta_{ij}A_k^-, \quad \{A_i^{\pm}, A_j^{\pm}\} = 0. \tag{19}$$

It is easy to verify that (19) implies indeed (18). The main observation of [1] is that the operators $A_k^{\pm}$ ($k = 1,2,3$), subject to the above relations (19), are odd elements generating the Lie superalgebra $\mathfrak{sl}(1|3)$. They are sometimes referred to as creation and annihilation operators (CAOs) of $\mathfrak{sl}(1|3)$.

The operators $\hat{\mathbf{R}} = (\hat{R}_1, \hat{R}_2, \hat{R}_3)$ and $\hat{\mathbf{P}} = (\hat{P}_1, \hat{P}_2, \hat{P}_3)$ of the WQO are interpreted and referred to as the position and momentum operators. We introduce one more physical notion, namely the angular momentum, to be defined as $\hat{\mathbf{M}} = (\hat{M}_1, \hat{M}_2, \hat{M}_3)$ with [2]

$$\hat{M}_j = -\frac{1}{\hbar}\sum_{k,l=1}^{3}\varepsilon_{jkl}\{\hat{R}_k, \hat{P}_l\} = -i\sum_{k,l=1}^{3}\varepsilon_{jkl}\{A_k^+, A_l^-\}, \tag{20}$$

where ε_{jkl} is the familiar antisymmetric tensor. The components $\hat{M}_j$ ($j = 1,2,3$) generate an $\mathfrak{so}(3)$ algebra, and they commute with the Hamiltonian.

In order to investigate physical properties of the WQO, we need to consider state spaces in which the operators act. According to the postulates, the state space W is a Hilbert space, and $(A_j^\pm)^\dagger = A_j^\mp$. The last condition implies that the operators $A_j^\pm$ act in "unitary" (i.e. star) representations of $\mathfrak{sl}(1|3)$. Instead of considering all star representations, we shall study only one class of them, namely the so-called Fock type representations [6] of $\mathfrak{sl}(1|3)$. The advantage of these representations is the fact that all operators have a simple action in a particular basis. The Fock representations W are characterized by a positive integer p, and are determined by the relations $A_j^-|0\rangle = 0$ and $A_j^- A_k^+|0\rangle = p\delta_{jk}|0\rangle$, where $|0\rangle$ is some generating vector. $W = W(p)$ is a finite dimensional covariant irreducible representation (irrep) with highest weight $(p,0,0)$. It is a *typical* irrep when $p \geq 3$ and *atypical* when $p < 3$ [7]. A basis vector of $W(p)$ is characterized by a string of 0's and 1's, $\Theta \equiv (\theta_1, \theta_2, \theta_3)$, with $\theta_i \in \{0,1\}$. Then, an orthonormal basis of $W(p)$ is given by the vectors

$$|p;\Theta\rangle \equiv |p;\theta_1,\theta_2,\theta_3\rangle = \sqrt{\frac{(p-q)!}{p!}}(A_1^+)^{\theta_1}(A_2^+)^{\theta_2}(A_3^+)^{\theta_3}|0\rangle, \tag{21}$$

where $q \equiv \sum_{i=1}^3 \theta_i$ must satisfy: $0 \leq q \leq \min(p,3)$. The action of the CAOs reads:

$$A_i^-|p;\Theta\rangle = \theta_i(-1)^{\psi_i}\sqrt{p-q+1}\,|p;\Theta\rangle_{\bar{i}}, \tag{22}$$

$$A_i^+|p;\Theta\rangle = (1-\theta_i)(-1)^{\psi_i}\sqrt{p-q}\,|p;\Theta\rangle_{\bar{i}}, \tag{23}$$

where $\psi_i = \sum_{j<i}\theta_j$, and $|p;\Theta\rangle_{\bar{i}}$ stands for the state obtained from $|p;\Theta\rangle$ after the replacement of θ_i by $\overline{\theta}_i = (1-\theta_i)$. Some physical properties can now be deduced from the following relations [7]:

$$\hat{H}\,|p;\Theta\rangle = \frac{\hbar\omega}{2}\,(3p-2q)\,|p;\Theta\rangle, \tag{24}$$

$$\hat{\mathbf{M}}^2|p;\Theta\rangle = \begin{cases} 0 & \text{if } \theta_1=\theta_2=\theta_3 \\ 2|p;\Theta\rangle & \text{otherwise} \end{cases}, \tag{25}$$

$$\hat{\mathbf{R}}^2|p;\Theta\rangle = \frac{\hbar}{2m\omega}(3p-2q)|p;\Theta\rangle, \tag{26}$$

$$\hat{R}_k^2|p;\Theta\rangle = \frac{\hbar}{2m\omega}(p-q+\theta_k)|p;\Theta\rangle \tag{27}$$

So the energy levels are equidistant, in steps of $\hbar\omega/2$. The angular momentum of the particle is 0 or 1. In fact, for $p > 2$ the decomposition with respect to $\mathfrak{so}(3)$ is $(1)\oplus(3)\oplus(3)\oplus(1)$. The most striking differences are a result of the non-commutativity of the position operators $\hat{R}_k$ (and similarly for the momentum operators). Quite surprisingly, however, the *squares* of the position (and momentum) operators do commute (and are diagonal in the basis considered). As a consequence of this and of the relations (26)-(27), if the system is in a fixed state $|p;\Theta\rangle$, the particle will be detected on one of the eight "nests" with coordinates $(\pm r_1, \pm r_2, \pm r_3)$, $r_k = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{(p-q+\theta_k)}$, on a sphere with radius $\sqrt{\hbar(3p-2q)/(2m\omega)}$ [7]. Note that by the non-commutativity of the coordinates the particle cannot be localised precisely: it is the choice of coordinates to be measured that leads to the observed values. The main feature however is that the spectrum of the operators $\hat{\mathbf{R}}^2$, $\hat{R}_k^2$ and $\hat{R}_k$ is finite (and thus discrete), quite different from the canonical solution.

It is interesting to compare properties of the canonical solution (in the Fock space) with the representation space $W = \oplus_{p=0}^{\infty} W(p)$ of the WQO. In Table 1, we list (according to increasing values of energy) the energy values (in units of $\hbar\omega$), the multiplicity (number of states with this energy), the angular momentum values M, and the expectation values of $\langle\hat{\mathbf{R}}^2\rangle$ in the stationary states (in units of $\hbar/m\omega$).

Table 1. Summary of some physical properties of the WQO compared to the CQO

WQO space W:				CQO Fock space:			
E	mult	M	$\langle\hat{\mathbf{R}}^2\rangle$	E	mult	M	$\langle\hat{\mathbf{R}}^2\rangle$
0	1	0	0	3/2	1	0	3/2
1/2	3	1	1/2	5/2	3	1	5/2
1	3	1	1	7/2	6	2, 0	7/2
3/2	2	0, 0	3/2	9/2	10	3, 1	9/2
2	3	1	2	11/2	15	4, 2, 0	11/2
5/2	3	1	5/2	13/2	21	5, 3, 1	13/2
3	2	0, 0	3	15/2	28	6, 4, 2, 0	15/2
⋮				⋮			

For the N-particle WQO [8] one introduces new operators $A^{\pm}_{\alpha k}$ ($k = 1,2,3$, $\alpha = 1,\ldots,N$), with relations similar to (19). These operators generate the Lie superalgebra $\mathfrak{sl}(1|3N)$, and provide a solution for the compatibility conditions for the N-particle case. The corresponding Fock spaces are given by representations $W(p)$ of $\mathfrak{sl}(1|3N)$, with basis states $|\,p;\Theta\rangle$, where Θ is now a string of $3N$ 0's or 1's. The energy spectrum is again equidistant. Each particle has angular momentum 0 or 1, and the angular momentum of the complete system follows from the branching rule of $\mathfrak{sl}(1|3N)$ to $\mathfrak{so}(3)$. The configuration of the system can be seen as a superposition of the single particle positions (with the previously described properties such as a discrete spectrum of position operators and the consequences of non-commutativity). The atypical representations (those with $p < 3N$) give rise to certain exclusion phenomena.

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References

[1] Palev T D 1982 *J. Math. Phys.* **23** 1778

[2] Palev T D and Stoilova N I 1997 *J. Math. Phys.* **38** 2506

[3] Wigner E P 1950 *Phys. Rev.* **77** 711

[4] Chihara T S 1978 *An introduction to orthogonal polynomials* (New York: Gorden and Breach)

[5] Mukunda N, Sudarshan E C G, Sharma J K and Mehta C L 1980 *J. Math. Phys.* **21** 2386

[6] Palev T D 1980 *J. Math. Phys.* **21** 1283

[7] King R C, Palev T D, Stoilova N I and Van der Jeugt J 2003 *J. Phys. A: Math. Gen.* **36** 4337

[8] King R C, Palev T D, Stoilova N I and Van der Jeugt J 2003 *J. Phys. A: Math. Gen.* **36** 11999