Realizations of coupled vectors in the tensor product of representations of $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)$

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Abstract

Using the realization of positive discrete series representations of $\mathfrak{su}(1,1)$ in terms of a complex variable z, we give an explicit expression for coupled basis vectors in the tensor product of $\nu+1$ representations as polynomials in $\nu+1$ variables $z_1,\ldots,z_{\nu+1}$. These expressions use the terminology of binary coupling trees (describing the coupled basis vectors), and are explicit in the sense that there is no reference to the Clebsch-Gordan coefficients of $\mathfrak{su}(1,1)$. In general, these polynomials can be written as (terminating) multiple hypergeometric series. For $\nu = 2$, these polynomials are triple hypergeometric series, and a relation between the two binary coupling trees yields a relation between two triple hypergeometric series. The case of $\mathfrak{su}(2)$ is discussed next. Also here the polynomials are determined explicitly in terms of a known realization; they yield an efficient way of computing coupled basis vectors in terms of uncoupled basis vectors.

Key words: multiple hypergeometric series, tensor products, realizations, coupling coefficient
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1 Introduction and notation

The relation between so-called coupling coefficients of the Lie algebras $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$ and orthogonal polynomials is well established [1–5]. The main activity in this area was stimulated by applications in quantum theory of angular momentum [6,5,7], as the angular momentum algebra coincides with the

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Lie algebra $\mathfrak{su}(2)$. The notion of "coupling of angular momentum" is equivalent to taking tensor products of irreducible representations of $\mathfrak{su}(2)$. The intertwining or coupling coefficients appearing in tensor products of two or three representations have been studied considerably. The $3j$-coefficients can be expressed in terms of a ${}_3F_2$ hypergeometric series of unit argument, leading to their relation with Hahn polynomials [1,3,7]. The $6j$-coefficients are expressed in terms of a balanced ${}_4F_3$ hypergeometric series of unit argument, due to Racah [8], and led to a new class of orthogonal polynomials [4] now referred to as Racah polynomials [9]. For the Lie algebra $\mathfrak{su}(1,1)$, the $3j$- and $6j$-coefficients for the positive discrete series representations are closely related to those of $\mathfrak{su}(2)$ [1,10,11].

The tensor product of an arbitrary number of representations has been given less attention in the literature. For the tensor product of $\nu+1$ representations of $\mathfrak{su}(2)$, the standard method is that of binary coupling [5]. Most of the attention went to the study of related $3\nu j$-coefficients for $\mathfrak{su}(2)$, for which a powerful graphical method was developed by Yutsis et al [12]. The method of binary coupling however, has the advantage that it works both for $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ [13,14]. There are a few papers dealing with the relation between $\nu+1$ fold tensor products (or $3\nu j$-coefficients) and orthogonal polynomials or hypergeometric series in several variables. For $9j$-coefficients of $\mathfrak{su}(2)$, a relation with a triple hypergeometric series was given by Ališauskas [7,15–17]. Orthogonal polynomials in several variables, appearing in the context of binary couplings for $\mathfrak{su}(1,1)$ representations, appear in the work of Rosengren [18] and in [19].

In the present paper we reconsider the classical realization of positive discrete series representations of $\mathfrak{su}(1,1)$, and study the corresponding realization of coupled vectors (defined by the method of binary coupling) in the tensor product of $\nu+1$ such representations. Our main result is a general expression for such a coupled vector (Theorem 1). This expression is a polynomial in $\nu+1$ variables, and can be written as a multiple hypergeometric series. These polynomials are also orthogonal as complex functions on the multidisc, for a standard weight function.

The structure of the paper is as follows. In the remainder of this section, we recall some standard properties for the Lie algebra $\mathfrak{su}(1,1)$, its positive discrete series representations, and their tensor products. In particular, we introduce the notion of binary coupling schemes and their notation. In Section 2 we consider the classical realization of these representations, and derive the expression for arbitrary coupled vectors in this realization. In Section 3 we consider an application : for the tensor product of three representations, the realization of coupled vectors is written as a triple hypergeometric series. The equation between two types of coupled vectors, related through an $\mathfrak{su}(1,1)$ Racah coefficient, leads to an identity relating such triple hypergeometric series by means of a ${}_4F_3$ series. Finally, in Section 4 we consider the case of $\nu+1$ tensor products of $\mathfrak{su}(2)$ representations and their realizations. The situation is very similar to that of $\mathfrak{su}(1,1)$. The expression for coupled basis vectors can be useful in applications if one needs an explicit expansion of such vectors in terms of uncoupled basis vectors.

The Lie algebra $\mathfrak{su}(1,1)$ is generated by $J_0, J_\pm$ subject to the relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0, \tag{1}$$

with the conditions $J_0^\dagger = J_0$ and $J_\pm^\dagger = J_\mp$. The positive discrete representations [1] $\mathcal{D}_k$ are labelled by a positive real number k. The representation space is $\ell^2(\mathbb{Z}_+)$, with orthonormal basis vectors denoted by $e_n^{(k)}$, with $n = 0, 1, 2, \ldots$ (sometimes denoted by $|k, n\rangle$). $\mathcal{D}_k$ is an irreducible $\star$-representation of $\mathfrak{su}(1,1)$, with inner product $\langle e_{n_1}^{(k)}, e_{n_2}^{(k)}\rangle = \delta_{n_1,n_2}$, and with action given by :

$$\begin{aligned} J_0 e_n^{(k)} &= (n+k) e_n^{(k)}, \\ J_+ e_n^{(k)} &= \sqrt{(n+1)(2k+n)}\, e_{n+1}^{(k)}, \\ J_- e_n^{(k)} &= \sqrt{n(2k+n-1)}\, e_{n-1}^{(k)}. \end{aligned} \tag{2}$$

The tensor product of two positive discrete series representations is completely reducible, and the decomposition is given by [1] :

$$\mathcal{D}_{k_1} \otimes \mathcal{D}_{k_2} = \sum_{j=0}^{\infty} \mathcal{D}_{k_1+k_2+j}. \tag{3}$$

The standard (orthonormal) basis vectors of $\mathcal{D}_k$ ($k = k_1 + k_2 + j$) in (3) are called the coupled basis vectors and related to the uncoupled basis vectors of $\mathcal{D}_{k_1} \otimes \mathcal{D}_{k_2}$ through the Clebsch-Gordan coefficients of $\mathfrak{su}(1,1)$. This is written as :

$$e_n^{(k_1,k_2)k} = \sum_{n_1,n_2} C_{n_1,n_2,n}^{k_1,k_2,k}\; e_{n_1}^{(k_1)} \otimes e_{n_2}^{(k_2)}, \tag{4}$$

$$k = k_1 + k_2 + j \;\; (j \in \mathbb{N}), \qquad k + n = k_1 + n_1 + k_2 + n_2, \tag{5}$$

where [1,10,11] :

$$\begin{aligned} C_{n_1,n_2,n}^{k_1,k_2,k} = & \left[\frac{(2k_1)_{n_1}(2k_2)_{n_2}(2k_1)_j}{n! n_1! n_2! j! (2k_1+2k_2+2j)_n (2k_2)_j (2k_1+2k_2+j-1)_j} \right]^{1/2} \\ & \times (j+n)!\; {}_3F_2\left({2k_1+2k_2+j-1, -n_1, -j \atop 2k_1, -n-j} ; 1 \right). \end{aligned} \tag{6}$$

Herein, $(a)_n$ is the classical notation for the Pochhammer symbol, and ${}_3F_2$ is a (terminating) generalized hypergeometric series [20–22]. Later on we shall also write $(a, b, \ldots)_n$ for $(a)_n(b)_n \ldots$.

We shall be concerned in this paper with the tensor product of $\nu + 1$ positive discrete series representations $\mathcal{D}_{k_1} \otimes \cdots \otimes \mathcal{D}_{k_{\nu+1}}$. Orthonormal basis vectors in such a tensor product can be constructed by means of binary coupling schemes. For example, two sets of basis vectors in the tensor product of three representations $\mathcal{D}_{k_1} \otimes \mathcal{D}_{k_2} \otimes \mathcal{D}_{k_3}$, according to the "coupling" $(\mathcal{D}_{k_1} \otimes \mathcal{D}_{k_2}) \otimes \mathcal{D}_{k_3}$ or $\mathcal{D}_{k_1} \otimes (\mathcal{D}_{k_2} \otimes \mathcal{D}_{k_3})$, are given by :

$$\begin{aligned} e_n^{((k_1,k_2)k_{12},k_3)k} &= \sum_{n_{12},n_3} C^{k_{12},k_3,k}_{n_{12},n_3,n}\; e^{(k_1,k_2)k_{12}}_{n_{12}} \otimes e^{(k_3)}_{n_3} \\ &= \sum_{n_1,n_2,n_3,n_{12}} C^{k_1,k_2,k_{12}}_{n_1,n_2,n_{12}} C^{k_{12},k_3,k}_{n_{12},n_3,n}\; e^{(k_1)}_{n_1} \otimes e^{(k_2)}_{n_2} \otimes e^{(k_3)}_{n_3}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} e_n^{(k_1,(k_2,k_3)k_{23})k} &= \sum_{n_1,n_{23}} C^{k_1,k_{23},k}_{n_1,n_{23},n}\; e^{(k_1)}_{n_1} \otimes e^{(k_2,k_3)k_{23}}_{n_{23}} \\ &= \sum_{n_1,n_2,n_3,n_{23}} C^{k_2,k_3,k_{23}}_{n_2,n_3,n_{23}} C^{k_1,k_{23},k}_{n_1,n_{23},n}\; e^{(k_1)}_{n_1} \otimes e^{(k_2)}_{n_2} \otimes e^{(k_3)}_{n_3}. \end{aligned} \tag{8}$$

Following (5), the above representation labels are such that

$$\begin{aligned} &k_{12} = k_1 + k_2 + j_{12}, \quad k_{23} = k_2 + k_3 + j_{23}, \\ &k = k_{12} + k_3 + j = k_1 + k_{23} + j', \\ &j_{12}, j, j_{23}, j' \in \mathbb{Z}_+, \text{ and } j_{12} + j = j_{23} + j'. \end{aligned} \tag{9}$$

More generally, a binary coupling scheme on a twofold tensor product $\mathcal{D}_k \otimes \mathcal{D}_{k'}$ is defined as

$$\sum_{n,n'} C^{k,k',K}_{n,n',N}\; e^{(k)}_n \otimes e^{(k')}_{n'} \qquad \text{or} \qquad \sum_{n,n'} C^{k',k,K}_{n',n,N}\; e^{(k)}_n \otimes e^{(k')}_{n'}. \tag{10}$$

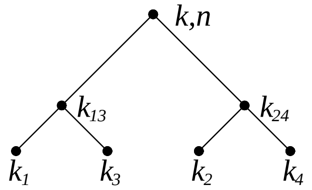
A binary coupling scheme $T = T(K, N)$ on the tensor product of $(\nu + 1)$ representations $\mathcal{D}_{k_1} \otimes \mathcal{D}_{k_2} \otimes \cdots \otimes \mathcal{D}_{k_{\nu+1}}$ is defined (recursively) as

$$\sum_{n,n'} C^{k,k',K}_{n,n',N}\; \sigma(T_1(k,n) \otimes T_2(k',n')), \tag{11}$$

where $T_1(k, n)$ is a binary coupling scheme on the tensor product of l representations and $T_2(k', n')$ is a binary coupling scheme on the tensor product of the remaining $\nu+1-l$ representations. The map σ reshuffles the order of the components in the tensor product such that they belong to $\mathcal{D}_{k_1}\otimes\mathcal{D}_{k_2}\otimes\cdots\otimes\mathcal{D}_{k_{\nu+1}}$. The notation for the vectors corresponding to a binary coupling scheme is $e_N^{(\cdots)K}$, where $(\cdots)$ is the binary bracketing determined by the binary coupling scheme; see for example (7) and (8). For instance, in $\mathcal{D}_{k_1}\otimes\mathcal{D}_{k_2}\otimes\mathcal{D}_{k_3}\otimes\mathcal{D}_{k_4}$,

$$\begin{aligned} e_n^{((k_1,k_3)k_{13},(k_2,k_4)k_{24})k} &= \sum_{n_{13},n_{24}} C^{k_{13},k_{24},k}_{n_{13},n_{24},n}\ \sigma(e_{n_{13}}^{(k_1,k_3)k_{13}}\otimes e_{n_{24}}^{(k_2,k_4)k_{24}}) \\ &= \sum_{n_{13},n_{24},n_1,n_3,n_2,n_4} C^{k_{13},k_{24},k}_{n_{13},n_{24},n} C^{k_1,k_3,k_{13}}_{n_1,n_3,n_{13}} C^{k_2,k_4,k_{24}}_{n_2,n_4,n_{24}} \\ &\quad \times\ e_{n_1}^{(k_1)}\otimes e_{n_2}^{(k_2)}\otimes e_{n_3}^{(k_3)}\otimes e_{n_4}^{(k_4)}. \end{aligned} \tag{12}$$

Such a binary coupling scheme is usually denoted by a binary tree, which is a graphical way of describing the order in which twofold tensor products are taken. For example, the binary tree corresponding to (12) is given by :



Finally, let us introduce the notation for the $\mathfrak{su}(1,1)$ Racah coefficients. These are by definition the transition coefficients between the basis vectors (7) and (8) :

$$e_n^{(k_1,(k_2,k_3)k_{23})k} = \sum_{k_{12}=k_1+k_2}^{k-k_3} U^{k_1,k_2,k_{12}}_{k_3,k,k_{23}}\ e_n^{((k_1,k_2)k_{12},k_3)k}. \tag{13}$$

An explicit expression for the Racah coefficients $U^{k_1,k_2,k_{12}}_{k_3,k,k_{23}}$ is given by [1,10,11] :

$$\begin{aligned} U^{k_1,k_2,k_{12}}_{k_3,k,k_{23}} &= \binom{j+j_{12}}{j_{23}} \frac{(2k_2)_{j_{12}}(2k_3)_j(2k_1+2k_2+2k_3+j+j_{12}-1)_{j_{23}}}{(2k_3,2k_2+2k_3+j_{23}-1)_{j_{23}}(2k_2+2k_3+2j_{23})_{j'}} \\ &\times\left(\frac{j'!(2k_1,2k_{23},2k_1+2k_{23}+j'-1)_{j'}\ j_{23}!(2k_2,2k_3,2k_2+2k_3+j_{23}-1)_{j_{23}}}{j!(2k_{12},2k_3,2k_{12}+2k_3+j-1)_j\ j_{12}!(2k_1,2k_2,2k_1+2k_2+j_{12}-1)_{j_{12}}}\right)^{1/2} \\ &\times\ {}_4F_3\left(\begin{matrix}2k_1+2k_2+j_{12}-1,2k_2+2k_3+j_{23}-1,-j_{12},-j_{23}\\ 2k_2,2k_1+2k_2+2k_3+j+j_{12}-1,-j-j_{12}\end{matrix};1\right), \end{aligned} \tag{14}$$

with the labels determined by (9).

2 Realization of coupled basis vectors

The Lie algebra $\mathfrak{su}(1,1)$ and its representation $\mathcal{D}_k$ have a well known realization for $k > 1/2$, in the Hilbert space of analytic functions $f(z)$ ($z \in \mathbb{C}$) on the unit disc $|z| < 1$ with inner product [1]

$$\langle f_1, f_2 \rangle = \frac{2k-1}{\pi} \iint\limits_{|z|<1} f_1(z)\overline{f_2(z)}(1-|z|^2)^{2k-2} dx dy, \quad (z = x+iy). \tag{15}$$

In this realization, the orthonormal basis vectors are given by

$$e_n^{(k)} = \sqrt{\frac{(2k)_n}{n!}} z^n, \tag{16}$$

and the realization of the $\mathfrak{su}(1,1)$ basis elements reads

$$J_0 = z\frac{d}{dz} + k, \qquad J_- = \frac{d}{dz}, \qquad J_+ = z^2\frac{d}{dz} + 2kz. \tag{17}$$

It is easy to verify that the action of these operators on the basis (16) is indeed the same as in (2).

We shall now investigate the explicit expressions for coupled vectors in the realization of tensor products.

For a coupled vector in the tensor product of two representations, an expression was already given in [23, Formula (3.16)] :

$$\begin{aligned} e_n^{(k_1,k_2)k}(z_1,z_2) &= \sqrt{\frac{(2k_1)_j(2k_2)_j(2k)_n}{j!n!(2k_1+2k_2+j-1)_j}}(z_2-z_1)^j z_1^n \\ &\quad \times {}_2F_1\left(\begin{matrix}-n, 2k_2+j\\ 2k\end{matrix}; 1-z_2/z_1\right) \\ &= \sqrt{\frac{(2k_1)_j(2k_2)_j}{j!n!(2k_1+2k_2+j-1)_j(2k)_n}}(z_2-z_1)^j z_1^n (2k_1+j)_n \\ &\quad \times {}_2F_1\left(\begin{matrix}-n, 2k_2+j\\ 1-n-2k_1-j\end{matrix}; \frac{z_2}{z_1}\right). \end{aligned} \tag{18}$$

The second expression is derived from the first one using the (easily verified) transformation

$${}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}; x\right) = \frac{(c-b)_n}{(c)_n}\, {}_2F_1\left(\begin{matrix}-n, b\\ b-c-n+1\end{matrix}; 1-x\right). \tag{19}$$

After expanding the ${}_2F_1$ in (18), this can be written in the following way :

$$e_n^{(k_1,k_2)k}(z_1,z_2) = \frac{(-1)^j}{\sqrt{j!n!(2k_1)_j(2k_2)_j(2k_1+2k_2+j-1)_j(2k)_n}} \times \sum_{\alpha_{12}=j} j!(2k_1)_{\alpha_1}(2k_2)_{\alpha_2}\frac{(z_1-z_2)^{\alpha_{12}}}{\alpha_{12}!} \times \sum_{h_1+h_2=n}\binom{n}{h_1,h_2}(2k_1+\alpha_1)_{h_1}(2k_2+\alpha_2)_{h_2}z_1^{h_1}z_2^{h_2}, \quad (20)$$

with $\alpha_1=\alpha_2=\alpha_{12}$. The reason for this awkward rewriting will become clear when considering the tensor product of an arbitrary number of representations.

Next, we consider the tensor product of three representations $\mathcal{D}_{k_1}\otimes\mathcal{D}_{k_2}\otimes\mathcal{D}_{k_3}$. For the realization of coupled basis vector of lowest weight (i.e. the vector (7) with $n=0$), an expression was derived in [18, Theorem 7.2] :

$$e_0^{((k_1,k_2)k_{12},k_3)k} = A \sum_{\substack{\alpha_{12}=j_{12}\\ \alpha_{13}+\alpha_{23}=j}} j_{12}!j!(2k_1)_{\alpha_1}(2k_2)_{\alpha_2}(2k_3)_{\alpha_3} \times \frac{(z_1-z_2)^{\alpha_{12}}}{\alpha_{12}!}\frac{(z_1-z_3)^{\alpha_{13}}}{\alpha_{13}!}\frac{(z_2-z_3)^{\alpha_{23}}}{\alpha_{23}!}, \quad (21)$$

where A is a normalization constant, $j_{12}=k_{12}-k_1-k_2$, $j=k-k_{12}-k_3$ and

$$\alpha_1=\alpha_{12}+\alpha_{13}, \quad \alpha_2=\alpha_{12}+\alpha_{23}, \quad \alpha_3=\alpha_{13}+\alpha_{23}.$$

Note that this vector depends only on the differences z_i-z_j and that it is a homogeneous polynomial in (z_1,z_2,z_3) of degree $j+j_{12}$.

From (2) is is clear that one can determine $e_n^{((k_1,k_2)k_{12},k_3)k}$ by n times applying J_+ to $e_0^{((k_1,k_2)k_{12},k_3)k}$:

$$J_+^n e_0^{((k_1,k_2)k_{12},k_3)k} = \sqrt{n!(2k)_n}\; e_n^{((k_1,k_2)k_{12},k_3)k}. \quad (22)$$

In the current realization J_+ takes the following form :

$$J_+ = z_1^2\partial_{z_1}+z_2^2\partial_{z_2}+z_3^2\partial_{z_3}+2k_1z_1+2k_2z_2+2k_3z_3; \quad (23)$$

thus $e_n^{((k_1,k_2)k_{12},k_3)k}$ is a homogeneous polynomial in (z_1,z_2,z_3) of degree $j_{12}+j+n$.

Applying J_+ a couple of times suggests the following form :

$$e_n^{((k_1,k_2)k_{12},k_3)k} = \frac{A}{\sqrt{n!(2k)_n}}$$
$$\times \sum_{\substack{\alpha_{12}=j_{12}\\ \alpha_{13}+\alpha_{23}=j}} j_{12}!j!(2k_1)_{\alpha_1}(2k_2)_{\alpha_2}(2k_3)_{\alpha_3} \frac{(z_1-z_2)^{\alpha_{12}}}{\alpha_{12}!}\frac{(z_1-z_3)^{\alpha_{13}}}{\alpha_{13}!}\frac{(z_2-z_3)^{\alpha_{23}}}{\alpha_{23}!}$$
$$\times \sum_{h_1+h_2+h_3=n} \binom{n}{h_1,h_2,h_3}(2k_1+\alpha_1)_{h_1}(2k_2+\alpha_2)_{h_2}(2k_3+\alpha_3)_{h_3} z_1^{h_1}z_2^{h_2}z_3^{h_3}. \tag{24}$$

The validity of this form follows from Theorem 1.

Now we turn our attention to the determination of the normalization constant A in (21) or (24). From Rosengren [18, page 33] and the fact that the polynomials (24) are normalized to unity, it follows that the square A^2 of the normalization constant is given by :

$$A^2 = (j_{12}!j!(2k_1)_{j_{12}}(2k_2)_{j_{12}}(2k_{12})_j(2k_3)_j(2k_1+2k_2+j_{12}-1)_{j_{12}}$$
$$\times(2k_{12}+2k_3+j-1)_j)^{-1}. \tag{25}$$

The phase factor of A depends on the choice of the phase factor for the $\mathfrak{su}(1,1)$ Clebsch-Gordan coefficients and the phase factor of (16). Computing the coefficient of $z_1^{j_{12}+j}$ in both left and right hand side of

$$e_0^{((k_1,k_2)k_{12},k_3)k} = \sum_{n_{12}+n_3=j} C_{n_{12},n_3,0}^{k_{12},k_3,k}\; e_{n_{12}}^{(k_1,k_2)k_{12}} e_{n_3}^{k_3}, \tag{26}$$

one finds that the phase factor of A equals $(-1)^{j_{12}+j}$. Thus we have :

$$A = (-1)^{j_{12}+j}/(j_{12}!j!(2k_1)_{j_{12}}(2k_2)_{j_{12}}(2k_{12})_j(2k_3)_j$$
$$\times(2k_1+2k_2+j_{12}-1)_{j_{12}}(2k_{12}+2k_3+j-1)_j)^{1/2}. \tag{27}$$

After these special examples, let us consider the tensor product of $\nu+1$ representations, and determine an explicit expression for the standard coupled basis vectors. We shall use here the terminology of binary coupling schemes, and of the related binary coupling trees. In order to fix some notation and terminology, let us consider an example in $\mathcal{D}_{k_1}\otimes\mathcal{D}_{k_2}\otimes\mathcal{D}_{k_3}\otimes\mathcal{D}_{k_4}\otimes\mathcal{D}_{k_5}$ $(\nu=4)$, namely

$$T(k,n) = (\;(((k_1,k_2)k_{12},k_3)k_{123},(k_4,k_5)k_{45})k,n). \tag{28}$$

The corresponding binary coupling tree has $\nu+1$ leaves, each with a representation label k_i, whilst the internal nodes are labelled by the intermediate representations. Figure 1(a) shows this binary coupling tree. Note that by the $\mathfrak{su}(1,1)$ tensor product rule (3), one can associate with every internal node, say v_i, of such a binary tree a nonnegative integer j_i.

From [18, Theorem 7.2] one can deduce an explicit expression for $T(k,0)$ for an arbitrary binary coupling scheme. To this end, we introduce $\binom{\nu+1}{2}$ summation variables α_{ij} associated with each pair of leaves k_i and k_j, with $i < j$. For conciseness of notation, we introduce also variables α_m which are the sum of all α_{ij} where either i or j equals m. Compare this with the conventions adopted in (21).

We say that two leaves *meet* at a node v_i if v_i is the first common ancestor of those two leaves. So, with every internal node one can associate a set of pairs of leaves meeting at that particular node, and hence a set of variables α_{ij}.

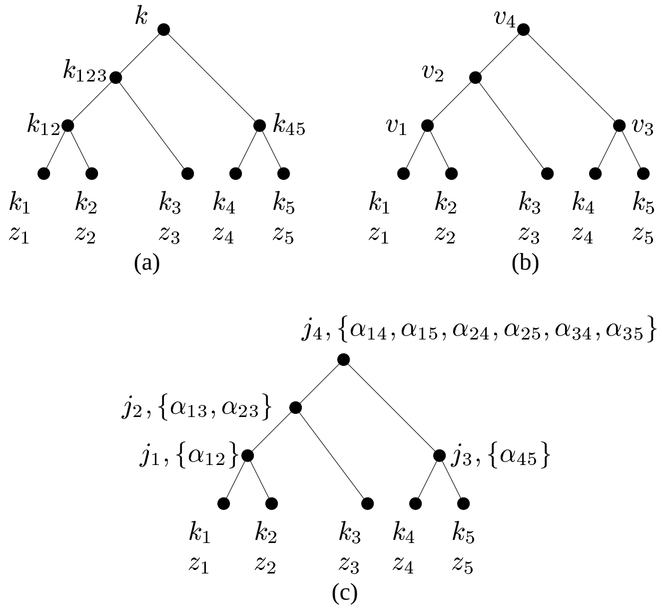


Fig. 1. example tree

Figure 1(b) gives a name v_i to the internal nodes in postorder. For the example tree, we have the following :

node	meeting pairs of leaves
v_1	(k_1,k_2)
v_2	(k_1,k_3), (k_2,k_3)
v_3	(k_4,k_5)
v_4	(k_1,k_4), (k_1,k_5), (k_2,k_4), (k_2,k_5), (k_3,k_4), (k_3,k_5)

This is indicated in Figure 1(c).

Then, one can write the following :

$$T(k,0) = A \sum_{\alpha_{ij}\in V} j_1!\cdots j_\nu!(2k_1)_{\alpha_1}\cdots(2k_{\nu+1})_{\alpha_{\nu+1}} \prod_{1\le i<j\le \nu+1} \frac{(z_i-z_j)^{\alpha_{ij}}}{\alpha_{ij}!}, \tag{29}$$

where V is a set of ν linear constraints to be satisfied by the variables α_{ij}. There is one constraint associated with each internal node of the coupling scheme, namely : the sum of the variables α_{ij} associated with an internal node v_m should be equal to j_m. Note that these constraints are in agreement with those in (20) and (21).

As before, it follows from Rosengren [18, page 33] that the square A^2 of the normalization constant is given by :

$$A^2 = \left(\prod_{i=1}^{\nu} j_i!(a_i+1)_{j_i}(b_i+1)_{j_i}(a_i+b_i+j_i+1)_{j_i}\right)^{-1}, \tag{30}$$

where

$$\begin{aligned} a_i &= \sum_{\substack{\text{left leaves}\\ \text{of } v_i}} 2k_l + \sum_{\substack{\text{nodes in left}\\ \text{subtree of } v_i}} 2j_l - 1, \\ b_i &= \sum_{\substack{\text{right leaves}\\ \text{of } v_i}} 2k_l + \sum_{\substack{\text{nodes in right}\\ \text{subtree of } v_i}} 2j_l - 1. \end{aligned} \tag{31}$$

We now come to the main result of this paper :

Theorem 1 *In the tensor product of $\nu+1$ representations $\mathcal{D}_{k_1}\otimes\mathcal{D}_{k_2}\otimes\cdots\otimes\mathcal{D}_{k_{\nu+1}}$, the realization of a coupled vector defined by a binary coupling scheme $T(k,n)$ is given by :*

$$\begin{aligned} T(k,n) &= (-1)^{|j|}\sqrt{\frac{j_1!\cdots j_\nu!}{n!(a_\nu+b_\nu+j_\nu+2)_n \prod_{i=1}^{\nu}(a_i+1,b_i+1,a_i+b_i+j_i+1)_{j_i}}} \\ &\times \sum_{\alpha_{ij}\in V}\sum_{|h|=n}\binom{n}{h_1,\ldots,h_{\nu+1}}(2k_1)_{\alpha_1+h_1}\cdots(2k_{\nu+1})_{\alpha_{\nu+1}+h_{\nu+1}} \\ &\times \prod_{1\le i<j\le\nu+1}\frac{(z_i-z_j)^{\alpha_{ij}}}{\alpha_{ij}!} z_1^{h_1}\cdots z_{\nu+1}^{h_{\nu+1}}. \end{aligned} \tag{32}$$

Herein, $|j| = j_1+\cdots+j_\nu$, $|h| = h_1+\cdots+h_{\nu+1}$, V is the set of ν linear constraints satisfied by the α_{ij}, α_m is the sum over all α_{ij} with either i or j equal to m, and a_i and b_i are determined by the binary coupling tree and given in (31).

Proof. Let us write (32) as

$$T(k,n) = B \sum_{\alpha_{ij}\in V} \sum_{|h|=n} \binom{n}{h_1,\ldots,h_{\nu+1}} \prod_{i=1}^{\nu+1} (2k_i)_{\alpha_i+h_i} \prod_{i<j} \frac{(z_i-z_j)^{\alpha_{ij}}}{\alpha_{ij}!} \prod_{i=1}^{\nu+1} z_i^{h_i}, \quad (33)$$

where B is some constant. It is clear that in the case $n=0$ (33) coincides with (29), so we shall use induction on n to prove (33). In the present realization, J_+ has the following form :

$$J_+ = \sum_{l=1}^{\nu+1} z_l^2 \partial_{z_l} + 2k_l z_l. \tag{34}$$

Since J_+ is a linear operator we can write :

$$\begin{aligned} J_+T(k,n) = B \sum_{\alpha_{ij}\in V} \sum_{|h|=n} J_+ \Bigg(\binom{n}{h_1,\ldots,h_{\nu+1}} \prod_{i=1}^{\nu+1} (2k_i)_{\alpha_i+h_i} \\ \times \prod_{i<j} \frac{(z_i-z_j)^{\alpha_{ij}}}{\alpha_{ij}!} \prod_{i=1}^{\nu+1} z_i^{h_i} \Bigg) = B \sum_{\alpha_{ij}\in V} \sum_{|h|=n} J_+ (Z), \end{aligned} \tag{35}$$

where the meaning of Z is obvious. We first concentrate on J_+Z.

$$\begin{aligned} J_+Z &= \sum_{l=1}^{\nu+1} z_l^2 \left(\sum_{j>l} \alpha_{lj} \frac{Z}{(z_l-z_j)} - \sum_{j<l} \alpha_{jl} \frac{Z}{(z_j-z_l)} + h_l \frac{Z}{z_l} \right) + 2k_l z_l Z \\ &= \sum_{l=1}^{\nu+1} \sum_{j=l+1}^{\nu+1} (z_l^2 - z_j^2) \alpha_{lj} \frac{Z}{z_l - z_j} + (2k_l + h_l) z_l Z \\ &= \sum_{l=1}^{\nu+1} \sum_{j=l+1}^{\nu+1} (z_l + z_j) \alpha_{lj} Z + (2k_l + h_l) z_l Z \\ &= \sum_{l=1}^{\nu+1} (2k_l + \alpha_l + h_l) z_l Z. \end{aligned}$$

Plugging in the explicit form for Z, we get :

$$\begin{aligned} J_+Z &= \left(\prod_{i<j} \frac{(z_i-z_j)^{\alpha_{ij}}}{\alpha_{ij}!} \right) \sum_{l=1}^{\nu+1} \binom{n}{h_1,\ldots,h_{\nu+1}} \left(\prod_{i=1}^{\nu+1} (2k_i)_{\alpha_i+h_i} \right) \\ &\times (2k_l + \alpha_l + h_l) \left(\prod_{i=1}^{\nu+1} z_i^{h_i} \right) z_l, \end{aligned}$$

and thus, when introducing the sum over the variables h

$$
\begin{aligned}
\sum_{|h|=n} J_+ Z &= \left(\prod_{i<j} \frac{(z_i - z_j)^{\alpha_{ij}}}{\alpha_{ij}!}\right) \sum_{|h|=n} \sum_{l=1}^{\nu+1} \binom{n}{h_1, \ldots, h_{\nu+1}} \\
&\quad \times \left(\prod_{i\neq l} (2k_i)_{\alpha_i+h_i}\right) (2k_l)_{\alpha_l+h_l+1} \left(\prod_{i\neq l} z_i^{h_i}\right) z_l^{h_l+1} \\
&= \left(\prod_{i<j} \frac{(z_i - z_j)^{\alpha_{ij}}}{\alpha_{ij}!}\right) \sum_{|h|=n+1} \sum_{l=1}^{\nu+1} \binom{n}{h_1, \ldots, h_l - 1, \ldots, h_{\nu+1}} \\
&\quad \times \prod_{i=1}^{\nu+1} (2k_i)_{\alpha_i+h_i} \prod_{i=1}^{\nu+1} z_i^{h_i} \\
&= \left(\prod_{i<j} \frac{(z_i - z_j)^{\alpha_{ij}}}{\alpha_{ij}!}\right) \sum_{|h|=n+1} \binom{n+1}{h_1, \ldots, h_{\nu+1}} \prod_{i=1}^{\nu+1} (2k_i)_{\alpha_i+h_i} \prod_{i=1}^{\nu+1} z_i^{h_i},
\end{aligned}
$$

where, in the last line we have used the fact that

$$
\sum_{l=1}^{\nu+1} \binom{n}{h_1, \ldots, h_l - 1, \ldots, h_{\nu+1}} = \binom{n+1}{h_1, \ldots, h_{\nu+1}}, \tag{36}
$$

when $|h| = n+1$. This proves (33). To fix the (absolute value of the) coefficient in (33), use the explicit action of J_+ and the knowledge of A^2 in (29). To fix the sign of this coefficient, use induction and

$$
T(k, 0) = \sum_{n_l+n_r=0} C^{k_l,k_r,k}_{n_l,n_r,0} \, \sigma(T_l(k_l, n_l) \otimes T_r(k_r, n_r)), \tag{37}
$$

and consider the coefficient of $z_1^{|j|}$ in the left and right hand side of this equation. $\square$

The uncoupled basis vectors $e^{(k_1)}_{n_1} \otimes \cdots \otimes e^{(k_{\nu+1})}_{n_{\nu+1}}$, in their realization (16), form an orthonormal basis in the Hilbert space $\mathcal{H}$ of analytic complex functions $f(z_1, \ldots, z_{\nu+1})$ on the multidisc $\mathcal{D} = (|z_1| < 1, \ldots, |z_{\nu+1}| < 1)$ with an inner product that is the generalization of (15) :

$$
\langle f_1, f_2 \rangle = \left(\prod_{i=1}^{\nu+1} \frac{2k_i - 1}{\pi}\right) \int_{\mathcal{D}} f_1 \overline{f_2} \left(\prod_{i=1}^{\nu+1} (1 - |z_i|^2)^{2k_i-2}\right) dx_1 dy_1 \cdots dx_{\nu+1} dy_{\nu+1}, \tag{38}
$$

provided all $k_i > 1/2$. As a consequence, we have

Corollary 2 *Let $T(k, n)$ and $T'(k', n')$ be two binary coupling schemes in the tensor product of $\nu + 1$ representations $\mathcal{D}_{k_1} \otimes \mathcal{D}_{k_2} \otimes \cdots \otimes \mathcal{D}_{k_{\nu+1}}$ with the same binary coupling tree. Then*

$$
\langle T(k, n), T'(k', n') \rangle = \delta_{j_1,j'_1} \cdots \delta_{j_\nu,j'_\nu} \delta_{n,n'},
$$

where the inner product is given by (38), $j_1, \cdots, j_\nu$ are the internal labels of $T(k,n)$, and $j'_1, \cdots, j'_\nu$ are those of $T'(k',n')$. For a fixed binary coupling tree, the binary coupling schemes $T(k,n)$ form an orthonormal basis in $\mathcal{H}$.

If $T(k,n)$ and $T'(k,n)$ are two binary coupling schemes with different underlying binary coupling trees, then $\langle T(k,n), T'(k,n)\rangle$ is usually defined as a generalized $3\nu j$-coefficient.

3 Coupled basis vectors as triple hypergeometric series

Upon inspecting the summations in (32) for specific binary coupling schemes, one finds that these summations can be rewritten in terms of multiple hypergeometric series. We consider one such example in this section, where a triple hypergeometric series [24, Section 1.5] appears. We use a notation close to that of Srivastava and Karlsson, and denote

$$F^{(3)}\left(\begin{matrix} a \\ e \end{matrix} :: \frac{b}{g}; \frac{b'}{g'}; \frac{b''}{g''}; \frac{c}{h}; \frac{c'}{h'}; \frac{c''}{h''}; x, y, z\right) = \sum_{m,n,p} \Lambda(m,n,p) \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{39}$$

with

$$\Lambda(m,n,p) = \frac{(a)_{m+n+p}(b)_{m+n}(b')_{n+p}(b'')_{p+m}(c)_m(c')_n(c'')_p}{(e)_{m+n+p}(g)_{m+n}(g')_{n+p}(g'')_{p+m}(h)_m(h')_n(h'')_p}. \tag{40}$$

In the previous a stands for $a_1, \ldots, a_A$ and $(a)_{m+n+p}$ denotes the product $(a_1)_{m+n+p} \cdots (a_A)_{m+n+p}$. The same applies for the other parameters of the $F^{(3)}$.

After some manipulations of the Pochhammer symbols and a reversal of one of the summation variables, we obtain for (7) :

$$\begin{aligned} e_n^{((k_1,k_2)k_{12},k_3)k} &= \frac{A}{\sqrt{n!(2k)_n}} (2k_1)_{j_{12}} (2k_2)_{j_{12}+j+n} (2k_3)_j (z_1 - z_2)^{j_{12}} (z_2 - z_3)^j z_2^n \\ &\times F^{(3)}\left(\begin{matrix} - \\ 1 - 2k_2 - j_{12} - j - n \end{matrix} :: \frac{2k_1 + j_{12}}{-}; \frac{-n}{-}; -; \frac{-j}{-}; -; \frac{2k_3 + j}{-}; \right. \\ &\qquad \left. \frac{z_1 - z_3}{z_2 - z_3}, \frac{z_1}{z_2}, \frac{z_3}{z_2}\right). \end{aligned} \tag{41}$$

The $F^{(3)}$ appearing in (41) is listed in [24, Table 4 – entry 13a], and was denoted as F_{13} resp. F_T by Lauricella [25] resp. Saran [26].

A similar expression for $e_n^{(k_1,(k_2,k_3)k_{23})k}$ is easily derived. Indeed, $e_n^{((k_2,k_3)k_{23},k_1)k}$ is immediately obtained from (41) by cyclic permutation of the indices 1, 2 and 3 (and hence replacing j by j'). Furthermore, $e_n^{(k_1,(k_2,k_3)k_{23})k}$ and $e_n^{((k_2,k_3)k_{23},k_1)k}$ only differ by a phase factor $(-1)^{j'}$. We thus find the following :

$$
\begin{aligned}
e_n^{(k_1,(k_2,k_3)k_{23})k} &= \frac{A'}{\sqrt{n!(2k)_n}}(2k_2)_{j_{23}}(2k_3)_{j_{23}+j'+n}(2k_1)_{j'}(z_2-z_3)^{j_{23}}(z_1-z_3)^{j'}z_3^n \\
&\times F^{(3)}\left({- \atop 1-2k_3-j_{23}-j'-n} :: {2k_2+j_{23} \atop -} ; {-n \atop -} ; - ; {-j' \atop -} ; - ; {2k_1+j' \atop -} ; \right. \\
&\qquad \left. \frac{z_1-z_2}{z_1-z_3}, \frac{z_2}{z_3}, \frac{z_1}{z_3} \right).
\end{aligned} \tag{42}
$$

Substituting the expressions (42), (14) and (41) in (13) gives an identity between (terminating) triple hypergeometric series. A simple renaming of the variables yields the following :

$$
\begin{aligned}
& F^{(3)}\left({- \atop a} :: {b \atop -} ; {-n \atop -} ; - ; {-j \atop -} ; - ; {c \atop -} ; x, y, \frac{y-x}{1-x} \right) = \sum_{l=0}^{m+j} (-1)^{m+j+n+l} \\
& \times \frac{(-m-j)_l(-a+b+c-m-j-n)_l(b-m)_{m+j+n}(1-x)^{j-l}x^l y^n}{l!(c-j+b-m+l-1)_l(1-c-b-l)_{m+j-l}(a)_{n+l}} \\
& \times {}_4F_3\left({-m, -l, c+b-m-j+l-1, -a+b-n-m-j \atop b-m, -a+b+c-n-m-j, -m-j} ; 1 \right) \\
& \times F^{(3)}\left({- \atop 1-b-n-j} :: {c-j+l \atop -} ; {-n \atop -} ; - ; {-m-j+l \atop -} ; - ; {1-a-n-l \atop -} ; \right. \\
& \qquad \left. \frac{1}{1-x}, \frac{y-x}{y(1-x)}, \frac{1}{y} \right).
\end{aligned} \tag{43}
$$

Using easy manipulations on single hypergeometric series, one proves the following transformation for the triple hypergeometric series on the right side :

$$
\begin{aligned}
& F^{(3)}\left({- \atop 1-b-n-j} :: {c-j+l \atop -} ; {-n \atop -} ; - ; {-m-j+l \atop -} ; - ; {1-a-n-l \atop -} ; \right. \\
& \left. \frac{1}{1-x}, \frac{y-x}{y(1-x)}, \frac{1}{y} \right) = \frac{(-1)^n(a+l)_n(1-b-c-l)_{m+j-l}(1-x)^{-m-j+l}y^{-n}}{(b+j)_n(1-b-j)_{m+j-l}} \\
& \times F^{(3)}\left({- \atop b+c-j-m+2l} :: {b-m+l \atop -} ; {-n, b+c+l \atop a+l} ; - ; \right. \\
& \left. {-m-j+l \atop -} ; - ; {c-j+l \atop -} ; x, y, \frac{y-x}{1-x} \right).
\end{aligned} \tag{44}
$$

This allows us to rewrite (43) as (where we have multiplied by the factor $(1-x)^m$) :

$$
\begin{aligned}
&(1-x)^m F^{(3)}\left(\begin{matrix} - \\ a \end{matrix} :: \begin{matrix} b \\ - \end{matrix} ; \begin{matrix} -n \\ - \end{matrix} ; - ; \begin{matrix} -j \\ - \end{matrix} ; - ; \begin{matrix} c \\ - \end{matrix} ; x, y, \frac{y-x}{1-x}\right) \\
&= \sum_{l=0}^{\infty} \frac{(-m-j)_l(-a+b+c-m-j-n)_l(b-m)_l x^l}{l!(c-j+b-m+l-1)_l(a)_l} \\
&\times {}_4F_3\left(\begin{matrix} -m, -l, c+b-m-j+l-1, -a+b-n-m-j \\ b-m, -a+b+c-n-m-j, -m-j \end{matrix} ; 1\right) \\
&\times F^{(3)}\left(\begin{matrix} - \\ b+c-j-m+2l \end{matrix} :: \begin{matrix} b-m+l \\ - \end{matrix} ; \begin{matrix} -n, b+c+l \\ a+l \end{matrix} ; - ;\right. \\
&\qquad\qquad \left.\begin{matrix} -m-j+l \\ - \end{matrix} ; - ; \begin{matrix} c-j+l \\ - \end{matrix} ; x, y, \frac{y-x}{1-x}\right). \qquad (45)
\end{aligned}
$$

Until now, the parameters j, m and n in (45), are assumed to be nonnegative integers, while a, b, and c are real numbers. Provided both sides converge, formula (45) is also valid for real j, m and n. Indeed, consider the coefficient of $x^p y^q$ on the left side of (45); this is :

$$
\begin{aligned}
&\sum_{h_1,h_2,h_3} (-1)^{h_3}(b)_{h_1+h_2}(-n)_{h_2+h_3}(-j)_{h_1}(c)_{h_3}(-m+h_3)_{p+q-h_1-h_2-h_3} \\
&\times(-h_3)_{q-h_2}/((a)_{h_1+h_2+h_3}h_1!h_2!h_3!(p+q-h_1-h_2-h_3)!(q-h_2)!). \quad (46)
\end{aligned}
$$

For nonnegative integer values of j, m and n, this is equal to the coefficient of $x^p y^q$ on the right side, by (45). But (46) is a rational function in j, m and n because the sum is terminating due to the appearance of the last two factorials in the denominator. It follows that (45) is also valid for real values of j, m and n.

Proposition 1 *Provided both the left and right hand side converge, we have*

$$
\begin{aligned}
&F^{(3)}\left(\begin{matrix} - \\ a \end{matrix} :: \begin{matrix} b \\ - \end{matrix} ; \begin{matrix} c \\ - \end{matrix} ; - ; \begin{matrix} d \\ - \end{matrix} ; - ; \begin{matrix} e \\ - \end{matrix} ; x, y, \frac{y-x}{1-x}\right) \\
&= \sum_{l=0}^{\infty} \frac{(d+f)_l(-a+b+c+d+e+f)_l(b+f)_l(1-x)^f x^l}{l!(b+d+e+f+l-1)_l(a)_l} \\
&\times {}_4F_3\left(\begin{matrix} f, -l, b+d+e+f+l-1, -a+b+c+d+f \\ b+f, -a+b+c+d+e+f, d+f \end{matrix} ; 1\right) \\
&\times F^{(3)}\left(\begin{matrix} - \\ b+d+e+f+2l \end{matrix} :: \begin{matrix} b+f+l \\ - \end{matrix} ; \begin{matrix} c, b+e+l \\ a+l \end{matrix} ; - ; \begin{matrix} d+f+l \\ - \end{matrix} ; - ;\right. \\
&\qquad\qquad \left.\begin{matrix} d+e+l \\ - \end{matrix} ; x, y, \frac{y-x}{1-x}\right). \qquad (47)
\end{aligned}
$$

For $c=0$, the two triple hypergeometric series in (47) reduce to the Gauss hypergeometric series and we get the special case :

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} b,d \\ a \end{matrix};x\right) &= \sum_{l=0}^{\infty} \frac{(d+f)_l(-a+b+d+e+f)_l(b+f)_l(1-x)^f x^l}{l!(b+d+e+f+l-1)_l(a)_l} \\
&\times {}_4F_3\left(\begin{matrix} f,-l,b+d+e+f+l-1,-a+b+d+f \\ b+f,-a+b+d+e+f,d+f \end{matrix};1\right) \\
&\times {}_2F_1\left(\begin{matrix} b+f+l,d+f+l \\ b+d+e+f+2l \end{matrix};x\right),
\end{aligned}
\tag{48}
$$

which is similar to Chaundy's formula [27].

4 Coupled basis vectors in $\mathfrak{su}(2)$

The Lie algebra $\mathfrak{su}(2)$ is generated by J_0, $J_\pm$ subject to the relations

$$
[J_0,J_\pm]=\pm J_\pm, \qquad [J_+,J_-]=2J_0,
\tag{49}
$$

with the conditions $J_0^\dagger=J_0$, $J_\pm^\dagger=J_\mp$. The finite-dimensional irreducible representations D_l are labelled by a positive halfinteger l (i.e. $2l$ is a nonnegative integer). The representation space D_l has orthonormal basis vectors $e_m^{(l)}$ with m running from $-l$ up to l in steps of 1; so the dimension is $2l+1$. The action of the $\mathfrak{su}(2)$ generators on these basis vectors is given by

$$
\begin{aligned}
J_0 e_m^{(l)} &= m e_m^{(l)}, \\
J_+ e_m^{(l)} &= \sqrt{(l-m)(l+m+1)}\, e_{m+1}^{(l)}, \\
J_- e_m^{(l)} &= \sqrt{(l+m)(l-m+1)}\, e_{m-1}^{(l)}.
\end{aligned}
\tag{50}
$$

These representations appear in the quantum theory of angular momentum, where the label l, resp. m of the basis vector $e_m^{(l)}$ stands for the total angular momentum, resp. its projection, of a particle or a system. The tensor product of representations then corresponds to the coupling of angular momenta. The tensor product of two such representation is given by [5,6]

$$
D_{l_1}\otimes D_{l_2} = \sum_{l=|l_1-l_2|}^{l_1+l_2} D_l.
\tag{51}
$$

The coefficients relating the coupled basis vectors to the uncoupled basis vectors are again called the Clebsch-Gordan coefficients of $\mathfrak{su}(2)$; their theory and properties are well known.

We consider here the following realization of $\mathfrak{su}(2)$ which is a formal analogue of the realization (17) of $\mathfrak{su}(1,1)$. Let l be a positive halfinteger,

$$J_0 = z\partial_z - l, \quad J_- = \partial_z, \quad J_+ = -z^2\partial_z + 2lz. \tag{52}$$

The realization of the basis vectors $e_m^{(l)}$ are given by :

$$e_m^{(l)} = \sqrt{\frac{(2l)!}{(l+m)!(l-m)!}}\, z^{l+m}. \tag{53}$$

Using the generating function of the Clebsch-Gordan coefficients $C^{l_1,l_2,l}_{m_1,m_2,m}$ in $\mathfrak{su}(2)$ [28, Section 8.8, Eq. 5], one determines an explicit form of the coupled vectors $e_m^{(l_1,l_2)l}$, where

$$e_m^{(l_1,l_2)l} = \sum_{m_1+m_2=m} C^{l_1,l_2,l}_{m_1,m_2,m}\; e^{(l_1)}_{m_1} \otimes e^{(l_2)}_{m_2}. \tag{54}$$

One finds :

$$\begin{aligned} e_m^{(l_1,l_2)l} = {} & \Delta(l_1,l_2,l)\sqrt{(2l_1)!(2l_2)!(l-m)!(l+m)!(2l+1)}\,\frac{(z_1-z_2)^{l_1+l_2-l}}{(l_1+l_2-l)!} \\ & \times \sum_{h_1+h_2=m+l} \frac{1}{(l_1-l_2+l-h_1)!(-l_1+l_2+l-h_2)!}\,\frac{z_1^{h_1}}{h_1!}\,\frac{z_2^{h_2}}{h_2!}, \end{aligned} \tag{55}$$

where, following [28],

$$\Delta(a,b,c) = \sqrt{\frac{(-a+b+c)!(a-b+c)!(a+b-c)!}{(a+b+c+1)!}}. \tag{56}$$

Our purpose here is to give the explicit expression for the realization of coupled vectors in the tensor product of $\nu+1$ representations of $\mathfrak{su}(2)$. The situation is very close to that of $\mathfrak{su}(1,1)$, so we shall not give any details of the proofs here. Furthermore, we shall also use the same terminology for binary coupling schemes and binary coupling trees.

To find the explicit form, note that (55) can be rewritten as

$$e_m^{(l_1,l_2)l} = \Delta(l_1,l_2,l)\sqrt{(2l_1)!(2l_2)!(l-m)!(l+m)!(2l+1)}$$

$$\times \sum_{\alpha_{12}=l_1+l_2-l} \frac{(z_1-z_2)^{\alpha_{12}}}{\alpha_{12}!} \sum_{h_1+h_2=m+l} \frac{1}{(2l_1-\alpha_1-h_1)!(2l_2-\alpha_2-h_2)!} \frac{z_1^{h_1}}{h_1!} \frac{z_2^{h_2}}{h_2!}, \tag{57}$$

with $\alpha_1 = \alpha_2 = \alpha_{12}$.

The main result is :

Theorem 3 *In the tensor product of $\nu+1$ representations $D_{l_1} \otimes D_{l_2} \otimes \cdots \otimes D_{l_{\nu+1}}$, the realization of a coupled vector defined by a binary coupling scheme $T(l,m)$, with internal labels j_i $(i=1,\ldots,\nu)$, is given by :*

$$\frac{T(l,m)}{\sqrt{(l-m)!(l+m)!}} = \prod_{i=1}^{\nu} \Delta(j_i) \sqrt{\prod_{i=1}^{\nu+1}(2l_i)! \prod_{i=1}^{\nu}(2j_i+1)}$$
$$\times \sum_{\alpha_{ij}\in V} \sum_{|h|=l+m} \prod_{1\le i<k\le \nu+1} \frac{(z_i-z_k)^{\alpha_{ik}}}{\alpha_{ik}!} \prod_{i=1}^{\nu+1} \frac{z_i^{h_i}}{h_i!(2l_i-\alpha_i-h_i)!}. \tag{58}$$

Herein, $|h| = h_1 + \cdots + h_{\nu+1}$, $\Delta(j_p) \equiv \Delta(left_p, right_p, j_p)$ is the function (56) applied to the internal representation label j_p and the representation labels of its left and right child, V is a set of ν linear constraints satisfied by the α_{ik}, and α_p is the sum over all α_{ik} with either i or k equal to p. The linear constraints are as follows : with each internal node with label j_p there is one constraint, namely the sum of all α_{ik} associated with the internal node j_p should be equal to $left_p + right_p - j_p$.

As an example, consider the tensor product of three representations, with a binary coupling scheme $T(l,m) = e_m^{((l_1,l_2)l_{12},l_3)l}$. We get the following :

$$\frac{e_m^{((l_1,l_2)l_{12},l_3)l}}{\sqrt{(l-m)!(l+m)!}} = \Delta(l_1,l_2,l_{12})\Delta(l_{12},l_3,l)\sqrt{(2l_1)!(2l_2)!(2l_3)!(2l_{12}+1)(2l+1)}$$
$$\times \sum_{\substack{\alpha_{12}=l_1+l_2-l_{12} \\ \alpha_{13}+\alpha_{23}=l_{12}+l_3-l}} \frac{(z_1-z_2)^{\alpha_{12}}}{\alpha_{12}!} \frac{(z_1-z_3)^{\alpha_{13}}}{\alpha_{13}!} \frac{(z_2-z_3)^{\alpha_{23}}}{\alpha_{23}!}$$
$$\times \sum_{h_1+h_2+h_3=m+l} \frac{1}{(2l_1-\alpha_1-h_1)!(2l_2-\alpha_2-h_2)!(2l_3-\alpha_3-h_3)!} \frac{z_1^{h_1}}{h_1!} \frac{z_2^{h_2}}{h_2!} \frac{z_3^{h_3}}{h_3!}. \tag{59}$$

Observe that this general result of Theorem 3 also has its use in quantum theory of angular momentum, as it gives quite explicitly the expansion of a coupled basis vector (or coupled state) in terms of uncoupled basis vectors (uncoupled states). Indeed, the expansion of (58) as a polynomial in the z_i

is quite straightforward; then using (53) this gives an expansion in terms of (tensor products of) the orthonormal basis vectors $e^{(l_i)}_{m_i}$ (usually denoted by $|j_i, m_i\rangle$ in the quantum theory context). We have observed that the computation of such an expansion is more efficient than the direct calculation using $\mathfrak{su}(2)$ Clebsch-Gordan coefficients.

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