Algebraic structures related to Racah doubles

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Abstract In [1], we classified all pairs of recurrence relations connecting two sets of Hahn, dual Hahn or Racah polynomials of the same type but with different parameters. We examine the algebraic relations underlying the Racah doubles and find that for a special case of Racah doubles with specific parameters this is given by the so-called Racah algebra.

1 Introduction

In [1], we classified all pairs of recurrence relations connecting two sets of Hahn, dual Hahn or Racah polynomials (classical finite and discrete hypergeometric orthogonal polynomials) of the same type but with different (shifted) parameters. This coupling of two polynomials was dubbed 'doubling' (yielding Hahn doubles, dual Hahn doubles and Racah doubles respectively) and it was seen to correspond to a specific pair of Christoffel-Geronimus transforms giving rise to a "new" system of orthogonal polynomials. By construction, for this new system the tridiagonal Jacobi matrix containing the coefficients of the three-term recurrence relation has zero diagonal. In the following, we will use the term 'two-diagonal' to refer to tridiagonal matrices with zero diagonal.

These Jacobi matrices could be interpreted as representation matrices of an algebra. For instance, the simplest case is that of the symmetric Krawtchouk polynomials. Here, the recurrence relation [8, (9.11.3)] for normalized Krawtchouk functions reads (for $n = 0, 1, \ldots, N$)

$$\sqrt{n(N-n+1)}\tilde{K}_{n-1}(x) + \sqrt{(n+1)(N-n)}\tilde{K}_{n+1}(x) = (N-2x)\tilde{K}_n(x), \qquad (1)$$ Kraw-recur

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where $\tilde{K}_n(x)$ stands for the normalized Krawtchouk functions $\tilde{K}_n(x) \sim K_n(x; \frac{1}{2}, N)$ which are scaled Krawtchouk polynomials $K_n(x; p, N)$ [8, 9] with $p = 1/2$. As the Krawtchouk polynomials are self-dual, the same relation holds when interchanging n and x. This bispectrality can then be encoded in an algebraic structure. Writing down (1) for $n = 0, 1, \ldots, N$, and putting this in matrix form, the coefficient matrix of the left hand side of (1) is just the two-diagonal $(N+1) \times (N+1)$ matrix

$$M = \begin{pmatrix} 0 & M_0 & & & \\ M_0 & 0 & M_1 & & \\ & M_1 & 0 & M_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2}$$ MK

with $M_n = \sqrt{(n+1)(N-n)}$. Here we recognise the action of J_+ and J_- appearing in the context of unitary representations of the Lie algebra $\mathfrak{su}(2)$. The simple two-diagonal structure of M makes it particularly interesting as a model for a finite quantum oscillator, namely the $\mathfrak{su}(2)$ oscillator model [3, 4].

Similarly, the bispectrality of the Racah polynomials is encoded in the algebraic structure known as the Racah algebra. The Racah algebra is a unital associative algebra over $\mathbb{C}$ with three generators K_1, K_2 and K_3 which obey the following relations in the generic presentation,

$$\begin{aligned} [K_1, K_2] &= K_3, \\ [K_2, K_3] &= a_2 K_2^2 + a_1\{K_1, K_2\} + c_1 K_1 + dK_2 + e_1, \\ [K_3, K_1] &= a_1 K_1^2 + a_2\{K_1, K_2\} + c_2 K_2 + dK_1 + e_2. \end{aligned} \tag{3}$$ alg-R

For the realization on the space of Racah polynomials, the matrix representation of the element K_2 is the tridiagonal Jacobi matrix of the Racah recurrence relation and K_1 is the diagonal matrix containing the quadratic expression $n(n+\alpha+\beta+1)$ for $n = 0, \ldots, N$ (which corresponds to the right hand side of the Racah difference equation). In this case the coefficients in (3) are functions of the Racah parameters $\alpha, \beta, \gamma, \delta$, given explicitly in [2]. This algebra also appears in the context of the Racah problem of $\mathfrak{su}(2)$ to derive the symmetry group of the $6j$-symbols [10].

We are now interested in the algebraic structures underlying the two-diagonal Jacobi matrices obtained through the doubling process. From [1], we observe that the dual Hahn and Racah doubles reduce to symmetric Krawtchouk polynomials when setting all parameters to trivial values and leaving no parameter unspecified. For general parameters, however, it is less trivial. In [1], the algebraic structures behind the matrices of the three cases of dual Hahn doubles were examined explicitly and were found to be extensions of $\mathfrak{su}(2)$. Such extensions consist of the addition of a parity operator P, while the standard $\mathfrak{su}(2)$ relations are altered to

$$\begin{aligned} &P^2 = 1, \qquad PJ_0 = J_0 P, \qquad PJ_\pm = -J_\pm P, \\ &[J_0, J_\pm] = \pm J_\pm, \\ &[J_+, J_-] = 2J_0 + 2(\gamma+\delta+1)J_0 P - (2j+1)(\gamma-\delta)P + (\gamma-\delta)I, \end{aligned} \tag{4}$$ alg-I

for the case **dual Hahn I**. The appearing parameters γ,δ here are the ones occurring also in the dual Hahn doubles, while the factor $(2j+1)$ can be interpreted as the addition of a central element. When $\gamma=\delta=-1/2$, the equations coincide with the $\mathfrak{su}(2)$ relations. These parameter values correspond to a reduction to the symmetric Krawtchouk case where the recurrence relations reduce to (1). The algebraic relations for the other dual Hahn cases are similar to (4) and can be found in [1, section 7]. While the general algebras have two parameters, special cases with only one parameter are of importance for the construction of finite oscillator models [6, 7].

The remaining question is of course whether the two-diagonal matrices arising from the Hahn and Racah doubles can also be interpreted as representation matrices of an algebra, and whether this is related to (an extension of) a known (Lie) algebra. In the current work, we shall consider a special case of a Racah double and show that for this special case the underlying algebraic structure is related to the Racah algebra. Hereto, in section 2 we will briefly summarize the definition and properties of the Racah polynomials.

2 Racah polynomials

Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ of degree n $(n=0,1,\ldots,N)$ in the variable $\lambda(x)=x(x+\gamma+\delta+1)$ are defined by [8, 9]

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta)={}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix};1\right),$$

where one of the denominator parameters should be $-N$. Herein, the function ${}_4F_3$ is the generalized hypergeometric series [5].

For the (discrete) orthogonality relation (depending on the choice of which parameter relates to $-N$) we refer to [8, (9.2.2)]. Under certain restrictions for the parameters, such that the weight function w and the squared norm h_n of the orthogonality relation are positive, we can define orthonormal Racah functions as follows

$$\tilde{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta)\equiv\sqrt{w(x;\alpha,\beta,\gamma,\delta)}\,R_n(\lambda(x);\alpha,\beta,\gamma,\delta)\;/\;\sqrt{h_n(\alpha,\beta,\gamma,\delta)}.$$

We now turn to a result obtained in [1, Appendix] corresponding to the case Racah I:

Proposition 1. *Let $\alpha+1=-N$, and suppose that $\gamma,\delta>-1$ and $\beta>N+\gamma$ or $\beta<-N-\delta-1$. Consider two $(2N+2)\times(2N+2)$ matrices U and M, defined as follows. U has elements ($n,x\in\{0,1,\ldots,N\}$):*

$$U_{2n,N-x}=U_{2n,N+x+1}=\frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta+1),$$
$$U_{2n+1,N-x}=-U_{2n+1,N+x+1}=-\frac{(-1)^n}{\sqrt{2}}\tilde{R}_n(\lambda(x);\alpha,\beta+1,\gamma+1,\delta);$$

M is the two-diagonal $(2N+2)\times(2N+2)$-matrix of the form (2) *with*

$$M_{2k} = \sqrt{\frac{(N-\beta-k)(\gamma+1+k)(N+\delta+1-k)(k+\beta+1)}{(N-\beta-2k)(2k-N+1+\beta)}},$$
$$M_{2k+1} = \sqrt{\frac{(\gamma+N-\beta-k)(k+1)(N-k)(k+\delta+\beta+2)}{(N-\beta-2k-2)(2k-N+1+\beta)}}. \qquad (5) \boxed{\texttt{M\_k-RI}}$$

Then U is orthogonal, and the columns of U are the eigenvectors of M, i.e. $MU = UD$, where D is a diagonal matrix containing the eigenvalues of M:

$$D = \mathrm{diag}(-\varepsilon_N,\dots,-\varepsilon_1,-\varepsilon_0,\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N), \quad \varepsilon_k = \sqrt{(k+\gamma+1)(k+\delta+1)}. \quad (6) \boxed{\texttt{eigenv-even}}$$

In short, the pair of polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta+1)$ and $R_n(\lambda(x);\alpha,\beta+1,\gamma+1,\delta)$ form a “Racah double”, and the relation $MU = UD$ governs the corresponding recurrence relations with M taking the role of a Jacobi matrix [1].

3 Algebraic structure

We start by taking M to be the Jacobi matrix of the first Racah double, i.e. the $(2N+2)\times(2N+2)$ matrix of the form (2) with entries given by (5). We now propose that M can be interpreted as the representation matrix of an algebra, and set out to determine the relations governing this algebra. A natural candidate for this algebraic structure is the Racah algebra (3). As such, we define $K_2 = M$. Inspired by the realization on the space of the ordinary Racah polynomials, we take the generator K_1 to be a diagonal matrix on the Racah double, containing a general quadratic expression in the degree n, say $K_1 = \mathrm{diag}[(pn^2+qn+r);n=0,\dots,2N+1]$. We set out to determine the coefficients p,q,r for which the commutation relations are for the form (3).

A direct computation shows that defining $K_3 = [K_1,K_2]$ we have the following commutation relation

$$[K_3,K_1] = -2p\{K_1,K_2\} + (p^2-q^2+4pr)K_2,$$

which is a quadratic relation of the same form as the Racah algebra relations (3). This relation actually holds for general matrices K_2 of the form (2) and K_1 as above, relying on the symmetric two-diagonal structure of K_2.

Next, we examine whether the remaining commutator $[K_2,K_3]$ can be cast in the same form as (3). However, in general this does not seem to be the case for K_1 of the above form. By direct computation we find that is possible, but only for specific values of the Racah parameters γ and δ, namely when both are equal to either $-1/2$ or $-N-3/2$. Only for one of these parameter values, the choice $q = 2p(\beta-N)$ yields the following relation

$$[K_2,K_3] = -2pK_2^2 - K_1 + (2N+1)(\beta+1)p + r\,.$$

There are two arbitrary constants left, namely p and r. If one chooses the value $p=-1/2$, we arrive at the following commutation relations for K_1,K_2 and K_3

$$\begin{aligned}
[K_1,K_2] &= K_3\\
[K_2,K_3] &= K_2^2 - K_1 - \frac{1}{2}(2N+1)(\beta+1) + r\\
[K_3,K_1] &= \{K_1,K_2\} + (\frac{1}{4} - (\beta-N)^2 - 2r)K_2.
\end{aligned}$$

Note that by means of the constant r, corresponding just to an affine transformation of K_1, one can fix the coefficient of K_2 in the third relation.

For the $(2N+1)\times(2N+1)$ matrices of another Racah double, namely Racah III [1, Appendix] similar results hold, only yielding the Racah algebra for either $\gamma=\delta=-1/2$ or $\gamma=\beta-1/2$ and $\delta=-\beta-1/2$. For the special parameter values $\gamma=\delta=-1/2$ the Jacobi matrices of those two Racah doubles can actually be unified in a single expression, matching the appropriate Racah double for even and for odd dimensions. For this special case, the spectrum of these matrices also reduces to equidistant integers or half-integers, as seen from (6), making them interesting candidates for finite quantum oscillator models. We will pursue the in-depth study of this oscillator model and the unification of the two Racah doubles in a separate paper.

We believe that for general parameter values of the Racah doubles the algebraic structure is also related to the Racah algebra. The matrix representation of K_1, however, will be more complicated and should follow from investigating the bispectrality of these polynomials. Only for the specific parameter values obtained here, does K_1 reduce to the same form as for the ordinary Racah polynomials. We intend to analyse this in further work.

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