A class of unitary irreducible representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$

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Abstract

Using the equivalence of the defining relations of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ to the defining triple relations of n pairs of parabose operators $b_i^{\pm}$ we construct a class of unitary irreducible (infinite-dimensional) lowest weight representations $V(p)$ of $\mathfrak{osp}(1|2n)$. We introduce an orthogonal basis of $V(p)$ in terms of Gelfand-Zetlin patterns, where the subalgebra $\mathfrak{u}(n)$ of $\mathfrak{osp}(1|2n)$ plays a crucial role and we present explicit actions of the $\mathfrak{osp}(1|2n)$ generators.

Following some physical ideas we construct a class of infinite dimensional unitary irreducible representations of the Lie superalgebra (LS) $\mathfrak{osp}(1|2n)$ [1] in an explicit form. In 1953 Green [2] introduced the so called parabose operators (PBOs) $b_j^{\pm}$ $(j = 1, 2, \ldots)$ satisfying

$$[\{b_j^{\xi}, b_k^{\eta}\}, b_l^{\epsilon}] = (\epsilon - \xi)\delta_{jl} b_k^{\eta} + (\epsilon - \eta)\delta_{kl} b_j^{\xi}, \tag{1}$$

$j, k, l \in \{1, 2, \ldots\}$ and $\eta, \epsilon, \xi \in \{+, -\}$ (to be interpreted as $+1$ and -1 in the algebraic expressions $\epsilon - \xi$ and $\epsilon - \eta$) as a generalization of the ordinary Bose operators. The Fock space $V(p)$ of n pairs of PBOs is a Hilbert space with a vacuum $|0\rangle$, defined by means of $(j, k = 1, 2, \ldots, n)$

$$\langle 0|0\rangle = 1, \quad b_j^-|0\rangle = 0, \quad (b_j^{\pm})^{\dagger} = b_j^{\mp}, \quad \{b_j^-, b_k^+\}|0\rangle = p\,\delta_{jk}\,|0\rangle, \tag{2}$$

and by irreducibility under the action of the algebra spanned by the elements b_j^+, b_j^- $(j = 1, \ldots, n)$, subject to (1). The parameter p is referred to as the order of the paraboson system. However the structure of the parabose Fock space is not known, also a proper basis has not been introduced. We solve these problems using the relation between n pairs of PBOs and the defining relations of the LS $\mathfrak{osp}(1|2n)$, discovered by Ganchev and Palev [3]. The orthosymplectic superalgebra $\mathfrak{osp}(1|2n)$ [1] consists of matrices of the form

$$\left(\begin{array}{ccc} 0 & a & a_1 \\ a_1^t & b & c \\ -a^t & d & -b^t \end{array}\right), \tag{3}$$

where a and a_1 are $(1 \times n)$-matrices, b is any $(n \times n)$-matrix, and c and d are symmetric $(n \times n)$-matrices. The even elements have $a = a_1 = 0$ and the odd elements are those with

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$b = c = d = 0$. Denote the row and column indices running from 0 to $2n$ and by e_{ij} the matrix with zeros everywhere except a 1 on position (i, j). Then as a basis in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(1|2n)$ consider

$$h_j = e_{jj} - e_{n+j,n+j} \qquad (j = 1, \ldots, n). \tag{4}$$

In terms of the dual basis δ_j of $\mathfrak{h}^*$, the root vectors and corresponding roots of $\mathfrak{osp}(1|2n)$ are given by:

$$\begin{array}{ll} e_{0,k} - e_{n+k,0} \leftrightarrow -\delta_k, & e_{0,n+k} + e_{k,0} \leftrightarrow \delta_k, \quad k = 1, \ldots, n, \text{ odd}, \\ e_{j,n+k} + e_{k,n+j} \leftrightarrow \delta_j + \delta_k, & e_{n+j,k} + e_{n+k,j} \leftrightarrow -\delta_j - \delta_k, \ j \leq k = 1, \ldots, n, \text{ even}, \\ e_{j,k} - e_{n+k,n+j} \leftrightarrow \delta_j - \delta_k, & j \neq k = 1, \ldots, n, \text{ even}. \end{array}$$

Introduce the following multiples of the odd root vectors

$$b_k^+ = \sqrt{2}(e_{0,n+k} + e_{k,0}), \qquad b_k^- = \sqrt{2}(e_{0,k} - e_{n+k,0}) \qquad (k = 1, \ldots, n). \tag{5}$$

Then the following holds [3]

Theorem 1 (Ganchev and Palev). *As a Lie superalgebra defined by generators and relations, $\mathfrak{osp}(1|2n)$ is generated by $2n$ odd elements $b_k^\pm$ subject to the parabose relations (1).*

From (4) and (5) it follows that $\{b_j^-, b_j^+\} = 2h_j, j = 1, \ldots, n$ and using (2) we have:

Corollary 2. *The parabose Fock space $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(1|2n)$ with lowest weight $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$.*

We can construct the representation $V(p)$ [4] using an induced module construction with an appropriate chain of subalgebras.

Proposition 3. *A basis for the even subalgebra $\mathfrak{sp}(2n)$ of $\mathfrak{osp}(1|2n)$ is given by the $2n^2 + n$ elements $\{b_j^\pm, b_k^\pm\}, 1 \leq j \leq k \leq n$, $\{b_j^+, b_k^-\}, 1 \leq j, k \leq n$. The n^2 elements $\{b_j^+, b_k^-\}, j, k = 1, \ldots, n$ are a basis for the $\mathfrak{sp}(2n)$ subalgebra $\mathfrak{u}(n)$.*

The subalgebra $\mathfrak{u}(n)$ can be extended to a parabolic subalgebra $\mathcal{P} = \text{span}\{\{b_j^+, b_k^-\}, b_j^-, \{b_j^-, b_k^-\}, j, k = 1, \ldots, n\}$ [4] of $\mathfrak{osp}(1|2n)$. Recall that $\{b_j^-, b_k^+\}|0\rangle = p\,\delta_{jk}\,|0\rangle$, with $\{b_j^-, b_j^+\} = 2h_j$. Then the space spanned by $|0\rangle$ is a trivial one-dimensional $\mathfrak{u}(n)$ module $\mathbb{C}|0\rangle$ of weight $(\frac{p}{2}, \ldots, \frac{p}{2})$. Since $b_j^-|0\rangle = 0$, the module $\mathbb{C}|0\rangle$ can be extended to a one-dimensional $\mathcal{P}$ module. Now we define the induced $\mathfrak{osp}(1|2n)$ module $\overline{V}(p)$ with lowest weight $(\frac{p}{2}, \ldots, \frac{p}{2})$: $\overline{V}(p) = \text{Ind}_{\mathcal{P}}^{\mathfrak{osp}(1|2n)}\mathbb{C}|0\rangle$.

By the Poincaré-Birkhoff-Witt theorem [1, 4], it is easy to give a basis for $\overline{V}(p)$:

$$(b_1^+)^{k_1} \cdots (b_n^+)^{k_n} (\{b_1^+, b_2^+\})^{k_{12}} (\{b_1^+, b_3^+\})^{k_{13}} \cdots (\{b_{n-1}^+, b_n^+\})^{k_{n-1,n}} |0\rangle,$$

where $k_1, \ldots, k_n, k_{12}, k_{13}, \ldots, k_{n-1,n} \in \mathbb{Z}_+$. However in general $\overline{V}(p)$ is not a simple module and let $M(p)$ be the maximal nontrivial submodule of $\overline{V}(p)$. Then the simple module (irreducible module), corresponding to the paraboson Fock space, is $V(p) = \overline{V}(p)/M(p)$. The purpose is now to determine the vectors belonging to $M(p)$ and also to find explicit matrix elements of the $\mathfrak{osp}(1|2n)$ generators $b_j^\pm$ in an appropriate basis of $V(p)$.

From the basis in $\overline{V}(p)$, it is easy to write down the character of $\overline{V}(p)$:

$$\text{char}\,\overline{V}(p) = \frac{(x_1 \cdots x_n)^{p/2}}{\prod_{i=1}^n (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_j x_k)}. \tag{6}$$

Such expressions have an interesting expansion in terms of Schur functions.

Proposition 4 (Cauchy, Littlewood)**.** *Let $x_1, \ldots, x_n$ be a set of n variables. Then [5]*

$$\frac{1}{\prod_{i=1}^{n}(1-x_i)\prod_{1\leq j<k\leq n}(1-x_jx_k)} = \sum_{\lambda} s_\lambda(x_1,\ldots,x_n) = \sum_{\lambda} s_\lambda(x). \tag{7}$$

Herein the sum is over all partitions λ and $s_\lambda(x)$ is the Schur symmetric function [6].

The characters of finite dimensional $\mathfrak{u}(n)$ representations are given by such Schur functions $s_\lambda(x)$. For such finite dimensional $\mathfrak{u}(n)$ representations labelled by a partition λ, there is a known basis: the Gelfand-Zetlin basis (GZ) [7]. We shall use the $\mathfrak{u}(n)$ GZ basis vectors as our new basis for $\overline{V}(p)$. Thus the new basis of $\overline{V}(p)$ consists of vectors of the form

$$|m) \equiv |m)^n \equiv \left| \begin{array}{lllll} m_{1n} & \cdots & \cdots & m_{n-1,n} & m_{nn} \\ m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & \\ \vdots & \cdot^{\cdot^{\cdot}} & & & \\ m_{11} & & & & \end{array} \right) = \left| \begin{array}{l} [m]^n \\ |m)^{n-1} \end{array} \right), \tag{8}$$

where the top line of the pattern, also denoted by the n-tuple $[m]^n$, is any partition λ (consisting of non increasing nonnegative numbers) with $\ell(\lambda) \leq n$. The label p itself is dropped in the notation of $|m)$. The remaining $n-1$ lines of the pattern will sometimes be denoted by $|m)^{n-1}$. So all m_{ij} in the above GZ-pattern are nonnegative integers, satisfying the *betweenness conditions* $m_{i,j+1} \geq m_{ij} \geq m_{i+1,j+1}, 1 \leq i \leq j \leq n-1$. Note that, since the weight of $|0\rangle$ is $(\frac{p}{2},\ldots,\frac{p}{2})$, the weight of the above vector is determined by

$$h_k|m) = \left(\frac{p}{2} + \sum_{j=1}^{k} m_{jk} - \sum_{j=1}^{k-1} m_{j,k-1}\right)|m). \tag{9}$$

The triple relations (1), imply that $(b_1^+, b_2^+, \ldots, b_n^+)$ is a standard $\mathfrak{u}(n)$ tensor of rank $(1,0,\ldots,0)$. Therefore we can attach a unique GZ-pattern with top line $10\cdots 0$ to every b_j^+, corresponding to the weight $+\delta_j$. Explicitly:

$$b_j^+ \sim \begin{array}{l} 10\cdots 000 \\ 10\cdots 00 \\ \cdots \\ 0\cdots 0 \\ \cdots \\ 0 \end{array}, \tag{10}$$

where the pattern consists of $j-1$ zero rows at the bottom, and the first $n-j+1$ rows are of the form $10\cdots 0$. The tensor product rule in $\mathfrak{u}(n)$ reads $([m]^n)\otimes(10\cdots 0) = ([m]^n_{+1})\oplus([m]^n_{+2})\oplus \cdots \oplus([m]^n_{+n})$ where $([m]^n) = (m_{1n}, m_{2n}, \ldots, m_{nn})$ and a subscript $\pm k$ indicates an increment of the kth label by ± 1: $([m]^n_{\pm k}) = (m_{1n}, \ldots, m_{kn} \pm 1, \ldots, m_{nn})$. A general matrix element of b_j^+ can now be written as follows:

$$(m'|b_j^+|m) = \left(\begin{array}{l} [m]^n_{+k} \\ |m')^{n-1} \end{array} \right| b_j^+ \left| \begin{array}{l} [m]^n \\ |m)^{n-1} \end{array} \right) = \left(\begin{array}{l} [m]^n \\ |m)^{n-1} \end{array} ; \begin{array}{l} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \right| \left. \begin{array}{l} [m]^n_{+k} \\ |m')^{n-1} \end{array} \right) ([m]^n_{+k}||b^+||[m]^n).$$

The first factor in the right hand side is a $\mathfrak{u}(n)$ Clebsch-Gordan coefficient [8], the second factor is a reduced matrix element. By the tensor product rule, the first line of $|m')$ has to be $[m']^n = [m]^n_{+k}$ for some k-value.

The special $\mathfrak{u}(n)$ CGCs appearing here are well known, and have fairly simple expressions. They can be found, e.g. in [8, 9]. The actual problem is now converted into finding expressions for the reduced matrix elements, i.e. for the functions $F_k([m]^n)$, for arbitrary n-tuples of non increasing nonnegative integers $[m]^n = (m_{1n}, m_{2n}, \dots, m_{nn})$:

$$F_k([m]^n) = F_k(m_{1n}, m_{2n}, \dots, m_{nn}) = ([m]^n_{+k}||b^+||[m]^n). \tag{11}$$

So one can write:

$$b_j^+|m) = \sum_{k,m'} \left(\begin{array}{c} [m]^n \\ |m)^{n-1} \end{array} ; \begin{array}{c} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \middle| \begin{array}{c} [m]^n_{+k} \\ |m')^{n-1} \end{array} \right) F_k([m]^n) \left| \begin{array}{c} [m]^n_{+k} \\ |m')^{n-1} \end{array} \right), \tag{12}$$

$$b_j^-|m) = \sum_{k,m'} \left(\begin{array}{c} [m]^n_{-k} \\ |m')^{n-1} \end{array} ; \begin{array}{c} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{array} \middle| \begin{array}{c} [m]^n \\ |m)^{n-1} \end{array} \right) F_k([m]^n_{-k}) \left| \begin{array}{c} [m]^n_{-k} \\ |m')^{n-1} \end{array} \right). \tag{13}$$

For $j = n$, the CGCs in (12)-(13) take a simple form [8], and we have

$$b_n^+|m) = \sum_{i=1}^{n} \left(\frac{\prod_{k=1}^{n-1}(m_{k,n-1} - m_{in} - k + i - 1)}{\prod_{k\neq i=1}^{n}(m_{kn} - m_{in} - k + i)} \right)^{1/2} F_i(m_{1n}, m_{2n}, \dots, m_{nn})\ |m)_{+in}; \tag{14}$$

$$b_n^-|m) = \sum_{i=1}^{n} \left(\frac{\prod_{k=1}^{n-1}(m_{k,n-1} - m_{in} - k + i)}{\prod_{k\neq i=1}^{n}(m_{kn} - m_{in} - k + i + 1)} \right)^{1/2} F_i(m_{1n}, \dots, m_{in} - 1, \dots, m_{nn})\ |m)_{-in}. \tag{15}$$

In order to determine the n unknown functions F_k, one can start from the following action:

$$\{b_n^-, b_n^+\}|m) = 2h_n|m) = (p + 2(\sum_{j=1}^{n} m_{jn} - \sum_{j=1}^{n-1} m_{j,n-1}))|m). \tag{16}$$

Expressing the left hand side by means of (14)-(15), one finds a system of coupled recurrence relations for the functions F_k. Taking the appropriate boundary conditions into account, we have been able to solve this system of relations [9].

Proposition 5. *The reduced matrix elements F_k appearing in the actions of $b_j^\pm$ on vectors $|m)$ of $\overline{V}(p)$ are given by:*

$$F_k(m_{1n}, m_{2n}, \dots, m_{nn}) = (-1)^{m_{k+1,n}+\cdots+m_{nn}}(m_{kn} + n + 1 - k + \mathcal{E}_{m_{kn}}(p - n))^{1/2} \\ \times \prod_{j\neq k=1}^{n} \left(\frac{m_{jn} - m_{kn} - j + k}{m_{jn} - m_{kn} - j + k - \mathcal{O}_{m_{jn}-m_{kn}}} \right)^{1/2}, \tag{17}$$

where $\mathcal{E}$ and $\mathcal{O}$ are the even *and* odd *functions defined by $\mathcal{E}_j = 1$ if j is even and 0 otherwise, $\mathcal{O}_j = 1$ if j is odd and 0 otherwise.*

The proof consists of verifying that all triple relations (1) hold when acting on any vector $|m)$. Each such verification leads to an algebraic identity in n variables $m_{1n}, \dots, m_{nn}$. In these computations, there are some intermediate verifications: e.g. the action $\{b_j^+, b_k^-\}|m)$ should leave the top row of the GZ-pattern $|m)$ invariant (since $\{b_j^+, b_k^-\}$ belongs to $\mathfrak{u}(n)$). Furthermore, it must yield (up to a factor 2) the known action of the standard $\mathfrak{u}(n)$ matrix elements E_{jk} in the classical GZ-basis. Consider now the factor $(m_{kn} + n + 1 - k + \mathcal{E}_{m_{kn}}(p - n))$ in the expression of $F_k([m]^n)$. This is the only factor in the right hand side of (17) that may become zero. If this factor is zero or negative, the assigned vector $|m')$ belongs to $M(p)$. Recall that the integers m_{jn} satisfy $m_{1n} \geq m_{2n} \geq \cdots \geq m_{nn} \geq 0$. If $m_{kn} = 0$ (its smallest possible value), then this factor in F_k takes the value $(p - k + 1)$. So the p-values $1, 2, \dots, n - 1$ play a special role leading to the following result [9]:

Theorem 6. *The* $\mathfrak{osp}(1|2n)$ *representation* $V(p)$ *with lowest weight* $(\frac{p}{2},\ldots,\frac{p}{2})$ *is a unirrep if and only if* $p \in \{1,2,\ldots,n-1\}$ *or* $p > n-1$*. For* $p > n-1$*,* $V(p) = \overline{V}(p)$ *and* $\mathrm{char}\, V(p) = \frac{(x_1\cdots x_n)^{p/2}}{\prod_i(1-x_i)\prod_{j<k}(1-x_jx_k)} = (x_1\cdots x_n)^{p/2}\sum_\lambda s_\lambda(x)$*. For* $p \in \{1,2,\ldots,n-1\}$*,* $V(p) = \overline{V}(p)/M(p)$ *with* $M(p) \neq 0$*. The structure of* $V(p)$ *is determined by* $\mathrm{char}\, V(p) = (x_1\cdots x_n)^{p/2}\sum_{\lambda,\ \ell(\lambda)\leq p} s_\lambda(x)$ *where* $\ell(\lambda)$ *is the* length *of the partition* λ*.*

The explicit action of the $\mathfrak{osp}(1|2n)$ generators in $V(p)$ is given by (12)-(13), and the basis is orthogonal and normalized. For $p \in \{1,2,\ldots,n-1\}$ this action remains valid, provided one keeps in mind that all vectors with $m_{p+1,n} \neq 0$ must vanish.

Note that the first line of Theorem 6 can also be deduced from [10, 11], where all unirreps of $\mathfrak{osp}(1|2n)$ are classified.

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